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I. S. Duff[2], S. Gratton[3], X. Pinel[3] and X. Vasseur[3]

ABSTRACT

We study methods for the numerical solution of the Helmholtz equation for two-dimensional applications in geophysics. The common framework of the iterative methods in our study is a combination of an inner iteration with a geometric multigrid method used as a preconditioner and an outer iteration with a Krylov subspace method. The preconditioning system is based on either a pure or shifted Helmholtz operator. A multigrid iteration is used to approximate the inverse of this operator. The proposed solution methods are evaluated on a complex benchmark in geophysics involving highly variable coefficients and high wavenumbers. We compare this preconditioned iterative method with a direct method and a hybrid method that combines our iterative approach with a direct method on a reduced problem. We see that the hybrid method outperforms both the iterative and the direct approach.

Keywords: Helmholz, hybrid solver, direct method, iterative method, preconditioning, multigrid.

AMS(MOS) subject classifications: 65F05, 65F50.

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[1]Current reports available by anonymous ftp to `ftp.numerical.rl.ac.uk` in directory pub/reports. This report is in files duffRAL2007002.pdf and duffRAL2007002.ps.gz. The report also available through the URL `www.numerical.rl.ac.uk/reports/reports.html`.

[2]i.s.duff@rl.ac.uk. This work was supported by the EPSRC Grant GR/S42170.

[3]CERFACS, 42 Ave G Coriolis, Toulouse 31057, France.

Computational Science and Engineering Department
Atlas Centre
Rutherford Appleton Laboratory
Oxon OX11 0QX

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Contents

1 Introduction

The target application of this study is a frequency-domain migration in seismics (Claerbout 1985). At a given frequency, a source is triggered at a certain position on the Earth's surface. As a consequence, a pressure wave propagates from the source. When a wave encounters discontinuities of elastic or density moduli between layers, it is scattered and propagated back to the surface. The pressure field is then recorded at several receiver locations located on the Earth's surface. This experimental process is repeated over a given range of frequencies. The main target of the numerical simulation is thus to reproduce these wave propagation phenomena occurring in the heterogeneous medium. This leads to an interpretative map of the subsoil that helps to detect both the location and the thickness of the reflecting layers. The resulting frequency-domain problem is then solved using, for example, techniques similar to those discussed in this paper. After that is done, a fast Fourier transformation is employed to deduce the time-domain solution from the frequency-domain solutions. This time-domain solution is of great importance in oil exploration for predicting correctly the structure of the subsurface. For practical applications, this requires the accurate computation of the wave propagation in an inhomogeneous medium. The wave propagation is modelled by the Helmholtz equation with absorbing boundary conditions. A key point for an efficient migration is thus a robust and fast solution method for the Helmholtz problem described in this paper both for large wavenumbers and for highly variable coefficients due to strong variations in velocities in the inhomogeneous medium.

The finite-difference discretization of the Helmholtz problem at high wavenumbers leads to a linear system $Ax = b$ where A is a large sparse matrix. This matrix is complex symmetric, indefinite, and generally ill-conditioned. For some years there has been considerable interest in multigrid methods (Brandt 1977, Trottenberg, Oosterlee and Schüller 2000) for Helmholtz problems (see for example, Elman, Ernst and O'Leary, 2001, Kim and Kim, 2002 and the references therein). Nevertheless the indefiniteness of the Helmholtz problem has prevented multigrid methods from being as efficient as for symmetric positive-definite problems. Multigrid methods encounter difficulties both in the smoothing procedure and in the coarse grid correction. On the one hand, standard smoothers become unstable for indefinite problems due to amplification of smooth error components. On the other hand, on coarse or very coarse meshes, the approximation of the discrete Helmholtz operator is relatively poor and this creates a difficulty for the coarse grid correction. Remedies have been proposed and analysed for homogeneous problems (Elman et al. 2001, Kim and Kim 2002). Recently a novel multigrid method has been proposed for the numerical solution of the Helmholtz equation (Erlangga, Oosterlee and Vuik 2006). The multigrid method is not directly applied to the discrete Helmholtz operator but to a complex shifted one. This shift avoids both the indefiniteness and the coarse grid correction problems (Erlangga et al. 2006). Thus it is possible to build a robust multigrid method with standard multigrid components that is used as a preconditioner for a Krylov subspace method. This solution method has been evaluated on model and realistic geophysical applications involving highly variable coefficients and relatively high wavenumbers. Nevertheless the complexity of the solution method for pure Helmholtz problems was found to be relatively high, see for example the recent analysis of a realistic dataset in geophysics (Riyanti, Erlangga, Plessix, Mulder, Vuik and Oosterlee 2006).

Direct methods have also been considered for the numerical solution of the Helmholtz equation in geophysics (Hustedt, Operto and Virieux 2004). They are currently limited by memory requirements especially for three-dimensional applications, although current algorithmic developments (use of out-of-core techniques) will soon extend such methods to much larger problems. The main idea of our work is to propose an algorithm where direct and iterative methods can be combined to solve the large sparse linear systems coming from geophysical applications. A combination that will be specifically considered is the use of a Krylov method preconditioned by a two-grid cycle involving a direct method. This two-grid cycle will approximate the inverse of the original unshifted Helmholtz operator. Our intention is to use a two-level hierarchy to avoid both smoothing and coarse grid correction difficulties and simultaneously to benefit from the robustness and computational efficiency of modern sparse direct solvers.

The goal of this paper is thus to present a numerical comparison between the robust multigrid method proposed in (Erlangga et al. 2006) and a hybrid approach of a direct-iterative procedure on a complex dataset in geophysics. The outline of the paper is as follows. In Section 2, the Helmholtz problem is introduced. We then describe the iterative solution methods: the multigrid method used as a preconditioner proposed by Erlangga et al. (2006) and the two-level hybrid preconditioner. Numerical experiments on a two-dimensional heterogeneous problem are presented in Section 3. Conclusions and perspectives are presented in Section 4.

2 Solution of the Helmholtz equation for seismic wave propagation

2.1 Problem setting

We consider the Helmholtz equation for a wave propagation problem in a two-dimensional region Ω:

$$-\Delta u - (1 - i\,\alpha)\,k^2 u = f \quad \text{in} \quad \Omega \tag{2.1}$$

satisfying either the first-order radiation condition:

$$\frac{\partial u}{\partial n} - iku = 0 \quad \text{on} \quad \delta\Omega \tag{2.2}$$

or the second-order radiation condition:

$$\frac{\partial u}{\partial n} - iku - \frac{i}{2k}\frac{\partial^2 u}{\partial^2 \tau} = 0 \quad \text{on} \quad \delta\Omega \tag{2.3}$$

with u the pressure wavefield, f the source term, n the unit outward normal to $\delta\Omega$, τ the unit tangent to $\delta\Omega$ and i the imaginary unit ($i^2 = -1$). The wavenumber is defined as $k = \frac{2\pi f}{c}$ where f is the frequency and c is the speed of sound. Note that for an inhomogeneous medium c is space-dependent (for example, Figure 3.1). Thus the wavenumber k is also space-dependent. The positive real coefficient α corresponds to the fraction of damping in the medium. For geophysical applications, α can be as large as 0.05. The wavelength l is defined as $l = \frac{c}{f}$.

We consider a standard finite-difference discretization of the Helmholtz equation (2.1) using an $O(h^2)$ 5-point discretization scheme (Cohen 2002). The resulting matrix A has complex entries due to the discrete boundary operator (2.2) or (2.3) and the damping mechanism in (2.1), if any. This leads to a sparse complex symmetric matrix, whose order is large for high wavenumbers. This large order is mainly due to an accuracy requirement for standard second order discretizations; for example, the number of points per wavelength $n_w = \frac{c}{h\,f}$ should be at least 10 to 12 (Harari and Turkel 1995, Cohen 2002) where h denotes the stepsize. Using the lowest value for the speed of sound $c = 1500\,m/s$ from the Marmousi problem presented in Section 3.1 (see also Figure 3.1 for the velocity profile) the condition $n_w > 12$ is fulfilled when the product of the dimensional stepsize h and the frequency f is equal to 120. This would give a value of 4 m for h if f was 30 Hz which is small relative to the domain dimensions of 9192 m by 2904 m.

2.2 Iterative Solution methods

The general framework for both iterative solution methods is first introduced. Then we present the robust multigrid method introduced by Erlangga et al. (2006) and a two-grid procedure combining iterative and direct methods.

2.2.1 General framework: multigrid used as a preconditioner

Our common framework for the iterative methods is a Krylov subspace approach. We will use preconditioning from the right. Each preconditioning step implies the solution of the preconditioning system $M\,\psi = d$ where M is an approximation to the discrete Helmholtz operator A. One cycle of a geometric multigrid method is used to approximate the inverse of M. Let C^{-1} denote this approximation. The convergence of the Krylov subspace method is thus related to the spectrum of the matrix $A\,C^{-1}$. If only one cycle of multigrid is performed, the iteration matrix of the preconditioning phase is equal to the iteration matrix of the multigrid procedure, that is:

$$T = (I - C^{-1}\,M) \quad \text{or} \quad C^{-1}\,M = I - T \tag{2.4}$$

where T denotes the multigrid iteration matrix (Trottenberg et al. 2000). When only two levels are used for the multigrid, it is possible to write T as:

$$T = S^{\nu_2} \, (I \; - P \, \overline{M}^{-1} \, R \, M) \, S^{\nu_1} \tag{2.5}$$

where S denotes the iteration matrix of the smoothing procedure on the fine grid, ν_1 and ν_2 the number of pre- and post-relaxations, P, R and $\overline{M}$ the prolongation from the coarse to the fine grid, the restriction from the fine to coarse grid and the coarse grid operator respectively. The multigrid iteration matrix for a hierarchy with more than two levels can be deduced recursively from the relation (2.5) (Trottenberg et al. 2000). From equation (2.4) the following relation can be deduced:

$$A \, C^{-1} = A \, (I - T) \, M^{-1} \, . \tag{2.6}$$

2.2.2 Multigrid method for complex shifted Helmholtz

Recently a robust geometric multigrid preconditioned Krylov subspace method was proposed for the solution of the two-dimensional Helmholtz problem for geophysical applications (Erlangga et al. 2006, Riyanti et al. 2006). The resulting method is a combination of an inner iteration involving a geometric multigrid method with an outer iteration involving a Krylov subspace method, namely BiCGSTAB. The key point of this novel solution method lies in the choice of the operator M that is used for the preconditioning. It is not chosen as the discrete Helmholtz operator but rather as the following complex operator (called a complex shifted Helmholtz operator):

$$-\Delta u - (\beta_1 - i\beta_2) \, k^2 u = f \quad \text{in} \quad \Omega \tag{2.7}$$

with real-valued β_1 and β_2 parameters. Boundary conditions for the preconditioning operator are set to either (2.2) or (2.3). The choice $(\beta_1, \beta_2) = (1., 0.5)$ ensures that the spectrum of $A \, C^{-1}$ is a favourably clustered for the convergence of the Krylov subspace method as was shown empirically by Erlangga et al. (2006) on model problems. Due to the complex shift, it is possible to avoid both smoothing and coarse grid correction difficulties as shown by Erlangga et al. (2006) who design a convergent multigrid method even at high wavenumbers for problems with homogeneous or heterogeneous coefficients. For problems with heterogeneous coefficients, a Galerkin coarse grid discretization is used to build the coarse grid operators of the multigrid hierarchy recursively. The coarse grid operator $\overline{M}$ is defined as:

$$\overline{M} = R \, M \, P. \tag{2.8}$$

Operator-dependent prolongation of de Zeeuw (1990) and Dendy Jr. (1983) will be used here, whereas full-weighting is used as a restriction operator for the Galerkin coarse grid discretization. Note that R and P are not adjoint to each other. As a consequence, the coarse grid operator is no longer complex symmetric. With this coarse grid construction it is still possible to use point relaxation methods to obtain a convergent geometric multigrid based method when dealing with variable coefficients. As advocated by Erlangga et al. (2006), the following choice of multigrid components leads to a robust multigrid preconditioner: a Galerkin coarse grid discretization with operator-dependent prolongation and full weighting restriction operators, with the smoothing procedure being based on damped Jacobi relaxation (with underrelaxation parameter $\omega = 0.5$). As a preconditioning step, one F-cycle of multigrid with $\nu_1 = 1$ and $\nu_2 = 1$ pre- and postrelaxations (noted F(1,1)) is used. This combination leads to a fixed preconditioner and BiCGSTAB (van der Vorst 1992) was used by Erlangga et al. (2006). As stated by Riyanti et al. (2006) this combination of multigrid used as a preconditioner allowed convergence for complex models with relatively high frequencies.

2.2.3 Hybrid two-level preconditioner

If the preconditioning operator M corresponds to the original Helmholtz operator A, it can be deduced from relation (2.6) that the spectrum of AC^{-1} is the same as that of $I - T$. An obvious idea would be to choose the multigrid components (smoothers on all levels, restriction and prolongation, number of relaxations) to minimize the spectral radius of the multigrid method. The immediate consequence would be a highly clustered spectrum around $(1., 0.)$ in the complex plane that is favourable for a Krylov subspace acceleration. The simplest strategy is adopted here: a two-grid procedure is presented where a direct solution method is used to solve the coarse problems. Both smoothing and coarse grid correction difficulties are avoided since the coarse grid is fine enough to represent the solution well.

The next question is how to choose the appropriate components of the two-grid procedure. For model problems with constant coefficients a local Fourier analysis (Trottenberg et al. 2000) can help to obtain estimates of this spectral radius and choose the right components of the two-grid cycle. However, for problems with strongly variable coefficients one must resort to numerical experiments to evaluate the potential effectiveness. This will be described and analysed in Section 3.

3 Applications

Two right preconditioned Krylov subspace methods for the solution of the Helmholtz problem are evaluated on a complex geophysical problem that we will present in Section 3.1. This application involves relatively large wavenumbers and heterogeneous velocities. We considered both BiCGSTAB and restarted GMRES (Saad and Schultz 1986). The implementation of GMRES presented by Frayssé, Giraud, Gratton and Langou (2005) has been used[1]. A restart value of 5 was chosen empirically. A larger value did not improve the convergence of GMRES significantly. Note that the memory cost of GMRES(5) is less than that of BiCGSTAB. A zero initial guess was used in all cases. Finally, the iterations

[1]Software available at http://www.cerfacs.fr/algor/Softs/GMRES/index.html

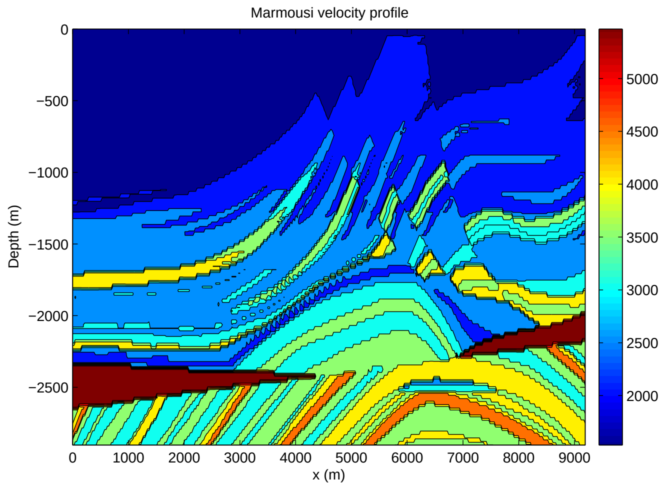

Figure 3.1: Velocity profile for the Marmousi dataset.

are terminated as soon as the norm of the residual at iteration k, $r^{(k)} = b - A\,x^{(k)}$, is reduced by 6 orders of magnitude in the L_2 discrete norm that is:

$$\frac{\|r^{(k)}\|_{2,h}}{\|r^{(0)}\|_{2,h}} \leq 10^{-6}\,. \tag{3.9}$$

3.1 Presentation of the Marmousi problem

The Marmousi dataset is a two-dimensional synthetic dataset generated by the Institut Français du Pétrole (IFP) (Bourgeois, Bourget, Lailly, Poulet, Ricarte and Versteeg 1991). The geometry and velocity models were created to produce complex seismic data which requires advanced processing techniques to obtain a correct earth image. The domain is rectangular of size $9192 \times 2904\,m^2$. A point source is located at $x = 6000\,m$ and 12 metres below the upper surface. The speed of sound is very inhomogeneous over the domain; see Figure 3.1. The velocities range from 1500 m/s to 5500 m/s. In this study, we will consider three different frequencies: 10, 20 and 30 Hz. The gridsize has been chosen such that the minimum number of points per wavelength n_w is greater than 12. This yields regular grids with stepsizes, h, of 12, 6, and 4 for frequencies 10, 20 and 30 Hz respectively. The resulting dimensions for the grids are thus 767×243, 1533×485 and 2299×727.

Figure 3.2 depicts the real and imaginary part of the pressure wavefield for a given frequency equal to 30Hz without any damping.

3.2 Numerical results

Sequential computations have been performed on a IBM pSeries 550Q machine (Power 5+ processor, 1.9 Ghz with 4 GB of RAM). All calculations have been performed on one processor with 64-bit floating-point arithmetic. Before presenting the numerical results of the iterative solution methods, we show the results when a direct solver is used to solve the Helmholtz problem.

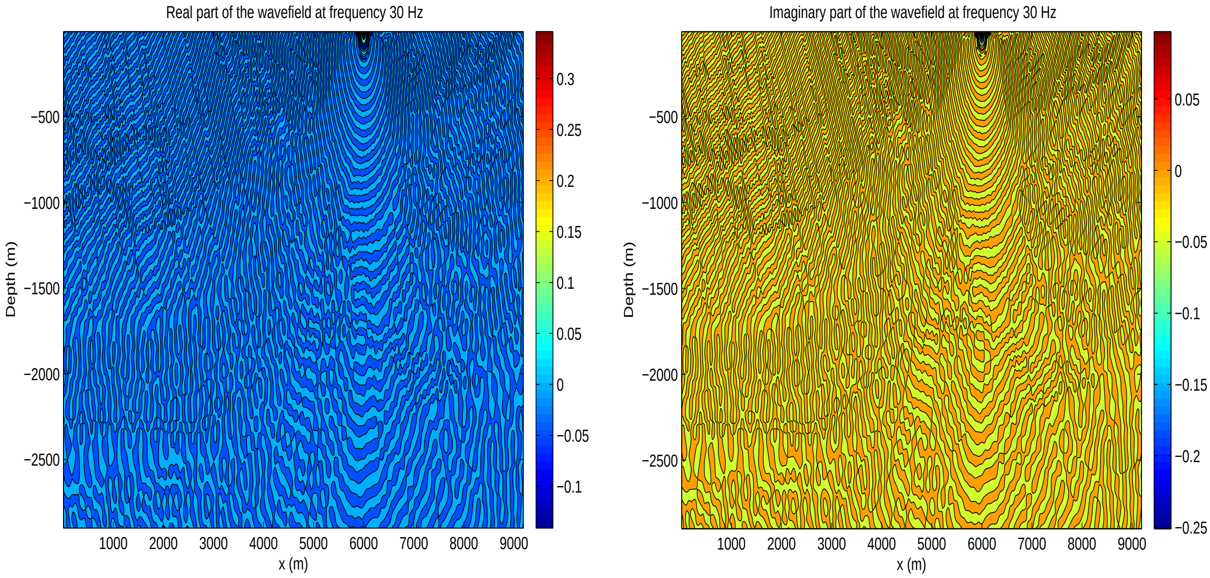

Figure 3.2: Real and imaginary parts of the wavefield at 30 Hz for the Marmousi model. No damping is considered. On the left real part, on the right, imaginary part.

3.2.1 Direct solver

Direct solver								
f	Grid	α	Mem$_f$	Mem	T_a	T_f	T_s	Time
10	767×243	0.0%	292	318	1.87	2.68	0.29	4.84
20	1533×485	0.0%	1355	1462	8.38	19.74	1.25	29.37
30	2299×727	0.0%	3206	3448	20.20	55.68	2.86	78.74
10	767×243	2.5%	292	318	1.88	2.81	0.30	4.99
20	1533×485	2.5%	1355	1462	8.37	20.38	1.29	30.04
30	2299×727	2.5%	3206	3448	20.21	57.17	2.88	80.26
10	767×243	5.0%	292	318	1.87	2.80	0.30	4.97
20	1533×485	5.0%	1355	1462	8.38	20.48	1.30	30.16
30	2299×727	5.0%	3206	3448	20.19	57.31	2.88	80.38

Table 3.1: Numerical results for the Marmousi problem with the MUMPS direct solver. Mem$_f$: amount of memory in Megabytes needed during factorization, Mem: total amount of memory in Megabytes, T_a, T_f, T_s: computational times in seconds for the analysis, factorization and solution phases respectively. Time is the overall computational time in seconds.

Table 3.1 shows the numerical results when a sparse direct solver is used to solve the original Helmholtz problem (2.1). The finite-difference discretization used leads to complex symmetric sparse matrices that are band structured with five bands of entries. The order of the matrices are $186\,381, 743\,505, 1\,671\,373$ for frequencies equal to $10, 20, 30$ respectively. The corresponding number of nonzero entries is $929\,885, 3\,713\,489, 8\,350\,813$ respectively. We have used the multifrontal sparse direct solver MUMPS [2] (Amestoy, Duff and L'Excellent 2000, Amestoy, Duff, Koster and L'Excellent 2001, Amestoy, Guermouche, L'Excellent and Pralet 2006) as the direct solver. The solution has three distinct phases:

- an analysis phase which computes both a reordering of the matrix to reduce the fill-in during factorization [the nested-dissection algorithm from Metis (Karypis and Kumar, 1998) has been chosen here] and a symbolic factorization
- the numerical factorization based on a multifrontal method (Duff and Reid 1983).
- the solution phase for a single right-hand side.

The computational times for these three phases are shown in Table 3.1 as well as the memory needed for the numerical factorization (Mem_f) and the total memory required (Mem). The total memory Mem corresponds to Mem_f and the memory required for storing both the Helmholtz operator triplet (i, j, a_{ij}) and the right-hand side of the problem. The use of damping in the original Helmholtz problem (2.1) does not change the structure of the matrix. Consequently, it does not modify the behaviour of the numerical factorization and solution phases. This is clear from the fact that both the time and memory requirements are very similar for the three different values of the damping coefficient α. The memory needed for the factorization (Mem_f) is found to grow as $n^{1.0905}$ where n denotes the dimension of the matrix. The total memory required by the direct solution method is approximately equal to 3.4 Gb for the 30 Hz frequency.

3.2.2 Complex shifted multigrid as a preconditioner

The results for the shifted multigrid preconditioner are displayed in Table 3.2 for frequencies ranging from 10 to 30 Hz on corresponding grid sizes. One multigrid preconditioning iteration consists of one $F(1, 1)$ cycle with point lexicographic Jacobi as a smoother, full-weighting as restriction, de Zeeuw's operator-dependent prolongation and Galerkin coarse grid discretization. The number of grids of the hierarchy is kept fixed to 9 whatever the frequency. The order of the coarsest matrix is thus $3, 12$, and 27 respectively. A direct solver is used on the coarsest grid. As a consequence, both memory and computing times for the analysis and numerical factorization of the coarsest matrix are negligible and thus are not reported here. The total number of iterations to satisfy the stopping criterion (3.9), the computational time, and the overall memory needed by the solution method are given both for BiCGSTAB and GMRES(5). The overall memory includes storage required for the preconditioning operator M on all the grids, the original operator A on the finest grid only, the weights for the operator-dependent prolongation, the factorization on the coarsest level and additional vectors for the chosen Krylov subspace method. As expected,

[2] Software available at http://mumps.enseeiht.fr/

the required memory (Mem) for the multigrid preconditioned Krylov subspace method is found to grow linearly with n where n denotes the dimension of the matrix.

The behaviour of the complex shifted preconditioner on this realistic test problem is found to be in agreement with previous numerical experiments on heterogeneous problems (Erlangga et al. 2006, Riyanti et al. 2006). When damping is present (for example when $\alpha = 5.0\%$) the number of iterations increases only very slowly for increasing frequencies. In this case, the use of GMRES(5) as a Krylov subspace method leads to a reduction in both time and memory requirements with respect to BiCGSTAB. However, this property is lost for the pure Helmholtz problem ($\alpha = 0.0\%$). The number of iterations grows almost linearly with the frequency for both Krylov subspace methods. This is a major drawback of the complex shifted preconditioner for pure Helmholtz problems. However, since the damping mechanism is present in actual geophysical applications, the complex shifted multigrid preconditioner is still attractive.

Complex shifted multigrid as a preconditioner								
			BiCGSTAB			GMRES(5)		
f	Grid	α	It	Time	Mem	It	Time	Mem
10	767×243	0.0%	75	15.77	141	153	16.34	136
20	1533×485	0.0%	156	164.77	563	307	165.02	540
30	2299×727	0.0%	224	493.63	1265	533	624.23	1214
10	767×243	2.5%	48	10.18	141	80	8.53	136
20	1533×485	2.5%	54	57.00	563	96	52.60	540
30	2299×727	2.5%	58	127.75	1265	109	127.74	1214
10	767×243	5.0%	31	6.58	141	54	5.71	136
20	1533×485	5.0%	39	41.76	563	59	31.95	540
30	2299×727	5.0%	33	73.94	1265	61	70.65	1214

Table 3.2: Numerical results for the Marmousi problem with the complex shifted multigrid as a preconditioner for two Krylov subspace methods. The number of iterations (It), the computational times in seconds (Time) and the overall amount of memory in Megabytes are shown for BiCGSTAB and GMRES(5).

3.2.3 Hybrid two-level preconditioner

Rather than considering the complex shifted operator, we focus on the unshifted Helmholtz operator and try to build effective multilevel preconditioners. Here we present numerical results when only two levels of grids are used. We study the combination of a mixed direct-iterative preconditioner for both BiCGSTAB and GMRES(5). The hybrid two-level preconditioning operation now consists of one V(1,1) cycle with red-black Jacobi as a smoother, full-weighting as the restriction operator, de Zeeuw's or Dendy's operator-dependent prolongation operator and Galerkin coarse grid discretization to build the coarse grid operator. The MUMPS direct solver is used on the coarse grid. Table 3.3 gives some statistics on the direct factorization for the two operator-dependent prolongations. Computational times and memory requirements are essentially the same since both coarse grid operators have the same sparsity and structure. The memory required for the factorization (Mem_f) is found to grow as $n^{1.112}$ where n denotes the dimension of the coarse grid matrix whereas the time for the factorization, T_f, grows as $n^{1.3282}$. Note that the Galerkin coarse grid discretization leads to a nine-point stencil.

Coarsest matrix factorization								
			De Zeeuw's operator			Dendy's operator		
f	Grid	α	Size	Mem_f	T_f	Size	Mem_f	T_f
10	767×243	0.0%	46848	93	0.59	46848	93	0.58
20	1533×485	0.0%	186381	444	3.82	186381	444	3.84
30	2299×727	0.0%	418600	1062	10.98	418600	1062	10.97
10	767×243	2.5%	46848	93	0.62	46848	93	0.63
20	1533×485	2.5%	186381	444	4.05	186381	444	4.02
30	2299×727	2.5%	418600	1062	11.57	418600	1062	11.53
10	767×243	5.0%	46848	93	0.62	46848	93	0.62
20	1533×485	5.0%	186381	444	4.03	186381	444	4.00
30	2299×727	5.0%	418600	1062	11.48	418600	1062	11.50

Table 3.3: Coarse problem informations for the Marmousi problem at various frequencies: size of the matrix, memory in Megabytes required during factorization and computational time in seconds needed for the factorization for two different operator-dependent prolongations.

Table 3.4 shows the numerical results when the De Zeeuw's operator-dependent prolongation is used. We only give the times for the Krylov subspace method in this table. The change in the preconditioning operator and the use of a two-level strategy lead to remarkable improvements both in terms of number of iterations and time compared with the complex shifted multigrid preconditioner used for the runs in Table 3.2. This is true whatever the frequency and the damping parameter α. Indeed, considering the largest grid size (for $f = 30$ Hz) and the total solution time including the computational cost for the numerical factorization on the coarse grid, the new solution method is $5.20, 3.21, 2.56$ times faster than the complex shifted solution multigrid method for damping parameters $0.0\%, 2.5\%, 5.0\%$ respectively.

Table 3.5 shows the numerical results when the Dendy's operator-dependent prolongation is used. This choice defines a new approximation of the coarse grid operator and does affect the convergence rate of the two-level procedure significantly as we can verify

Hybrid two-level preconditioner with De Zeeuw's prolongation								
			BiCGSTAB			GMRES(5)		
f	Grid	α	It	Time	Mem	It	Time	Mem
10	767×243	0.0%	18	4.85	226	30	4.34	220
20	1533×485	0.0%	25	32.10	976	42	28.46	953
30	2299×727	0.0%	39	115.21	2258	69	108.96	2208
10	767×243	2.5%	14	3.79	226	23	3.30	220
20	1533×485	2.5%	11	14.47	976	21	14.12	953
30	2299×727	2.5%	10	30.16	2258	18	28.20	2208
10	767×243	5.0%	9	2.50	226	18	2.56	220
20	1533×485	5.0%	7	9.21	976	12	8.13	953
30	2299×727	5.0%	6	18.24	2258	10	16.13	2208

Table 3.4: Numerical results for the Marmousi problem with the hybrid two-level preconditioner for two Krylov subspace methods. The number of iterations (It), the computational times in seconds (Time) and the overall amount of memory in Megabytes are shown for BiCGSTAB and GMRES(5).

for a model problem through a local Fourier analysis. A reduction in terms of number of iterations and time is observed for all the cases with respect to the results in Table 3.4. In eight cases out of nine GMRES(5) is the faster Krylov subspace method.

Hybrid two-level preconditioner with Dendy's prolongation								
			BiCGSTAB			GMRES(5)		
f	Grid	α	It	Time	Mem	It	Time	Mem
10	767×243	0.0%	16	4.37	226	27	3.87	220
20	1533×485	0.0%	16	20.99	976	29	19.73	953
30	2299×727	0.0%	19	57.41	2258	32	50.14	2208
10	767×243	2.5%	12	3.22	226	22	3.15	220
20	1533×485	2.5%	11	14.32	976	18	12.23	953
30	2299×727	2.5%	9	26.80	2258	14	22.02	2208
10	767×243	5.0%	9	2.47	226	18	2.56	220
20	1533×485	5.0%	7	9.19	976	10	6.88	953
30	2299×727	5.0%	6	18.10	2258	7	11.54	2208

Table 3.5: Numerical results for the Marmousi problem with the hybrid two-level preconditioner for two Krylov subspace methods. The number of iterations (It), the computational times in seconds (Time) and the overall amount of memory in Megabytes are shown for BiCGSTAB and GMRES(5).

3.3 Summary

Scaled Memory Requirement (SMR) and Scaled Timing (ST)								
			Direct		Multigrid		Two-level	
f	Grid	α	SMR	ST	SMR	ST	SMR	ST
10	767×243	0.0%	22.36	0.26	9.56	0.87	15.47	0.24
20	1533×485	0.0%	25.77	0.39	9.52	2.21	16.80	0.32
30	2299×727	0.0%	27.04	0.47	9.52	3.73	17.31	0.37
10	767×243	2.5%	22.36	0.27	9.56	0.45	15.47	0.20
20	1533×485	2.5%	25.77	0.40	9.52	0.71	16.80	0.22
30	2299×727	2.5%	27.04	0.48	9.52	0.76	17.31	0.20
10	767×243	5.0%	22.36	0.26	9.56	0.31	15.47	0.17
20	1533×485	5.0%	25.77	0.40	9.52	0.43	16.80	0.17
30	2299×727	5.0%	27.04	0.48	9.52	0.43	17.31	0.14

Table 3.6: Numerical results for the Marmousi problem. Comparison between three solution methods. The overall amount of memory in Megabytes divided by the amount of memory required to store the matrix (SMR) and the computational times in seconds divided by the number of unknowns multiplied by 10^4 (ST) are shown.

Table 3.6 summarizes the numerical results for the Marmousi problem by comparing memory requirements and computational times for the three solution methods that we have investigated. In this table, we show *scaled* quantities: the total memory divided by the memory required to store the matrix for the finest grid and the computational times divided by the number of unknowns on the finest grid. For the two-level procedure, the results corresponding to the GMRES(5) Krylov subspace method from Table 3.5 have been used.

From these results it is possible to draw the following conclusions:

- The memory requirements for the complete LU factorization grow only modestly with the problem size. This is due to the quality of the reordering (nested dissection) that reduces the fill-in for this band-structured matrix. For the analysis and solution phases, the complexity in time and memory is nearly independent of the damping coefficient. This independence is clearly an advantage that both iterative methods do not have.
- The memory requirements for the multigrid preconditioned Krylov subspace method is directly proportional to the number of unknowns. The memory required on the largest problem is approximately a factor of three less than for the direct solver. The robustness of this multigrid based solution method has been verified on this complex

dataset. Nevertheless its complexity is quite high especially when damping is not considered.

- The use of the two-level method brings spectacular improvements for the undamped case ($\alpha = 0.0\%$) compared to the shifted multigrid preconditioned Krylov subspace method. Clearly designing a two-grid method that tries to approximate the inverse of the original unshifted Helmholtz operator is a good strategy. When damping is present, the solution time is much better than for the iterative scheme. Although the accuracy of the solution is different, this leads to a solution method that is competitive with the direct solver. One drawback is of course the increase in memory due to the use of a direct solver on a coarse grid. Nevertheless memory requirements grow only modestly with the problem size. A reduction in memory requirement can be obtained if a three-level hierarchy is used. This is reported in Table 3.7 that shows numerical results for the case of the damping coefficient set to $\alpha = 5.0\%$. Note that the use of a three-level preconditioner slightly improves the computational times for this case although, in other experiments, we found no further improvement if we went to four levels. An analysis and a full set of numerical experiments will be reported elsewhere.

Hybrid preconditioner with Dendy's prolongation								
			Two-level			Three-level		
f	Grid	α	Time	Mem_f	Mem	Time	Mem_f	Mem
10	767×243	5.0%	3.18	93	220	1.88	21	154
20	1533×485	5.0%	12.88	444	953	9.92	93	624
30	2299×727	5.0%	23.04	1062	2208	21.62	233	1430

Table 3.7: Numerical results for the Marmousi problem with the hybrid two- and three-level preconditioner for the GMRES(5) subspace method for damping coefficient set to 5.0%. The total computational times (including factorization and solution phases) in seconds (Time), the memory required for the factorization of the coarsest grid matrix (Mem_f) and the overall amount of memory in Megabytes (Mem) are shown.

- The solution of Helmholtz problems with multiple sources is frequently required in geophysical applications. As this corresponds to a multiple right-hand sides problem, it is clear that direct methods should be preferred due to the inexpensive solution phases (see Table 3.1 column T_s). For single right-hand sides, however, if high accuracy on the physical solution is not required, the hybrid direct-iterative preconditioner can be competitive both in terms of memory and computational time.

4 Conclusions and perspectives

Iterative methods have been numerically investigated on a complex benchmark problem in geophysics: the Marmousi problem. A two-grid hybrid procedure has been proposed to solve the Helmholtz equation in two dimensions. A comparison with both a direct solution method and a robust multigrid method used as a preconditioner has been also presented. The combination of direct and iterative methods in the two-grid procedure has been shown to be robust and computationally efficient.

Realistic applications in geophysics require extensions of this work to three dimensional problems. Due to the large size of the linear systems, a parallel implementation of these algorithms must be considered. It could be noticed that the components of the two-level procedure (smoother, restriction, prolongation, coarse grid solver) have been chosen so that the global solution method is easily parallelizable. We will be investigating this in the near future.

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