Parametric quadratic programming, revisited

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14th December 2004

1 Introduction

In this paper, we consider the parametric quadratic programming problem, namely to

$$\begin{array}{rll} \mathrm{QP}(\theta): & \underset{x\in\mathbb{R}^n}{\text{minimize}} & q(x,\theta) = \frac{1}{2}x^THx + x^T(g+\theta\delta g) \\ & \text{subject to} & A_{\mathcal{E}}x = b_{\mathcal{E}} + \theta\delta b_{\mathcal{E}} \\ & \text{and} & A_{\mathcal{I}}x \geq b_{\mathcal{I}} + \theta\delta b_{\mathcal{I}} \end{array}$$

for all $\theta \in [0,\theta^{\mathrm{U}}]$. Here H is symmetric, $A_{\mathcal{E}}$ is full rank, $\mathcal{E} = \{1,\ldots,m_{\mathcal{E}}\}$ and $\mathcal{I} = \{m_{\mathcal{E}}+1,\ldots,m_{\mathcal{E}}+m_{\mathcal{I}}\}$, and any/all of $b_{\mathcal{I}}$, $\delta b_{\mathcal{I}}$ and θ^{U} may be infinite—for consistency we require that each pair $(b_{\mathcal{I}},\delta b_{\mathcal{I}})$ be finite or infinite together. Simple bound constraints

$$x^{\mathrm{L}} + \theta\delta x^{\mathrm{L}} \leq x \leq x^{\mathrm{U}} + \theta\delta x^{\mathrm{U}}$$

are certainly allowed, but for simplicity of exposition, these will be treated as general constraints—well-designed software will always exploit their special Jacobian structure. Likewise, two-sided inequality constraints

$$b_{\mathcal{I}}^{\mathrm{L}} + \theta\delta b_{\mathcal{I}}^{\mathrm{L}} \leq A_{\mathcal{I}}x \leq b_{\mathcal{I}}^{\mathrm{U}} + \theta\delta b_{\mathcal{I}}^{\mathrm{U}}$$

naturally fit into the general format given above, but would be handled specially by good software. However, since we do not insist that H be positive definite, we will only be concerned with parametric local solutions to QP(θ). We do not presume that QP(θ) has a solution for all $\theta \in [0,\theta^{\mathrm{U}}]$, but for simplicity we will assume that it has one when $\theta = 0$.

This work is heavily based on the algorithm proposed by Best [1][1] in the convex (that is H positive semi-definite) case. We have already broadly described how such a method may be extended in the non-convex case [3]), but here give the details. In particular, we describe how we cope with degeneracies encountered as the parametric solution evolves.

Our aim is to describe the fortran 90 package PQP from the GALAHAD library [4].

[1]This originally appeared as a University of Waterloo technical report, CORR 82-24, in 1982.

Notation

For any given $v \in \mathbb{R}^p$ and subset $\mathcal{S} \subseteq \{1, \ldots p\}$, we denote the sub-vector of v whose indices lie in $\mathcal{S}$ by $v_{\mathcal{S}}$; if v_k is dependent on an iteration counter k, we write $v_{k,\mathcal{S}}$ for the appropriate sub-vector. Similarly, if M (or M_k) $\in \mathbb{R}^{p \times n}$, $M_{\mathcal{S}}$ (or $M_{k,\mathcal{S}}$) will be the sub-matrix whose rows are indexed by $\mathcal{S}$. For brevity, we write $g(\theta) = g + \theta\delta g$ and $b(\theta) = b + \theta\delta b$, where $(b, \delta b)$ has components $(b_i, \delta b_i)$, $i \in \mathcal{E} \cup \mathcal{I}$.

2 Parametric QP

Let $x(\theta)$ be a (local) solution of QP(θ), and let

$$\mathcal{A}_\theta = \{i \in \mathcal{I} \mid a_i^T x(\theta) = b_i + \theta\delta b_i\} \;\text{ and }\; \mathcal{I}_\theta = \mathcal{I} \setminus \mathcal{A}_\theta$$

be the *active* and *inactive* sets of inequality constraints at $x(\theta)$. Suppose that a *working* set $\mathcal{W}_\theta \subseteq \mathcal{A}_\theta$ is chosen so that $\{a_i\}$, $i \in \mathcal{L}_\theta \stackrel{\text{def}}{=} \mathcal{E} \cup \mathcal{W}_\theta$, are linearly independent, and that $x(\theta)$ and Lagrange multipliers $y(\theta)$ satisfy the first-order optimality (KKT) conditions

$$\begin{pmatrix} H & A_{\mathcal{L}_\theta}^T \\ A_{\mathcal{L}_\theta} & 0 \end{pmatrix} \begin{pmatrix} x(\theta) \\ -y_{\mathcal{L}_\theta}(\theta) \end{pmatrix} = \begin{pmatrix} -g(\theta) \\ b_{\mathcal{L}_\theta}(\theta) \end{pmatrix} \tag{2.1}$$

along with

$$a_i^T x(\theta) \geq b_i(\theta) \;\text{ for all }\; i \in \mathcal{I}_\theta \;\text{ and }\; y_i(\theta) \geq 0 \;\text{ for all }\; i \in \mathcal{W}_\theta; \tag{2.2}$$

the multipliers $y_i(\theta), i \in \mathcal{I} \setminus \mathcal{W}_\theta$, for the remaining inequality constraints are zero. Suppose, furthermore that the second-order optimality condition

$$u^T H u > 0 \;\text{ for all vectors }\; u \neq 0 \;\text{ such that }\; A_{\mathcal{L}_\theta} u = 0 \tag{2.3}$$

holds.

Given a KKT pair $(x(\theta_k), y(\theta_k))$ for some $\theta_k \in [0, \theta^{\text{U}})$, we now investigate how $(x(\theta), y(\theta))$ evolves for $\theta > \theta_k$. For brevity, we write $(x_k, y_k, g_k, b_k) \stackrel{\text{def}}{=} (x(\theta_k), y(\theta_k), g(\theta_k), b(\theta_k))$, $A_k = A_{\mathcal{L}_k}$, $\mathcal{W}_k \stackrel{\text{def}}{=} \mathcal{W}_{\theta_k}$, $\mathcal{I}_k \stackrel{\text{def}}{=} \mathcal{I}_{\theta_k}$ and $\mathcal{L}_k \stackrel{\text{def}}{=} \mathcal{L}_{\theta_k}$.

We shall assume, for the time being, that

A1 $\quad a_i^T x_k > b_{k,i}$ for all $i \in \mathcal{I}_k$ and $y_{k,i} > 0$ for all $i \in \mathcal{W}_k$

and that

A2 $\quad$ the second-order condition (2.3) holds at θ_k.

Then $\mathcal{W}_\theta = \mathcal{W}_k$ (and consequently $\mathcal{L}_\theta = \mathcal{L}_k$) and

$$\begin{pmatrix} x(\theta) \\ y_{\mathcal{L}_\theta}(\theta) \end{pmatrix} = \begin{pmatrix} x_k \\ y_{k,\mathcal{L}_k} \end{pmatrix} + (\theta - \theta_k) \begin{pmatrix} \delta x_k \\ \delta y_{k,\mathcal{L}_k} \end{pmatrix}, \tag{2.4}$$

where

$$\begin{pmatrix} H & A_k^T \\ A_k & 0 \end{pmatrix} \begin{pmatrix} x_k \\ -y_{k,\mathcal{L}_k} \end{pmatrix} = \begin{pmatrix} -g_k \\ b_{k,\mathcal{L}_k} \end{pmatrix} \tag{2.5}$$

and

$$\begin{pmatrix} H & A_k^T \\ A_k & 0 \end{pmatrix} \begin{pmatrix} \delta x_k \\ -\delta y_{k,\mathcal{L}_k} \end{pmatrix} = \begin{pmatrix} -\delta g \\ \delta b_{\mathcal{L}_k} \end{pmatrix}, \tag{2.6}$$

so long as the primal feasibility requirements

$$a_i^T x(\theta) \geq b_i(\theta) \;\text{ for all }\; i \in \mathcal{I}_k \tag{2.7}$$

and the dual ones

$$y_i(\theta) \geq 0 \;\text{ for all }\; i \in \mathcal{W}_k \tag{2.8}$$

all hold.

Let $\Delta\theta = \theta - \theta_k$ and define

$$\Delta\theta_{i,k} \stackrel{\text{def}}{=} \begin{cases} \dfrac{b_{k,j} - a_i^T x_k}{a_i^T \delta x_k - \delta b_i} & \text{if } \; a_i^T \delta x_k < \delta b_i \;\text{ and }\; i \in \mathcal{I}_k, \\ -\dfrac{y_{k,i}}{\delta y_{k,i}} & \text{if } \; \delta y_{k,i} < 0 \;\text{ and }\; i \in \mathcal{W}_k, \;\text{ and} \\ \infty & \text{otherwise.} \end{cases}$$

Then (2.7) is satisfied so long as $\Delta\theta \leq \min_{i\in\mathcal{I}_k} \Delta\theta_{i,k}$ while the same is true of (2.8) provided that $\Delta\theta \leq \min_{i\in\mathcal{W}_k} \Delta\theta_{i,k}$. It follows from **A1** that

$$\Delta\theta_k \stackrel{\text{def}}{=} \min_{i\in\mathcal{I}_k\cup\mathcal{W}_k} \Delta\theta_{i,k} > 0$$

and that (2.4) provides a KKT point for QP(θ) for all $\theta \in [\theta_k, \theta_{k+1}]$, where $\theta_{k+1} = \theta_k + \Delta\theta_k$. Our task is thus to investigate how the parametric solution changes once $\Delta\theta$ exceeds $\Delta\theta_k$; of course if $\Delta\theta_k = \infty$ or indeed $\theta_{k+1} > \theta^{\text{U}}$ we have reached the end of the required parametric interval and need look no further.

In what follows we shall assume, at least for the time being, that

$\boxed{\textbf{A3}}$ $\quad j = \arg \min_{i\in\mathcal{I}_k\cup\mathcal{W}_k} \Delta\theta_{i,k}$ is unique.

2.1 Adding a constraint to the working set

Firstly, suppose that $j \in \mathcal{I}_k$, and thus that constraint j becomes active at θ_{k+1}, i.e., $a_j^T x_{k+1} = b_{k+1,j}$, where we have written $(x_{k+1}, y_{k+1}, g_{k+1}, b_{k+1}) \stackrel{\text{def}}{=} (x(\theta_{k+1}), y(\theta_{k+1}), g(\theta_{k+1}), b(\theta_{k+1}))$. Since this constraint would be violated if $x(\theta)$ took the form (2.4) and $\theta > \theta_{k+1}$, it follows that constraint j must be added to the working set.

To do so, we require that

$$K_{k+1} = \begin{pmatrix} H & A_k^T & a_j \\ A_k & 0 & 0 \\ a_j^T & 0 & 0 \end{pmatrix}$$

is invertible, or equivalently that the Schur complement

$$u_k^T H u_k \neq 0, \quad \text{where} \quad \begin{pmatrix} H & A_k^T \\ A_k & 0 \end{pmatrix} \begin{pmatrix} u_k \\ v_{k,\mathcal{L}_k} \end{pmatrix} = \begin{pmatrix} a_j \\ 0 \end{pmatrix}. \tag{2.9}$$

But since then $A_k u_k = 0$, the second-order optimality condition (2.3) implies that K_{k+1} is invertible if and only if $u_k \neq 0$.

2.1.1 The case $u_k \neq 0$

If $u_k \neq 0$, constraint j may be added to the working set, and the first and second-order optimaility conditions (2.1)–(2.3) will continue to be satisfied at θ_{k+1}. Let $\mathcal{W}_{k+1} = \mathcal{W}_k \cup \{j\}$, $\mathcal{L}_{k+1} = \mathcal{L}_k \cup \{j\}$ and $\mathcal{I}_{k+1} = \mathcal{I}_k \setminus \{j\}$. Then (2.4)–(2.6) (with $k+1$ replacing k) continue to hold so long as

$$a_i^T x(\theta) \geq b_i(\theta) \text{ for all } i \in \mathcal{I}_{k+1} \tag{2.10}$$

and

$$y_i(\theta) \geq 0 \text{ for all } i \in \mathcal{W}_{k+1} \tag{2.11}$$

remain true. Continuity of $(x(\theta), y(\theta))$ at θ_{k+1} and **A3** imply that (2.10) holds for some open interval containing θ_{k+1}, and that the same is true for (2.11) for $i \in \mathcal{W}_k$. In addition, the Lagrange multiplier for the added constraint is

$$y_j(\theta) = y_j(\theta_{k+1}) + (\theta - \theta_{k+1})\delta y_{k+1,j} = (\theta - \theta_{k+1})\delta y_{k+1,j} \tag{2.12}$$

But subtracting (2.4) for the $k+1$ case from that for the k one gives

$$\begin{pmatrix} H & A_k^T & a_j \\ A_k & 0 & 0 \\ a_j^T & 0 & 0 \end{pmatrix} \begin{pmatrix} \delta x_{k+1} - \delta x_k \\ \delta y_{k,\mathcal{L}_k} - \delta y_{k+1,\mathcal{L}_k} \\ -\delta y_{k+1,j} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \delta b_j - a_j^T \delta x_k \end{pmatrix},$$

from which it follows that

$$\delta y_{k+1,j} = \frac{(\delta x_{k+1} - \delta x_k)^T H (\delta x_{k+1} - \delta x_k)}{\delta b_j - a_j^T \delta x_k}, \quad \text{where} \quad A_k(\delta x_{k+1} - \delta x_k) = 0. \tag{2.13}$$

Since (2.3) ensures that $(\delta x_{k+1} - \delta x_k)^T H (\delta x_{k+1} - \delta x_k) \geq 0$ and as $\delta b_j > a_j^T \delta x_k$ because $\Delta\theta_k < \infty$, it follows from (2.13) that $\delta y_{k,j} > 0$, and thus from (2.12) that $y_j(\theta) > 0$ for all $\theta > \theta_{k+1}$. Thus both (2.10) and (2.11) hold for some open interval containing θ_{k+1}. Finally **A2** continues to hold at θ_{k+1} since the null-space of A_{k+1} is contained in that of A_k.

2.1.2 The case $u_k = 0$ and $v_{k,\mathcal{W}_k} \leq 0$

If $u_k = 0$, a_j is linearly dependent on A_k, and (2.9) gives

$$A_k^T v_{k,\mathcal{L}_k} = a_j. \tag{2.14}$$

In particular, since

$$\begin{aligned} A_k(x_{k+1} + (\theta - \theta_{k+1})\delta x_{k+1}) &= b_{k+1,\mathcal{L}_k} + (\theta - \theta_{k+1})\delta b_{\mathcal{L}_k} \\ \text{and } \; a_j^T(x_{k+1} + (\theta - \theta_{k+1})\delta x_{k+1}) &< b_{k+1,j} + (\theta - \theta_{k+1})\delta b_j \end{aligned}$$

for $\theta > \theta_{k+1}$,

$$\begin{aligned} v_{k,\mathcal{L}_k}^T(b_{k+1,\mathcal{L}_k} + (\theta - \theta_{k+1})\delta b_{\mathcal{L}_k}) &= v_{k,\mathcal{L}_k}^T A_k(x_{k+1} + (\theta - \theta_{k+1})\delta x_{k+1}) \\ &= a_j^T(x_{k+1} + (\theta - \theta_{k+1})\delta x_{k+1}) < b_{k+1,j} + (\theta - \theta_{k+1})\delta b_j \end{aligned} \tag{2.15}$$

for all such θ.

Now suppose that $v_{k,\mathcal{W}_k} \leq 0$, and that x is a feasible point for QP(θ) for some $\theta > \theta_{k+1}$. Then in particular

$$A_{\mathcal{E}}x = b_{k+1,\mathcal{E}} + (\theta - \theta_{k+1})\delta b_{\mathcal{E}} \;\text{ and }\; A_{\mathcal{W}_k}x \geq b_{k+1,\mathcal{W}_k} + (\theta - \theta_{k+1})\delta b_{\mathcal{W}_k},$$

and, on multiplying by $v_{k,\mathcal{L}_k}$,

$$a_j^T x = v_{k,\mathcal{L}_k}^T A_k x \leq v_{k,\mathcal{L}_k}^T(b_{k+1,\mathcal{L}_k} + (\theta - \theta_{k+1})\delta b_{\mathcal{L}_k}).$$

But then (2.15) implies that

$$a_j^T x < b_{k+1,j} + (\theta - \theta_k)\delta b_j$$

which contradicts the assumption that x is feasible. Thus the parametric solution ends at θ_{k+1} whenever $v_{k,\mathcal{W}_k} \leq 0$.

2.1.3 The case $u_k = 0$ and $v_{k,i} > 0$ for some $i \in \mathcal{W}_k$

The KKT conditions at θ_{k+1} imply that

$$Hx_{k+1} + g_{k+1} = A_k^T y_{\mathcal{L}_k}(\theta_{k+1}),$$

where **A3** implies that $y_{\mathcal{L}_k}(\theta_{k+1}) > 0$. But then it follows from (2.14) that

$$Hx_{k+1} + g_{k+1} = A_k^T(y_{\mathcal{L}_k}(\theta_{k+1}) - \lambda v_{k,\mathcal{L}_k}) + \lambda a_j.$$

Hence

$$\begin{pmatrix} y_{\mathcal{L}_k}(\theta_{k+1}) - \lambda v_{k,\mathcal{L}_k} \\ \lambda \end{pmatrix}$$

are valid alternative Lagrange multipliers at θ_{k+1} for all $\lambda \in [0, \lambda_{k+1}]$, where

$$\lambda_{k+1} \stackrel{\text{def}}{=} \min_{i \in \mathcal{W}_k} \lambda_{i,k+1} > 0 \;\text{ and }\; \lambda_{i,k+1} \stackrel{\text{def}}{=} \begin{cases} \dfrac{y_{k+1,i}}{v_{k,i}} & \text{if } \; v_{k,i} > 0 \; \text{ and } \; i \in \mathcal{W}_k, \;\text{ and} \\ \infty & \text{otherwise.} \end{cases}$$

To proceed, we shall assume, for the time being, that

$\boxed{\textbf{A4}}$ $\ell = \arg\min_{i\in\mathcal{W}_k} \lambda_{i,k+1}$ is unique.

We now show that it is possible to continue the parametric solution beyond θ_{k+1} by selecting $\mathcal{W}_{k+1} = \mathcal{W}_k \cup \{\ell\} \setminus \{j\}$, and choosing Lagrange multipliers

$$y_{k+1,i} = \begin{cases} y_i(\theta_{k+1}) - \lambda_{k+1} v_{k,i} & \text{if } i \in \mathcal{W}_k, \\ \lambda_{k+1} & \text{if } i = j, \text{ and} \\ 0 & \text{otherwise.} \end{cases} \tag{2.16}$$

To do so, first note that $\{a_i\}$, $i \in \mathcal{L}_{k+1}$, are linearly independent as $v_{k,\ell} \neq 0$. Secondly, (2.4)–(2.6) (with $k+1$ replacing k) give valid primal-dual solutions for $\theta > \theta_{k+1}$ so long as (2.7) and (2.8) continue to hold. Continuity of $(x(\theta), y(\theta))$ at θ_{k+1} and **A3** imply that (2.10) automatically holds for all $i \in \mathcal{I}_{k+1} \setminus \{\ell\}$ in some open interval containing θ_{k+1}, while **A4** and (2.16) guarantees that the same is true for (2.11) for all $i \in \mathcal{W}_{k+1}$. Thus it remains to show that

$$a_\ell^T x(\theta) \geq b_\ell(\theta) \tag{2.17}$$

in such an interval.

It follows immediately from (2.14) that

$$\begin{aligned} & a_\ell^T (x_{k+1} + (\theta - \theta_{k+1})\delta x_{k+1}) = \\ & \quad \frac{1}{v_{k,\ell}} \left[a_j^T (x_{k+1} + (\theta - \theta_{k+1})\delta x_{k+1}) - \sum_{i\in\mathcal{L}_k\setminus\{\ell\}} v_{k,i} a_i^T (x_{k+1} + (\theta - \theta_{k+1})\delta x_{k+1}) \right]. \end{aligned} \tag{2.18}$$

Since

$$a_i^T (x_{k+1} + (\theta - \theta_{k+1})\delta x_{k+1}) = b_{k+1,i} + (\theta - \theta_{k+1})\delta b_i \ \text{ for all } \ i \in \mathcal{L}_k \setminus \{\ell\},$$

(2.15) and (2.18) imply that

$$\begin{aligned} & a_\ell^T (x_{k+1} + (\theta - \theta_{k+1})\delta x_{k+1}) = \\ & \quad \frac{1}{v_{k,\ell}} \left[b_{k+1,j} + (\theta - \theta_{k+1})\delta b_j - v_{k,\mathcal{L}_k}^T (b_{k+1,\mathcal{L}_k} + (\theta - \theta_{k+1})\delta b_{\mathcal{L}_k}) \right] + b_{k+1,\ell} + (\theta - \theta_{k+1})\delta b_\ell \\ & \quad > b_{k+1,\ell} + (\theta - \theta_{k+1})\delta b_\ell \end{aligned}$$

for $\theta > \theta_{k+1}$, which gives (2.17). Finally, since A_k and A_{k+1} have the same null-space, **A2** continues to hold at θ_{k+1}.

2.2 Removing a constraint from the working set

Now suppose that $j \in \mathcal{W}_k$, and thus that the Lagrange multiplier for active constraint j becomes zero at θ_{k+1}, i.e., $y_j(\theta_{k+1}) = 0$. Since this constraint would be inactive if $y(\theta)$ took the form (2.4) and $\theta > \theta_{k+1}$, it follows that constraint j should be removed from the working set.

The second-order requirement (2.3) will continue to be satisfied once constraint j is removed if and only if

$$(0 \quad e_j^T)\begin{pmatrix} H & A_k^T \\ A_k & 0 \end{pmatrix}^{-1}\begin{pmatrix} 0 \\ e_j \end{pmatrix} < 0$$

[2, Lemma 7.2], or more succinctly if $w_{k,j} > 0$, where

$$\begin{pmatrix} H & A_k^T \\ A_k & 0 \end{pmatrix}\begin{pmatrix} t_k \\ w_{k,\mathcal{L}_k} \end{pmatrix} = -\begin{pmatrix} 0 \\ e_j \end{pmatrix}.$$

2.2.1 The case $w_{k,j} > 0$

If $w_{k,j} > 0$, constraint j may be removed from the working set—that is, $\mathcal{W}_{k+1} = \mathcal{W}_k \setminus \{j\}$, $\mathcal{L}_{k+1} = \mathcal{L}_k \setminus \{j\}$ and $\mathcal{I}_{k+1} = \mathcal{I}_k \cup \{j\}$—and the first and second-order optimaility conditions (2.1)–(2.3) will continue to be satisfied at θ_{k+1}. Then, as before, (2.4)–(2.6) (with $k+1$ replacing k) continue to hold so long as (2.10) and (2.11) remain true.

Continuity of $(x(\theta), y(\theta))$ at θ_{k+1} and **A3** imply that (2.11) holds for some open interval containing θ_{k+1}, and that the same is true for (2.10) for $i \in \mathcal{W}_k$. In addition

$$a_j^T x(\theta) - b_j(\theta) = (\theta - \theta_{k+1})(a_j^T \delta x_{k+1} - \delta b_j) \tag{2.19}$$

for $\theta \geq \theta_{k+1}$. But subtracting (2.4) for the $k+1$ case from that for the k one gives

$$\begin{pmatrix} H & A_{k+1}^T & a_j \\ A_{k+1} & 0 & 0 \\ a_j^T & 0 & 0 \end{pmatrix}\begin{pmatrix} \delta x_k - \delta x_{k+1} \\ \delta y_{k+1,\mathcal{L}_k} - \delta y_{k,\mathcal{L}_k} \\ \delta y_{k,j} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \delta b_j - a_j^T \delta x_{k+1} \end{pmatrix},$$

from which it follows that

$$a_j^T \delta x_{k+1} - \delta b_j = \frac{(\delta x_k - \delta x_{k+1})^T H (\delta x_k - \delta x_{k+1})}{(-\delta y_{k,j})}, \quad \text{where} \quad A_{k+1}(\delta x_k - \delta x_{k+1}) = 0. \tag{2.20}$$

Since (2.3) ensures that $(\delta x_k - \delta x_{k+1})^T H (\delta x_k - \delta x_{k+1}) \geq 0$ and as $\delta y_{k,j} < 0$ because $\Delta\theta_k < \infty$, it follows from (2.20) that $a_j^T \delta x_{k+1} > \delta b_j 0$, and thus from (2.19) that $a_j^T x(\theta) > b_j(\theta)$ for all $\theta > \theta_{k+1}$. Thus both (2.10) and (2.11) hold for some open interval containing θ_{k+1}.

2.2.2 The case $w_{k,j} \leq 0$

In this case it is unclear whether $x_k + \Delta\theta_k \delta x_k$ solves QP(θ_{k+1}) since the second-order sufficiency condition (2.3) is violated. Our remedy is simply to pick a θ_{k+2} very slightly larger than θ_{k+1}, to resolve the problem from scratch for this parameter, and then to retrace the parametric solution back from θ_{k+2} to θ_{k+1}.

We note in passing that it is possible to deal more effectively with the case $w_{k,j} = 0$ when H is positive semi-definite [1]. In particular, under these circumstances the analysis in Section 2.2.1 also holds so long as $w_{k,\mathcal{L}_k} \neq 0$. But if $w_{k,\mathcal{L}_k} = 0$, $x_k + \Delta\theta_k \delta x_k$ is not the unique solution to QP(θ_{k+1}) and either a (traceable) discontinuity of the parametric solution occurs or the problem is unbounded for all $\theta > \theta_{k+1}$.

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