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H S Dollar

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On Taylor series approximations for trust-region and regularized subproblems in optimization¹

H. Sue Dollar²

ABSTRACT

Dollar, Gould and Robinson have recently built on the pioneering work of Gay, Moré and Sorensen for solving trust-region and regularisation subproblems which arise in unconstrained optimization: enhancements include the use of high-order polynomial approximation. In this paper, we consider the Taylor-series expansions that Dollar *et al.* have used and give conditions for when these will overestimate or underestimate the function that is being considered. Full proofs are provided.

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² Computational Science and Engineering Department, Rutherford Appleton Laboratory, Chilton, Oxfordshire, OX11 0QX, England, EU.
Email: sue.dollar@stfc.ac.uk
Current reports available from "<http://www.numerical.rl.ac.uk/reports/>".

Computational Science and Engineering Department
Atlas Centre
Rutherford Appleton Laboratory
Oxfordshire OX11 0QX
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1 Introduction

Given a symmetric matrix $H \in \mathbb{R}^{n \times n}$, a vector $c \in \mathbb{R}^n$ and positive scalars Δ , σ and $p > 2$, we are interested in computing solutions of the optimization problems

$$\underset{x \in \mathbb{R}^n}{\text{minimize}} \quad q(x) \stackrel{\text{def}}{=} c^T x + \frac{1}{2} x^T H x \quad \text{subject to} \quad \|x\|_M \leq \Delta \quad (1.1)$$

and

$$\underset{x \in \mathbb{R}^n}{\text{minimize}} \quad r(x) \stackrel{\text{def}}{=} c^T x + \frac{1}{2} x^T H x + \frac{\sigma}{p} \|x\|_M^p, \quad (1.2)$$

where the M -norm of x is $\|x\|_M \stackrel{\text{def}}{=} \sqrt{x^T M x}$. In [1], we built on the pioneering work of Gay, Moré and Sorensen [2], methods which obtain the solution of a sequence of parametrized linear systems by factorization are used. Enhancements using high-order polynomial approximation and inverse iteration ensure that the resulting method is both globally and asymptotically at least superlinearly convergent in all cases, including in the notorious hard case. In this report, we give the full proof for Lemma B.6 in [1].

Let $x(\lambda)$ satisfy

$$(H + \lambda I)x(\lambda) = -c,$$

then

$$\pi(\lambda; \beta) := \|x(\lambda)\|^\beta = \left(\sum_{i=1}^n \frac{\gamma_i^2}{(\lambda + \lambda_i)^2} \right)^{\frac{\beta}{2}},$$

where $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$ are the eigenvalues of the generalized eigenvalue problem $Hx = \lambda Mx$, [1]. Let

$$\lambda_s \stackrel{\text{def}}{=} \max 0, -\lambda_1.$$

For problem (1.1) we often wish to find $\lambda \in (\lambda_s, \infty)$ which solves the ‘‘secular’’ equation

$$\pi(\lambda; \beta) = \Delta^\beta$$

and for problem (1.2) we often wish to find $\lambda \in (\lambda_s, \infty)$ which solves the nonlinear secular equation

$$\pi(\lambda; \beta) = (\lambda/\sigma)^{\frac{\beta}{p-2}},$$

where $\beta \neq 0$ for both these problems. One possibility is to approximate $\pi(\lambda; \beta)$ by Taylor series expansions of various degrees. To this end, we would like to know whether the Taylor series expansion overestimates or underestimates $\pi(\lambda; \beta)$. If we define the k -th order Taylor approximation

$$\pi_k(\delta; \beta) \stackrel{\text{def}}{=} \sum_{j=0}^k \frac{\pi^{(j)}(\lambda_C; \beta)}{j!} \delta^j$$

to $\pi(\lambda_C + \delta; \beta)$, we see from Taylor’s theorem that the error is

$$\pi(\lambda_C + \delta; \beta) - \pi_k(\delta; \beta) = \frac{1}{(k+1)!} \pi^{(k+1)}(\lambda_C + \psi; \beta) \delta^{k+1}$$

for some ψ strictly between 0 and δ so long as $\delta > \lambda_s - \lambda_C$. Thus, we are interested in knowing when $\pi^{(k+1)}(\lambda; \beta)$ is guaranteed to be positive or negative.

Differentiating $\pi(\lambda; \beta) = \|x(\lambda)\|_M^\beta \equiv [\pi(\lambda)]^{\frac{\beta}{2}}$ with respect to λ , using the chain rule, we obtain

$$\begin{aligned}\pi^{(1)}(\lambda; \beta) &= \frac{\beta}{2} [\pi(\lambda)]^{\frac{\beta}{2}-1} \pi^{(1)}(\lambda), \\ \pi^{(2)}(\lambda; \beta) &= \frac{\beta}{2} [\pi(\lambda)]^{\frac{\beta}{2}-1} \pi^{(2)}(\lambda) + \frac{\beta}{2} (\frac{\beta}{2} - 1) [\pi(\lambda)]^{\frac{\beta}{2}-2} [\pi^{(1)}(\lambda)]^2, \\ \pi^{(3)}(\lambda; \beta) &= \frac{\beta}{2} [\pi(\lambda)]^{\frac{\beta}{2}-3} \left([\pi(\lambda)]^2 \pi^{(3)}(\lambda) + 3 (\frac{\beta}{2} - 1) \pi(\lambda) \pi^{(1)}(\lambda) \pi^{(2)}(\lambda) \right. \\ &\quad \left. + (\frac{\beta}{2} - 1) (\frac{\beta}{2} - 2) [\pi^{(1)}(\lambda)]^3 \right)\end{aligned}\tag{1.3}$$

$$\begin{aligned}\text{and } \pi^{(4)}(\lambda; \beta) &= \frac{\beta}{2} [\pi(\lambda)]^{\frac{\beta}{2}-4} \left([\pi(\lambda)]^3 \pi^{(4)}(\lambda) + 4 (\frac{\beta}{2} - 1) [\pi(\lambda)]^2 \pi^{(1)}(\lambda) \pi^{(3)}(\lambda) \right. \\ &\quad \left. + 3 (\frac{\beta}{2} - 1) [\pi(\lambda)]^2 [\pi^{(2)}(\lambda)]^2 + 6 (\frac{\beta}{2} - 1) (\frac{\beta}{2} - 2) \pi(\lambda) [\pi^{(1)}(\lambda)]^2 \pi^{(2)}(\lambda) \right. \\ &\quad \left. + (\frac{\beta}{2} - 1) (\frac{\beta}{2} - 2) (\frac{\beta}{2} - 3) [\pi^{(1)}(\lambda)]^4 \right).\end{aligned}\tag{1.4}$$

From this we may deduce the following result.

Lemma 1.1 Let $\pi(\lambda; \beta) = \|x(\lambda)\|_M^\beta$, where $x(\lambda)$ satisfies $(H + \lambda I)x(\lambda) = -c$. Suppose that $\lambda > \lambda_s$. Then

- (i) $\pi^{(1)}(\lambda; \beta) < 0$ for all $\beta > 0$ while $\pi^{(1)}(\lambda; \beta) > 0$ for all $\beta < 0$;
- (ii) $\pi^{(2)}(\lambda; \beta) > 0$ for all $\beta > 0$ while $\pi^{(2)}(\lambda; \beta) < 0$ for all $\beta \in [-1, 0)$;
- (iii) $\pi^{(3)}(\lambda; \beta) < 0$ for all $\beta > 0$ while $\pi^{(3)}(\lambda; \beta) > 0$ for all $\beta \in [-\frac{2}{3}, 0)$; and
- (iv) $\pi^{(4)}(\lambda; \beta) > 0$ for all $\beta > 0$ while $\pi^{(4)}(\lambda; \beta) < 0$ for all $\beta \in [-\frac{2}{5}, 0)$.

The first two statements are proved within [1] but only a sketch of the methodology for proving the last two statements is given. In Section 2 we give the proof for the third statement and in Section 3 we provide the proof for the fourth statement.

Notation: $\text{perm}(i, j, k, l)$ is defined to be the set of distinct permutations of the arguments. For example,

$$\text{perm}(2, 2, 4, 4) = \{(2, 2, 4, 4), (2, 4, 2, 4), (2, 4, 4, 2), (4, 2, 2, 4), (4, 2, 4, 2), (4, 4, 2, 2)\}.$$

2 Third derivative proof

Since $\pi(\lambda) = \sum_{i=1}^n \gamma_i^2 / (\lambda + \lambda_i)^2$ it follows that

$$\pi^{(1)}(\lambda) = -2 \sum_{i=1}^n \frac{\gamma_i^2}{(\lambda + \lambda_i)^3} \quad (2.5)$$

$$\pi^{(2)}(\lambda) = 6 \sum_{i=1}^n \frac{\gamma_i^2}{(\lambda + \lambda_i)^4} \quad (2.6)$$

$$\pi^{(3)}(\lambda) = -24 \sum_{i=1}^n \frac{\gamma_i^2}{(\lambda + \lambda_i)^5} \quad (2.7)$$

$$\pi^{(4)}(\lambda) = 120 \sum_{i=1}^n \frac{\gamma_i^2}{(\lambda + \lambda_i)^6} \quad (2.8)$$

Lemma 2.1 Let $\beta > 0$, then $\pi^{(3)}(\lambda; \beta) < 0$ for all $\lambda \in (\lambda_s, \infty)$; let $\beta \in [-\frac{2}{3}, 0)$, then $\pi^{(3)}(\lambda; \beta) > 0$ for all $\lambda \in (\lambda_s, \infty)$;

Proof. Substituting (2.5)–(2.7) into (1.3) and rearranging we obtain

$$\pi^{(3)}(\lambda; \beta) = \beta [\pi(\lambda)]^{\beta-6} \eta(\lambda; \beta),$$

where

$$\begin{aligned} \eta(\lambda; \beta) = & (-\beta^2 + 6\beta - 8) \left(\sum_{i=1}^n \frac{\gamma_i^2}{(\lambda + \lambda_i)^3} \right)^3 - 12 \left(\sum_{i=1}^n \frac{\gamma_i^2}{(\lambda + \lambda_i)^2} \right)^2 \left(\sum_{i=1}^n \frac{\gamma_i^2}{(\lambda + \lambda_i)^5} \right) \\ & + (18 - 9\beta) \left(\sum_{i=1}^n \frac{\gamma_i^2}{(\lambda + \lambda_i)^2} \right) \left(\sum_{i=1}^n \frac{\gamma_i^2}{(\lambda + \lambda_i)^3} \right) \left(\sum_{i=1}^n \frac{\gamma_i^2}{(\lambda + \lambda_i)^4} \right). \end{aligned}$$

From Lemma 2.2 we deduce that $\pi^{(3)}(\lambda; \beta) < 0$ for all $\lambda \in (\lambda_s, \infty)$ and $\beta > 0$; $\pi^{(3)}(\lambda; \beta) > 0$ for all $\lambda \in (\lambda_s, \infty)$ and $\beta \in [-\frac{2}{3}, 0)$. This completes the proof. \square

Lemma 2.2 Let

$$\begin{aligned} \eta(\lambda; \beta) &= (-\beta^2 + 6\beta - 8) \left(\sum_{i=1}^n \frac{\gamma_i^2}{(\lambda + \lambda_i)^3} \right)^3 - 12 \left(\sum_{i=1}^n \frac{\gamma_i^2}{(\lambda + \lambda_i)^2} \right)^2 \left(\sum_{i=1}^n \frac{\gamma_i^2}{(\lambda + \lambda_i)^5} \right) \\ &\quad + (18 - 9\beta) \left(\sum_{i=1}^n \frac{\gamma_i^2}{(\lambda + \lambda_i)^2} \right) \left(\sum_{i=1}^n \frac{\gamma_i^2}{(\lambda + \lambda_i)^3} \right) \left(\sum_{i=1}^n \frac{\gamma_i^2}{(\lambda + \lambda_i)^4} \right). \end{aligned}$$

The expression $\eta(\lambda; \beta)$ is negative for all $\lambda \in (\lambda_s, \infty)$ and $\beta \geq -\frac{2}{3}$.

Proof. The proof is by induction. Let p be such that $\gamma_p \neq 0$ and $\gamma_i = 0$ for $i = 1, \dots, p-1$. Define

$$\begin{aligned} \eta(\lambda; \beta; q) &\stackrel{\text{def}}{=} (-\beta^2 + 6\beta - 8) \left(\sum_{i=p}^q \frac{\gamma_i^2}{(\lambda + \lambda_i)^3} \right)^3 - 12 \left(\sum_{i=p}^q \frac{\gamma_i^2}{(\lambda + \lambda_i)^2} \right)^2 \left(\sum_{i=p}^q \frac{\gamma_i^2}{(\lambda + \lambda_i)^5} \right) \\ &\quad + (18 - 9\beta) \left(\sum_{i=p}^q \frac{\gamma_i^2}{(\lambda + \lambda_i)^2} \right) \left(\sum_{i=p}^q \frac{\gamma_i^2}{(\lambda + \lambda_i)^3} \right) \left(\sum_{i=p}^q \frac{\gamma_i^2}{(\lambda + \lambda_i)^4} \right). \end{aligned}$$

Note that $\eta(\lambda; \beta, n) = \eta(\lambda; \beta)$.

Now $\eta(\lambda; \beta; p) = (-\beta^2 - 3\beta - 2) \frac{\gamma_p^6}{(\lambda + \lambda_p)^9} < 0$ for all $\lambda \in (\lambda_s, \infty)$ and $\beta \geq -\frac{2}{3}$.

Assume that $\eta(\lambda; \beta; k-1) < 0$ for all $\lambda \in (\lambda_s, \infty)$. Now

$$\begin{aligned} \eta(\lambda; \beta; k) &= \eta(\lambda; \beta; k-1) + (-\beta^2 - 3\beta - 2) \frac{\gamma_k^6}{(\lambda + \lambda_k)^9} \\ &\quad + \sum_{i=p}^{k-1} \frac{\gamma_k^4 \gamma_i^2 \eta_1(\lambda; \lambda_k; \lambda_i; \beta)}{(\lambda + \lambda_k)^7 (\lambda + \lambda_i)^5} \\ &\quad + \sum_{i,j=p}^{k-1} \frac{\gamma_k^2 \gamma_i^2 \gamma_j^2 \eta_2(\lambda; \lambda_k; \lambda_j; \lambda_i; \beta)}{(\lambda + \lambda_k)^5 (\lambda + \lambda_i)^5 (\lambda + \lambda_j)^5}, \end{aligned}$$

where

$$\begin{aligned}
\eta_1(\lambda; \lambda_k; \lambda_i; \beta) &= (-9\beta - 6)(\lambda + \lambda_i)^3 + (-3\beta^2 + 9\beta - 6)(\lambda + \lambda_i)^2(\lambda + \lambda_k) \\
&\quad + (18 - 9\beta)(\lambda + \lambda_i)(\lambda + \lambda_k)^2 - 12(\lambda + \lambda_k)^3, \\
\eta_2(\lambda; \lambda_k; \lambda_j; \lambda_i; \beta) &= \frac{18 - 9\beta}{2}(\lambda + \lambda_k)^3((\lambda + \lambda_i)(\lambda + \lambda_j)^2 + (\lambda + \lambda_i)^2(\lambda + \lambda_j)) \\
&\quad + \frac{18 - 9\beta}{2}(\lambda + \lambda_k)^2((\lambda + \lambda_i)(\lambda + \lambda_j)^3 + (\lambda + \lambda_i)^3(\lambda + \lambda_j)) \\
&\quad + \frac{18 - 9\beta}{2}(\lambda + \lambda_k)((\lambda + \lambda_i)^2(\lambda + \lambda_j)^3 + (\lambda + \lambda_i)^3(\lambda + \lambda_j)^2) \\
&\quad + (-3\beta^2 + 18\beta - 24)(\lambda + \lambda_i)^2(\lambda + \lambda_j)^2(\lambda + \lambda_k)^2 \\
&\quad - 12(\lambda + \lambda_i)^3(\lambda + \lambda_j)^3 - 12(\lambda + \lambda_i)^3(\lambda + \lambda_k)^3 \\
&\quad - 12(\lambda + \lambda_j)^3(\lambda + \lambda_k)^3.
\end{aligned}$$

From Lemmas A.3 and A.4 we deduce that $\eta(\lambda; \beta; k) < 0$ for all $\lambda \in (\lambda_s, \infty)$ and $\beta \geq -\frac{2}{3}$.

□

3 Fourth derivative proof

Lemma 3.1 Let $\beta > 0$, then $\pi^{(4)}(\lambda; \beta) > 0$ for all $\lambda \in (\lambda_s, \infty)$; let $\beta \in [-\frac{2}{5}, 0)$, then $\pi^{(4)}(\lambda; \beta) < 0$ for all $\lambda \in (\lambda_s, \infty)$;

Proof. Substituting (2.5)–(2.8) into (1.4) we obtain

$$\pi^{(4)}(\lambda; \beta) = \beta [\pi(\lambda)]^{\beta-8} \zeta(\lambda; \beta),$$

where

$$\begin{aligned}
\zeta(\lambda; \beta) = & (\beta - 2)(\beta - 4)(\beta - 6) \left(\sum_{i=1}^n \frac{\gamma_i^2}{(\lambda + \lambda_i)^3} \right)^4 \\
& + 60 \left(\sum_{i=1}^n \frac{\gamma_i^2}{(\lambda + \lambda_i)^2} \right)^3 \left(\sum_{i=1}^n \frac{\gamma_i^2}{(\lambda + \lambda_i)^6} \right) \\
& + 27(\beta - 2) \left(\sum_{i=1}^n \frac{\gamma_i^2}{(\lambda + \lambda_i)^2} \right)^2 \left(\sum_{i=1}^n \frac{\gamma_i^2}{(\lambda + \lambda_i)^4} \right)^2 \\
& + 48(\beta - 2) \left(\sum_{i=1}^n \frac{\gamma_i^2}{(\lambda + \lambda_i)^2} \right)^2 \left(\sum_{i=1}^n \frac{\gamma_i^2}{(\lambda + \lambda_i)^3} \right) \left(\sum_{i=1}^n \frac{\gamma_i^2}{(\lambda + \lambda_i)^5} \right) \\
& + 18(\beta - 2)(\beta - 4) \left(\sum_{i=1}^n \frac{\gamma_i^2}{(\lambda + \lambda_i)^2} \right) \left(\sum_{i=1}^n \frac{\gamma_i^2}{(\lambda + \lambda_i)^3} \right)^2 \left(\sum_{i=1}^n \frac{\gamma_i^2}{(\lambda + \lambda_i)^4} \right)
\end{aligned}$$

From Lemma 3.2 we deduce that $\pi^{(4)}(\lambda; \beta)$ is positive for all $\lambda \in (\lambda_s, \infty)$ and $\beta > 0$; $\pi^{(4)}(\lambda; \beta)$ is negative for all $\lambda \in (\lambda_s, \infty)$ and $\beta \in [-\frac{2}{5}, 0)$.

□

Lemma 3.2 Let

$$\begin{aligned}
\zeta(\lambda; \beta) = & (\beta - 2)(\beta - 4)(\beta - 6) \left(\sum_{i=1}^n \frac{\gamma_i^2}{(\lambda + \lambda_i)^3} \right)^4 \\
& + 60 \left(\sum_{i=1}^n \frac{\gamma_i^2}{(\lambda + \lambda_i)^2} \right)^3 \left(\sum_{i=1}^n \frac{\gamma_i^2}{(\lambda + \lambda_i)^6} \right) \\
& + 27(\beta - 2) \left(\sum_{i=1}^n \frac{\gamma_i^2}{(\lambda + \lambda_i)^2} \right)^2 \left(\sum_{i=1}^n \frac{\gamma_i^2}{(\lambda + \lambda_i)^4} \right)^2 \\
& + 48(\beta - 2) \left(\sum_{i=1}^n \frac{\gamma_i^2}{(\lambda + \lambda_i)^2} \right)^2 \left(\sum_{i=1}^n \frac{\gamma_i^2}{(\lambda + \lambda_i)^3} \right) \left(\sum_{i=1}^n \frac{\gamma_i^2}{(\lambda + \lambda_i)^5} \right) \\
& + 18(\beta - 2)(\beta - 4) \left(\sum_{i=1}^n \frac{\gamma_i^2}{(\lambda + \lambda_i)^2} \right) \left(\sum_{i=1}^n \frac{\gamma_i^2}{(\lambda + \lambda_i)^3} \right)^2 \left(\sum_{i=1}^n \frac{\gamma_i^2}{(\lambda + \lambda_i)^4} \right).
\end{aligned}$$

The expression $\zeta(\lambda; \beta)$ is positive for all $\lambda \in (\lambda_s, \infty)$ and $\beta \geq -\frac{2}{5}$.

Proof. The proof is by induction. Let p be such that $\gamma_p \neq 0$ and $\gamma_i = 0$ for $i = 1, \dots, p-1$. Define

$$\begin{aligned} \zeta(\lambda; \beta; q) &= (\beta - 2)(\beta - 4)(\beta - 6) \left(\sum_{i=p}^q \frac{\gamma_i^2}{(\lambda + \lambda_i)^3} \right)^4 \\ &\quad + 60 \left(\sum_{i=p}^q \frac{\gamma_i^2}{(\lambda + \lambda_i)^2} \right)^3 \left(\sum_{i=p}^q \frac{\gamma_i^2}{(\lambda + \lambda_i)^6} \right) \\ &\quad + 27(\beta - 2) \left(\sum_{i=p}^q \frac{\gamma_i^2}{(\lambda + \lambda_i)^2} \right)^2 \left(\sum_{i=p}^q \frac{\gamma_i^2}{(\lambda + \lambda_i)^4} \right)^2 \\ &\quad + 48(\beta - 2) \left(\sum_{i=p}^q \frac{\gamma_i^2}{(\lambda + \lambda_i)^2} \right)^2 \left(\sum_{i=p}^q \frac{\gamma_i^2}{(\lambda + \lambda_i)^3} \right) \left(\sum_{i=p}^q \frac{\gamma_i^2}{(\lambda + \lambda_i)^5} \right) \\ &\quad + 18(\beta - 2)(\beta - 4) \left(\sum_{i=p}^q \frac{\gamma_i^2}{(\lambda + \lambda_i)^2} \right) \left(\sum_{i=p}^q \frac{\gamma_i^2}{(\lambda + \lambda_i)^3} \right)^2 \left(\sum_{i=p}^q \frac{\gamma_i^2}{(\lambda + \lambda_i)^4} \right). \end{aligned}$$

Now $\zeta(\lambda; \beta; p) = (\beta^3 + 6\beta^2 + 11\beta + 6) \frac{\gamma_p^8}{(\lambda + \lambda_p)^{12}} > 0$ for all $\lambda \in (\lambda_s, \infty)$ and $\beta \geq -\frac{2}{5}$.

Assume that $\zeta(\lambda; \beta; k-1) > 0$ for all $\lambda \in (\lambda_s, \infty)$. Now

$$\begin{aligned} \zeta(\lambda; \beta; k) &= \zeta(\lambda; \beta; k-1) + (\beta^3 + 6\beta^2 + 11\beta + 6) \frac{\gamma_k^8}{(\lambda + \lambda_k)^{12}} \\ &\quad + \sum_{i=1}^{k-1} \frac{\gamma_k^6 \gamma_i^2 \zeta_1(\lambda; \lambda_k; \lambda_i; \beta)}{(\lambda + \lambda_k)^{10} (\lambda + \lambda_i)^6} \\ &\quad + \sum_{i,j=1}^{k-1} \frac{\gamma_k^4 \gamma_i^2 \gamma_j^2 \zeta_2(\lambda; \lambda_k; \lambda_j; \lambda_i; \beta)}{(\lambda + \lambda_k)^8 (\lambda + \lambda_i)^6 (\lambda + \lambda_j)^6} \\ &\quad + \sum_{i,j,l=1}^{k-1} \frac{\gamma_k^2 \gamma_i^2 \gamma_j^2 \gamma_l^2 \zeta_3(\lambda; \lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta)}{(\lambda + \lambda_k)^6 (\lambda + \lambda_i)^6 (\lambda + \lambda_j)^6} (\lambda + \lambda_l)^6, \end{aligned}$$

where

$$\begin{aligned}
\zeta_1(\lambda; \lambda_k; \lambda_i; \beta) &= (18\beta^2 + 42\beta + 24) (\lambda + \lambda_i)^4 + 60 (\lambda + \lambda_k)^4 \\
&\quad + (4\beta^3 - 12\beta^2 + 8\beta) (\lambda + \lambda_i)^3 (\lambda + \lambda_k) - 48 (2 - \beta) (\lambda + \lambda_i) (\lambda + \lambda_k)^3 \\
&\quad + (18\beta^2 - 54\beta + 36) (\lambda + \lambda_i)^2 (\lambda + \lambda_k)^2, \\
\zeta_2(\lambda; \lambda_k; \lambda_j; \lambda_i; \beta) &= (75\beta + 30) (\lambda + \lambda_i)^4 (\lambda + \lambda_j)^4 \\
&\quad + 90 (\lambda + \lambda_i)^4 (\lambda + \lambda_k)^4 + 90 (\lambda + \lambda_j)^4 (\lambda + \lambda_k)^4 \\
&\quad + (18\beta^2 - 60\beta + 48) (\lambda + \lambda_i)^3 (\lambda + \lambda_j)^4 (\lambda + \lambda_k) \\
&\quad + (18\beta^2 - 60\beta + 48) (\lambda + \lambda_i)^4 (\lambda + \lambda_j)^3 (\lambda + \lambda_k) \\
&\quad - 48 (2 - \beta) (\lambda + \lambda_i) (\lambda + \lambda_j)^4 (\lambda + \lambda_k)^3 \\
&\quad - 48 (2 - \beta) (\lambda + \lambda_i)^4 (\lambda + \lambda_j) (\lambda + \lambda_k)^3 \\
&\quad - 24 (2 - \beta) (\lambda + \lambda_i) (\lambda + \lambda_j)^3 (\lambda + \lambda_k)^4 \\
&\quad - 24 (2 - \beta) (\lambda + \lambda_i)^3 (\lambda + \lambda_j) (\lambda + \lambda_k)^4 \\
&\quad - 27 (2 - \beta) (\lambda + \lambda_i)^2 (\lambda + \lambda_j)^2 (\lambda + \lambda_k)^4 \\
&\quad + (9\beta^2 - 36) (\lambda + \lambda_i)^2 (\lambda + \lambda_j)^4 (\lambda + \lambda_k)^2 \\
&\quad + (9\beta^2 - 36) (\lambda + \lambda_i)^4 (\lambda + \lambda_j)^2 (\lambda + \lambda_k)^2 \\
&\quad + 18 (\beta^2 - 6\beta + 8) (\lambda + \lambda_i)^2 (\lambda + \lambda_j)^3 (\lambda + \lambda_k)^3 \\
&\quad + 18 (\beta^2 - 6\beta + 8) (\lambda + \lambda_i)^3 (\lambda + \lambda_j)^2 (\lambda + \lambda_k)^3 \\
&\quad + (6\beta^3 - 54\beta^2 + 156\beta - 144) (\lambda + \lambda_i)^3 (\lambda + \lambda_j)^3 (\lambda + \lambda_k)^2,
\end{aligned}$$

and

$$\begin{aligned}
\zeta_3(\lambda; \lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta) &= 60 \sum_{\substack{(n_1, n_2, n_3, n_4) \\ \in \text{perm}(0,4,4,4)}} (\lambda + \lambda_i)^{n_1} (\lambda + \lambda_j)^{n_2} (\lambda + \lambda_k)^{n_3} (\lambda + \lambda_l)^{n_4} \\
&\quad + (16\beta - 32) \sum_{\substack{(n_1, n_2, n_3, n_4) \\ \in \text{perm}(1,3,4,4)}} (\lambda + \lambda_i)^{n_1} (\lambda + \lambda_j)^{n_2} (\lambda + \lambda_k)^{n_3} (\lambda + \lambda_l)^{n_4} \\
&\quad + (18\beta - 36) \sum_{\substack{(n_1, n_2, n_3, n_4) \\ \in \text{perm}(2,2,4,4)}} (\lambda + \lambda_i)^{n_1} (\lambda + \lambda_j)^{n_2} (\lambda + \lambda_k)^{n_3} (\lambda + \lambda_l)^{n_4} \\
&\quad + (6\beta^2 - 36\beta + 48) \sum_{\substack{(n_1, n_2, n_3, n_4) \\ \in \text{perm}(2,3,3,4)}} (\lambda + \lambda_i)^{n_1} (\lambda + \lambda_j)^{n_2} (\lambda + \lambda_k)^{n_3} (\lambda + \lambda_l)^{n_4} \\
&\quad + (4\beta^3 - 48\beta^2 + 176\beta - 192) (\lambda + \lambda_k)^3 (\lambda + \lambda_i)^3 (\lambda + \lambda_j)^3 (\lambda + \lambda_l)^3.
\end{aligned}$$

From Lemmas B.5–B.7 we deduce that $\zeta(\lambda; \beta; k) > 0$ for all $\lambda \in (\lambda_s, \infty)$ and $\beta \geq -\frac{2}{5}$.

This completes the proof.

□

Appendix A

Lemma A.3 Let

$$\begin{aligned} \eta_1(\lambda; \lambda_k; \lambda_i; \beta) &= (-9\beta - 6)(\lambda + \lambda_i)^3 + (-3\beta^2 + 9\beta - 6)(\lambda + \lambda_i)^2(\lambda + \lambda_k) \\ &\quad + (18 - 9\beta)(\lambda + \lambda_i)(\lambda + \lambda_k)^2 - 12(\lambda + \lambda_k)^3 \end{aligned}$$

and assume that $\lambda_i \leq \lambda_k$. If $\lambda_i \leq 0$, the expression $\eta_1(\lambda; \lambda_k; \lambda_i; \beta)$ is non-positive for all $\lambda \in [-\lambda_i, \infty)$ and $\beta \geq -1$; if $\lambda_i > 0$, the expression $\eta_1(\lambda; \lambda_k; \lambda_i; \beta)$ is negative for all $\lambda \in [0, \infty)$ and $\beta > -1$;

Proof. Differentiating $\eta_1(\lambda; \lambda_k; \lambda_i; \beta)$ with respect to λ we obtain

$$\begin{aligned} \eta_1(\lambda; \lambda_k; \lambda_i; \beta) &= -12(\lambda + \lambda_k)^3 + (18 - 9\beta)(\lambda + \lambda_k)^2(\lambda + \lambda_i) \\ &\quad + (-3\beta^2 + 9\beta - 6)(\lambda + \lambda_k)(\lambda + \lambda_i)^2 \\ &\quad + (-9\beta - 6)(\lambda + \lambda_i)^3, \end{aligned} \tag{A.1}$$

$$\begin{aligned} \eta_1^{(1)}(\lambda; \lambda_k; \lambda_i; \beta) &= (-18 - 9\beta)(\lambda + \lambda_k)^2 + (24 - 6\beta^2)(\lambda + \lambda_k)(\lambda + \lambda_i) \\ &\quad + (-3\beta^2 - 18\beta - 24)(\lambda + \lambda_i)^2, \end{aligned} \tag{A.2}$$

$$\begin{aligned} \eta_1^{(2)}(\lambda; \lambda_k; \lambda_i; \beta) &= (-6\beta^2 - 18\beta - 12)(\lambda + \lambda_k) \\ &\quad + (-12\beta^2 - 36\beta - 24)(\lambda + \lambda_i), \end{aligned} \tag{A.3}$$

$$\eta_1^{(3)}(\lambda; \lambda_k; \lambda_i; \beta) = -18\beta^2 - 54\beta - 36. \tag{A.4}$$

Clearly $\eta_1^{(3)}(\lambda; \lambda_k; \lambda_i; \beta)$ is negative for all $\beta > -1$.

Consider the case $\lambda_i \leq 0$. Substituting in $\lambda = -\lambda_i$ into (A.1)–(A.3) we obtain

$$\begin{aligned} \eta_1(-\lambda_i; \lambda_k; \lambda_i; \beta) &= -12(\lambda_k - \lambda_i)^3 \leq 0, \\ \eta_1^{(1)}(-\lambda_i; \lambda_k; \lambda_i; \beta) &= (-18 - 9\beta)(\lambda_k - \lambda_i)^2 \leq 0 \text{ for all } \beta \geq -2 \\ \eta_1^{(2)}(-\lambda_i; \lambda_k; \lambda_i; \beta) &= (-6\beta^2 - 18\beta - 12)(\lambda_k - \lambda_i) \leq 0 \text{ for all } \beta > -1. \end{aligned}$$

Hence, $\eta_1(\lambda; \lambda_k; \lambda_i; \beta)$ is non-positive for all $\lambda \geq -\lambda_i$ and $\beta \geq -1$. Since $\lambda_s \geq -\lambda_i$, this completes the proof for the case $\lambda_i \leq 0$.

Consider the case $\lambda_i > 0$. Substituting in $\lambda = 0$ into (A.1)–(A.3) we obtain

$$\begin{aligned} \eta_1(0; \lambda_k; \lambda_i; \beta) &= -12\lambda_k^3 + (18 - 9\beta)\lambda_k^2\lambda_i + (-3\beta^2 + 9\beta - 6)\lambda_k\lambda_i^2 + (-9\beta - 6)\lambda_i^3, \\ \eta_1^{(1)}(0; \lambda_k; \lambda_i; \beta) &= (-18 - 9\beta)\lambda_k^2 + (24 - 6\beta^2)\lambda_k\lambda_i + (-3\beta^2 - 18\beta - 24)\lambda_i^2, \\ \eta_1^{(2)}(0; \lambda_k; \lambda_i; \beta) &= (-6\beta^2 - 18\beta - 12)\lambda_k + (-12\beta^2 - 36\beta - 24)\lambda_i. \end{aligned}$$

We wish to show that the above equations are all non-positive for all $\lambda_k \geq \lambda_i$ and $\beta \geq -1$.

Define $\eta_{10}(\lambda_k; \lambda_i; \beta) = \eta_1(0; \lambda_k; \lambda_i; \beta)$. Differentiating $\eta_{10}(\lambda_k; \lambda_i; \beta)$ with respect to λ_k we obtain

$$\begin{aligned} \eta_{10}(\lambda_k; \lambda_i; \beta) &= -12\lambda_k^3 + (18 - 9\beta)\lambda_k^2\lambda_i + (-3\beta^2 + 9\beta - 6)\lambda_k\lambda_i^2 \\ &\quad + (-9\beta - 6)\lambda_i^3, \end{aligned} \quad (\text{A.5})$$

$$\eta_{10}^{(1)}(\lambda_k; \lambda_i; \beta) = -36\lambda_k^2 + (-18\beta + 36)\lambda_k\lambda_i + (-3\beta^2 + 9\beta - 6)\lambda_i^2, \quad (\text{A.6})$$

$$\eta_{10}^{(2)}(\lambda_k; \lambda_i; \beta) = -72\lambda_k + (36 - 18\beta)\lambda_i, \quad (\text{A.7})$$

$$\eta_{10}^{(3)}(\lambda_k; \lambda_i; \beta) = -72.$$

Substituting $\lambda_k = \lambda_i$ into (A.5)–(A.7) we obtain

$$\begin{aligned} \eta_{10}(\lambda_i; \lambda_i; \beta) &= (-3\beta^2 - 9\beta - 6)\lambda_i^3 < 0 \text{ for all } \beta > -1, \\ \eta_{10}^{(1)}(\lambda_i; \lambda_i; \beta) &= (-3\beta^2 - 9\beta - 6)\lambda_i^2 < 0 \text{ for all } \beta > -1, \\ \eta_{10}^{(2)}(\lambda_i; \lambda_i; \beta) &= (-18\beta - 36)\lambda_i < 0 \text{ for all } \beta > -2. \end{aligned}$$

Hence, $\eta_1(0; \lambda_k; \lambda_i; \beta)$ is negative for all $\lambda_k \geq \lambda_i$ and $\beta > -1$.

Define $\eta_{11}(\lambda_k; \lambda_i; \beta) = \eta_1^{(1)}(0; \lambda_k; \lambda_i; \beta)$. Differentiating $\eta_{11}(\lambda_k; \lambda_i; \beta)$ with respect to λ_k we obtain

$$\eta_{11}(\lambda_k; \lambda_i; \beta) = (-18 - 9\beta)\lambda_k^2 + (24 - 6\beta^2)\lambda_k\lambda_i + (-3\beta^2 - 18\beta - 24)\lambda_i^2, \quad (\text{A.8})$$

$$\eta_{11}^{(1)}(\lambda_k; \lambda_i; \beta) = (-36 - 18\beta)\lambda_k + (24 - 6\beta)\lambda_i, \quad (\text{A.9})$$

$$\eta_{11}^{(2)}(\lambda_k; \lambda_i; \beta) = -36 - 18\beta \leq 0 \text{ for all } \beta \geq -2.$$

Substituting $\lambda_k = \lambda_i$ into (A.8) and (A.9) we obtain

$$\begin{aligned} \eta_{11}(\lambda_i; \lambda_i; \beta) &= (-9\beta^2 - 27\beta - 18)\lambda_i^2 < 0 \text{ for all } \beta > -1, \\ \eta_{11}^{(1)}(\lambda_i; \lambda_i; \beta) &= (-6\beta^2 - 18\beta - 12)\lambda_i < 0 \text{ for all } \beta > -1. \end{aligned}$$

Hence, $\eta_1^{(1)}(0; \lambda_k; \lambda_i; \beta)$ is negative for all $\lambda_k \geq \lambda_i$ and $\beta > -1$.

Define $\eta_{12}(\lambda_k; \lambda_i; \beta) = \eta_1^{(2)}(0; \lambda_k; \lambda_i; \beta)$. Differentiating $\eta_{12}(\lambda_k; \lambda_i; \beta)$ with respect to λ_k we obtain

$$\eta_{12}(\lambda_k; \lambda_i; \beta) = (-6\beta^2 - 18\beta - 12)\lambda_k + (-12\beta^2 - 36\beta - 24)\lambda_i, \quad (\text{A.10})$$

$$\eta_{12}^{(1)}(\lambda_k; \lambda_i; \beta) = -6\beta^2 - 18\beta - 12 < 0 \text{ for all } \beta > -1.$$

Substituting $\lambda_k = \lambda_i$ into (A.10) we obtain

$$\eta_{12}(\lambda_i; \lambda_i; \beta) = (-18\beta^2 - 54\beta - 36)\lambda_i < 0 \text{ for all } \beta > -1.$$

Hence, $\eta_1^{(2)}(0; \lambda_k; \lambda_i; \beta)$ is negative for all $\lambda_k \geq \lambda_i$ and $\beta \geq -1$. This implies that $\eta_1(\lambda; \lambda_k; \lambda_i; \beta)$ is negative for all $\lambda \geq 0$, $\lambda_k \geq \lambda_i$ and $\beta > -1$.

This concludes the proof. □

Lemma A.4 Let

$$\begin{aligned} \eta_2(\lambda; \lambda_k; \lambda_j; \lambda_i; \beta) &= \frac{18-9\beta}{2} (\lambda + \lambda_k)^3 ((\lambda + \lambda_i)(\lambda + \lambda_j)^2 + (\lambda + \lambda_i)^2(\lambda + \lambda_j)) \\ &\quad + \frac{18-9\beta}{2} (\lambda + \lambda_k)^2 ((\lambda + \lambda_i)(\lambda + \lambda_j)^3 + (\lambda + \lambda_i)^3(\lambda + \lambda_j)) \\ &\quad + \frac{18-9\beta}{2} (\lambda + \lambda_k) ((\lambda + \lambda_i)^2(\lambda + \lambda_j)^3 + (\lambda + \lambda_i)^3(\lambda + \lambda_j)^2) \\ &\quad + (-3\beta^2 + 18\beta - 24) (\lambda + \lambda_i)^2 (\lambda + \lambda_j)^2 (\lambda + \lambda_k)^2 \\ &\quad - 12 (\lambda + \lambda_i)^3 (\lambda + \lambda_j)^3 - 12 (\lambda + \lambda_i)^3 (\lambda + \lambda_k)^3 \\ &\quad - 12 (\lambda + \lambda_j)^3 (\lambda + \lambda_k)^3 \end{aligned}$$

and $\lambda_k \geq \max(\lambda_i, \lambda_j)$.

If $\lambda_i \leq 0$, the expression $\eta_2(\lambda; \lambda_k; \lambda_j; \lambda_i; \beta)$ is non-positive for all $\lambda \in [-\lambda_i, \infty)$ and $\beta \geq -\frac{2}{3}$; if $\lambda_i > 0$, the expression $\eta_2(\lambda; \lambda_k; \lambda_j; \lambda_i; \beta)$ is negative for all $\lambda \in [0, \infty)$ and $\beta \geq -\frac{2}{3}$.

Proof. Without loss of generality assume that $\lambda_i \leq \lambda_j$.

Differentiating $\eta_2(\lambda; \lambda_k; \lambda_j; \lambda_i; \beta)$ with respect to λ we obtain

$$\begin{aligned} \eta_2(\lambda; \lambda_k; \lambda_j; \lambda_i; \beta) &= \left(\frac{18-9\beta}{2} \right) \sum_{\substack{(n_1, n_2, n_3) \\ \in \text{perm}(1,2,3)}} (\lambda + \lambda_i)^{n_1} (\lambda + \lambda_j)^{n_2} (\lambda + \lambda_k)^{n_3} \quad (\text{A.11}) \\ &\quad + (-3\beta^2 + 18\beta - 24) (\lambda + \lambda_i)^2 (\lambda + \lambda_j)^2 (\lambda + \lambda_k)^2 \\ &\quad - 12 \sum_{\substack{(n_1, n_2, n_3) \\ \in \text{perm}(0,3,3)}} (\lambda + \lambda_i)^{n_1} (\lambda + \lambda_j)^{n_2} (\lambda + \lambda_k)^{n_3}, \end{aligned}$$

$$\begin{aligned} \eta_2^{(1)}(\lambda; \lambda_k; \lambda_j; \lambda_i; \beta) &= \left(\frac{-54-9\beta}{2} \right) \sum_{\substack{(n_1, n_2, n_3) \\ \in \text{perm}(0,2,3)}} (\lambda + \lambda_i)^{n_1} (\lambda + \lambda_j)^{n_2} (\lambda + \lambda_k)^{n_3} \quad (\text{A.12}) \\ &\quad + (36-18\beta) \sum_{\substack{(n_1, n_2, n_3) \\ \in \text{perm}(1,1,3)}} (\lambda + \lambda_i)^{n_1} (\lambda + \lambda_j)^{n_2} (\lambda + \lambda_k)^{n_3} \\ &\quad + (-6\beta^2 + 9\beta + 6) \sum_{\substack{(n_1, n_2, n_3) \\ \in \text{perm}(1,2,2)}} (\lambda + \lambda_i)^{n_1} (\lambda + \lambda_j)^{n_2} (\lambda + \lambda_k)^{n_3}, \end{aligned}$$

$$\begin{aligned}
\eta_2^{(2)}(\lambda; \lambda_k; \lambda_j; \lambda_i; \beta) &= (-18 - 27\beta) \sum_{\substack{(n_1, n_2, n_3) \\ \in \text{perm}(0,1,3)}} (\lambda + \lambda_i)^{n_1} (\lambda + \lambda_j)^{n_2} (\lambda + \lambda_k)^{n_3} \quad (\text{A.13}) \\
&+ (-6\beta^2 - 18\beta - 156) \sum_{\substack{(n_1, n_2, n_3) \\ \in \text{perm}(0,2,2)}} (\lambda + \lambda_i)^{n_1} (\lambda + \lambda_j)^{n_2} (\lambda + \lambda_k)^{n_3} \\
&+ (-24\beta^2 - 18\beta + 132) \sum_{\substack{(n_1, n_2, n_3) \\ \in \text{perm}(1,1,2)}} (\lambda + \lambda_i)^{n_1} (\lambda + \lambda_j)^{n_2} (\lambda + \lambda_k)^{n_3},
\end{aligned}$$

$$\begin{aligned}
\eta_2^{(3)}(\lambda; \lambda_k; \lambda_j; \lambda_i; \beta) &= (-36 - 54\beta) ((\lambda + \lambda_k)^3 + (\lambda + \lambda_j)^3 + (\lambda + \lambda_i)^3) \quad (\text{A.14}) \\
&+ (-144\beta^2 - 108\beta + 792)(\lambda + \lambda_i)(\lambda + \lambda_j)(\lambda + \lambda_k) \\
&+ (-36\beta^2 - 135\beta - 234) \sum_{\substack{(n_1, n_2, n_3) \\ \in \text{perm}(0,1,2)}} (\lambda + \lambda_i)^{n_1} (\lambda + \lambda_j)^{n_2} (\lambda + \lambda_k)^{n_3},
\end{aligned}$$

$$\begin{aligned}
\eta_2^{(4)}(\lambda; \lambda_k; \lambda_j; \lambda_i; \beta) &= (-72\beta^2 - 432\beta - 576) ((\lambda + \lambda_k)^2 + (\lambda + \lambda_j)^2 + (\lambda + \lambda_i)^2) \quad (\text{A.15}) \\
&+ (-288\beta^2 - 648\beta - 144) \sum_{\substack{(n_1, n_2, n_3) \\ \in \text{perm}(0,1,1)}} (\lambda + \lambda_i)^{n_1} (\lambda + \lambda_j)^{n_2} (\lambda + \lambda_k)^{n_3},
\end{aligned}$$

$$\eta_2^{(5)}(\lambda; \lambda_k; \lambda_j; \lambda_i; \beta) = (-720\beta^2 - 2160\beta - 1440) ((\lambda + \lambda_k) + (\lambda + \lambda_j) + (\lambda + \lambda_i)), \quad (\text{A.16})$$

$$\eta_2^{(6)}(\lambda; \lambda_k; \lambda_j; \lambda_i; \beta) = -2160\beta^2 - 6480\beta - 4320. \quad (\text{A.17})$$

Clearly $\eta_2^{(6)}(\lambda; \lambda_k; \lambda_j; \lambda_i; \beta)$ is negative for all $\beta > -1$.

Consider the case $\lambda_i \leq 0$. Substituting in $\lambda = -\lambda_i$ into (A.11)–(A.16) we obtain

$$\begin{aligned}
\eta_2(-\lambda_i; \lambda_k; \lambda_j; \lambda_i; \beta) &= -12(\lambda_k - \lambda_i)^3(\lambda_j - \lambda_i)^3 \\
&\leq 0, \\
\eta_2^{(1)}(-\lambda_i; \lambda_k; \lambda_j; \lambda_i; \beta) &= \left(\frac{-54 - 9\beta}{2}\right) ((\lambda_k - \lambda_i)^3(\lambda_j - \lambda_i)^2 + (\lambda_k - \lambda_i)^2(\lambda_j - \lambda_i)^3) 2 \\
&\leq 0 \text{ for all } \beta \geq -6, \\
\eta_2^{(2)}(-\lambda_i; \lambda_k; \lambda_j; \lambda_i; \beta) &= (-18 - 27\beta) ((\lambda_k - \lambda_i)(\lambda_j - \lambda_i)^3 + (\lambda_k - \lambda_i)^3(\lambda_j - \lambda_i)) 2 \\
&\quad + (-6\beta^2 - 18\beta - 156)(\lambda_k - \lambda_i)^2(\lambda_j - \lambda_i)^2 \\
&\leq 0 \text{ for all } \beta \geq -\frac{2}{3}, \\
\eta_2^{(3)}(-\lambda_i; \lambda_k; \lambda_j; \lambda_i; \beta) &= (-36 - 54\beta) ((\lambda_k - \lambda_i)^3 + (\lambda_j - \lambda_i)^3) \\
&\quad + (-36\beta^2 - 135\beta - 234) ((\lambda_k - \lambda_i)(\lambda_j - \lambda_i)^2 + (\lambda_k - \lambda_i)^2(\lambda_j - \lambda_i)) \\
&\leq 0 \text{ for all } \beta \geq -\frac{2}{3}, \\
\eta_2^{(4)}(-\lambda_i; \lambda_k; \lambda_j; \lambda_i; \beta) &= (-72\beta^2 - 432\beta - 576) ((\lambda_k - \lambda_i)^2 + (\lambda_j - \lambda_i)^2) \\
&\quad + (-288\beta^2 - 648\beta - 144)(\lambda_k - \lambda_i)(\lambda_j - \lambda_i), \\
\eta_2^{(5)}(-\lambda_i; \lambda_k; \lambda_j; \lambda_i; \beta) &= (-720\beta^2 - 2160\beta - 1440) ((\lambda_k - \lambda_i) + (\lambda_j - \lambda_i)) \\
&\leq 0 \text{ for all } \beta \geq -1.
\end{aligned}$$

It remains for us to show that $\eta_2^{(4)}(-\lambda_i; \lambda_k; \lambda_j; \lambda_i; \beta)$ is non-positive for all $\beta \geq -\frac{2}{3}$ when $\lambda_i \leq 0$.

Define $\hat{\eta}_2(\lambda_k; \lambda_j; \lambda_i; \beta) = \eta_2^{(4)}(-\lambda_i; \lambda_k; \lambda_j; \lambda_i; \beta)$. Differentiating $\hat{\eta}_2(\lambda_k; \lambda_j; \lambda_i; \beta)$ with respect to λ_k we obtain

$$\begin{aligned} \hat{\eta}_2(\lambda_k; \lambda_j; \lambda_i; \beta) &= (-72\beta^2 - 432\beta - 576) ((\lambda_k - \lambda_i)^2 + (\lambda_j - \lambda_i)^2) \quad (\text{A.18}) \\ &\quad + (-288\beta^2 - 648\beta - 144) (\lambda_k - \lambda_i) (\lambda_j - \lambda_i), \end{aligned}$$

$$\begin{aligned} \hat{\eta}_2^{(1)}(\lambda_k; \lambda_j; \lambda_i; \beta) &= (-144\beta^2 - 864\beta - 1152) (\lambda_k - \lambda_i) \quad (\text{A.19}) \\ &\quad + (-288\beta^2 - 648\beta - 144) (\lambda_j - \lambda_i), \end{aligned}$$

$$\begin{aligned} \hat{\eta}_2^{(2)}(\lambda_k; \lambda_j; \lambda_i; \beta) &= -144\beta^2 - 864\beta - 1152 \\ &\leq 0 \text{ for all } \beta \geq -2. \end{aligned}$$

Substituting $\lambda_k = \lambda_j$ into (A.18) and (A.19) we obtain

$$\begin{aligned} \hat{\eta}_2(\lambda_j; \lambda_j; \lambda_i; \beta) &= 2(-432\beta^2 - 1512\beta - 1296) (\lambda_j - \lambda_i)^2 \\ &\leq 0 \text{ for all } \beta \geq -\frac{3}{2}, \\ \hat{\eta}_2^{(1)}(\lambda_j; \lambda_j; \lambda_i; \beta) &= (-432\beta^2 - 1512\beta - 1296) (\lambda_j - \lambda_i) \\ &\leq 0 \text{ for all } \beta \geq -\frac{3}{2}. \end{aligned}$$

Hence, $\eta_2^{(4)}(-\lambda_i; \lambda_k; \lambda_j; \lambda_i; \beta)$ is non-positive for all $\beta \geq -\frac{2}{3}$ when $\lambda_i \leq 0$. From this we deduce that $\eta_2(\lambda; \lambda_k; \lambda_j; \lambda_i; \beta)$ is non-positive for all $\lambda \geq -\lambda_i$ and $\beta \geq -\frac{2}{3}$.

Consider the case $\lambda_i > 0$. Let us define the functions $\eta_{20}(\lambda_k; \lambda_j; \lambda_i; \beta)$, $\eta_{21}(\lambda_k; \lambda_j; \lambda_i; \beta)$, $\eta_{22}(\lambda_k; \lambda_j; \lambda_i; \beta)$, $\eta_{23}(\lambda_k; \lambda_j; \lambda_i; \beta)$, $\eta_{24}(\lambda_k; \lambda_j; \lambda_i; \beta)$ and $\eta_{25}(\lambda_k; \lambda_j; \lambda_i; \beta)$ as follows:

$$\begin{aligned} \eta_{20}(\lambda_k; \lambda_j; \lambda_i; \beta) &= \eta_2(0; \lambda_k; \lambda_j; \lambda_i; \beta), \\ \eta_{21}(\lambda_k; \lambda_j; \lambda_i; \beta) &= \eta_2^{(1)}(0; \lambda_k; \lambda_j; \lambda_i; \beta), \\ \eta_{22}(\lambda_k; \lambda_j; \lambda_i; \beta) &= \eta_2^{(2)}(0; \lambda_k; \lambda_j; \lambda_i; \beta), \\ \eta_{23}(\lambda_k; \lambda_j; \lambda_i; \beta) &= \eta_2^{(3)}(0; \lambda_k; \lambda_j; \lambda_i; \beta), \\ \eta_{24}(\lambda_k; \lambda_j; \lambda_i; \beta) &= \eta_2^{(4)}(0; \lambda_k; \lambda_j; \lambda_i; \beta), \\ \eta_{25}(\lambda_k; \lambda_j; \lambda_i; \beta) &= \eta_2^{(5)}(0; \lambda_k; \lambda_j; \lambda_i; \beta). \end{aligned}$$

We will show that the above equations are all negative for all $\lambda_k \geq \lambda_j$ and $\beta \geq -\frac{2}{3}$. This combined with $\eta_2^{(6)}(\lambda; \lambda_k; \lambda_j; \lambda_i; \beta) < 0$ for all $\beta \geq -\frac{2}{3}$ will prove that $\eta_2(\lambda; \lambda_k; \lambda_j; \lambda_i; \beta)$ is negative for all $\lambda \geq 0$ and $\beta \geq -\frac{2}{3}$.

Differentiating $\eta_{20}(\lambda_k; \lambda_j; \lambda_i; \beta)$ with respect to λ_k we obtain

$$\begin{aligned} \eta_{20}(\lambda_k; \lambda_j; \lambda_i; \beta) &= \left(\frac{18 - 9\beta}{2} \right) \sum_{\substack{(n_1, n_2, n_3) \\ \in \text{perm}(1,2,3)}} \lambda_i^{n_1} \lambda_j^{n_2} \lambda_k^{n_3} \\ &\quad + (-3\beta^2 + 18\beta - 24) \lambda_i^2 \lambda_j^2 \lambda_k^2 \\ &\quad - 12 \sum_{\substack{(n_1, n_2, n_3) \\ \in \text{perm}(0,3,3)}} \lambda_i^{n_1} \lambda_j^{n_2} \lambda_k^{n_3}, \\ \eta_{20}^{(1)}(\lambda_k; \lambda_j; \lambda_i; \beta) &= -36\lambda_k^2 (\lambda_j^3 + \lambda_i^3) + \left(\frac{54 - 27\beta}{2} \right) \lambda_k^2 \lambda_j \lambda_i (\lambda_j + \lambda_i) + (18 - 9\beta) \lambda_k (\lambda_j^3 + \lambda_i^3) \\ &\quad + \left(\frac{-18 - 9\beta}{2} \right) \lambda_j^2 \lambda_i^2 (\lambda_j + \lambda_i) + (-6\beta^2 + 36\beta - 48) \lambda_k \lambda_j^2 \lambda_i^2, \\ \eta_{20}^{(2)}(\lambda_k; \lambda_j; \lambda_i; \beta) &= -72\lambda_k (\lambda_j^3 + \lambda_i^3) + (54 - 27\beta) \lambda_k \lambda_j \lambda_i (\lambda_j + \lambda_i) \\ &\quad + (18 - 9\beta) \lambda_j \lambda_i (\lambda_j^2 + \lambda_i^2) + (-6\beta^2 + 36\beta - 48) \lambda_j^2 \lambda_i^2, \\ \eta_{20}^{(3)}(\lambda_k; \lambda_j; \lambda_i; \beta) &= -72 (\lambda_j^3 + \lambda_i^3) + (54 - 27\beta) \lambda_j \lambda_i (\lambda_j + \lambda_i). \end{aligned}$$

Define $\eta_{203}(\lambda_j; \lambda_i; \beta) = \eta_{20}^{(3)}(\lambda_k; \lambda_j; \lambda_i; \beta)$. We wish to show that η_{203} is non-positive for all $\lambda_j \geq \lambda_i$. Differentiating η_{203} with respect to λ_j gives

$$\begin{aligned} \eta_{203}(\lambda_j; \lambda_i; \beta) &= -72 (\lambda_j^3 + \lambda_i^3) + (54 - 27\beta) \lambda_j \lambda_i (\lambda_j + \lambda_i), \\ \eta_{203}^{(1)}(\lambda_j; \lambda_i; \beta) &= -216\lambda_j^2 + (-54\beta + 108) \lambda_j \lambda_i + (54 - 27\beta) \lambda_i^2, \\ \eta_{203}^{(2)}(\lambda_j; \lambda_i; \beta) &= -432\lambda_j + (108 - 54\beta) \lambda_i, \\ \eta_{203}^{(3)}(\lambda_j; \lambda_i; \beta) &= -432 < 0. \end{aligned}$$

Now

$$\begin{aligned} \eta_{203}(\lambda_i; \lambda_i; \beta) &= (-36 - 54\beta) \lambda_i^3 \leq 0 \text{ for all } \beta \geq -\frac{2}{3}, \\ \eta_{203}^{(1)}(\lambda_i; \lambda_i; \beta) &= (-81 - 54\beta) \lambda_i^2 \leq 0 \text{ for all } \beta \geq -\frac{3}{2}, \\ \eta_{203}^{(2)}(\lambda_i; \lambda_i; \beta) &= (-324 - 54\beta) \lambda_i \leq 0 \text{ for all } \beta \geq -6. \end{aligned}$$

Hence, $\eta_{20}^{(3)}(\lambda_k; \lambda_j; \lambda_i; \beta)$ is non-positive for all $\lambda_k \geq \lambda_j$ and $\beta \geq -\frac{2}{3}$.

Define $\eta_{202}(\lambda_j; \lambda_i; \beta) = \eta_{20}^{(2)}(\lambda_j; \lambda_j; \lambda_i; \beta)/\lambda_j$. We wish to show that η_{202} is non-positive for all $\lambda_j \geq \lambda_i$. Differentiating η_{202} with respect to λ_j gives

$$\begin{aligned} \eta_{202}(\lambda_j; \lambda_i; \beta) &= -72\lambda_j^3 + (72 - 36\beta) \lambda_j^2 \lambda_i + (-6\beta^2 + 9\beta + 6) \lambda_j \lambda_i^2 + (-54 - 9\beta) \lambda_i^3, \\ \eta_{202}^{(1)}(\lambda_j; \lambda_i; \beta) &= -216\lambda_j^2 + (144 - 72\beta) \lambda_j \lambda_i + (-6\beta^2 + 9\beta + 6) \lambda_i^2, \\ \eta_{202}^{(2)}(\lambda_j; \lambda_i; \beta) &= -432\lambda_j + (144 - 72\beta) \lambda_i, \\ \eta_{202}^{(3)}(\lambda_j; \lambda_i; \beta) &= -432 < 0. \end{aligned}$$

Now

$$\begin{aligned}\eta_{202}(\lambda_i; \lambda_i; \beta) &= (-6\beta^2 - 36\beta - 48) \lambda_i^3 \leq 0 \text{ for all } \beta \geq -2, \\ \eta_{202}^{(1)}(\lambda_i; \lambda_i; \beta) &= (-6\beta^2 - 63\beta - 66) \lambda_i^2 \leq 0 \text{ for all } \beta \geq -1.1803, \\ \eta_{202}^{(2)}(\lambda_i; \lambda_i; \beta) &= (-288 - 72\beta) \lambda_i \leq 0 \text{ for all } \beta \geq -4.\end{aligned}$$

Hence, $\eta_{20}^{(2)}(\lambda_j; \lambda_j; \lambda_i; \beta)$ is non-positive for all $\lambda_j \geq \lambda_i$ and $\beta \geq -\frac{2}{3}$.

Define $\eta_{201}(\lambda_j; \lambda_i; \beta) = \eta_{20}^{(1)}(\lambda_j; \lambda_j; \lambda_i; \beta)/\lambda_j^2$. We wish to show that η_{201} is non-positive for all $\lambda_j \geq \lambda_i$. Differentiating η_{201} with respect to λ_j gives

$$\begin{aligned}\eta_{201}(\lambda_j; \lambda_i; \beta) &= -36\lambda_j^3 + \left(\frac{90 - 45\beta}{2}\right) \lambda_j^2 \lambda_i + (-6\beta^2 + 18\beta - 12) \lambda_j \lambda_i^2 + \left(\frac{18 - 27\beta}{2}\right) \lambda_i^3, \\ \eta_{201}^{(1)}(\lambda_j; \lambda_i; \beta) &= -108\lambda_j^2 + (90 - 45\beta) \lambda_j \lambda_i + (-6\beta^2 + 18\beta - 12) \lambda_i^2, \\ \eta_{201}^{(2)}(\lambda_j; \lambda_i; \beta) &= -216\lambda_j + (90 - 45\beta) \lambda_i, \\ \eta_{201}^{(3)}(\lambda_j; \lambda_i; \beta) &= -216 < 0.\end{aligned}$$

Now

$$\begin{aligned}\eta_{201}(\lambda_i; \lambda_i; \beta) &= (-6\beta^2 - 18\beta - 12) \lambda_i^3 \leq 0 \text{ for all } \beta \geq -1, \\ \eta_{201}^{(1)}(\lambda_i; \lambda_i; \beta) &= (-6\beta^2 - 27\beta - 30) \lambda_i^2 \leq 0 \text{ for all } \beta \geq -2, \\ \eta_{201}^{(2)}(\lambda_i; \lambda_i; \beta) &= (-126 - 45\beta) \lambda_i \leq 0 \text{ for all } \beta \geq -2.8.\end{aligned}$$

Hence, $\eta_{20}^{(1)}(\lambda_j; \lambda_j; \lambda_i; \beta)$ is non-positive for all $\lambda_j \geq \lambda_i$ and $\beta \geq -\frac{2}{3}$.

Define $\eta_{200}(\lambda_j; \lambda_i; \beta) = \eta_{20}(\lambda_j; \lambda_j; \lambda_i; \beta)/\lambda_j^3$. We wish to show that η_{200} is negative for all $\lambda_j \geq \lambda_i$. Differentiating η_{200} with respect to λ_j gives

$$\begin{aligned}\eta_{200}(\lambda_j; \lambda_i; \beta) &= -12\lambda_j^3 + (18 - 9\beta) \lambda_j^2 \lambda_i + (-3\beta^2 + 9\beta - 6) \lambda_j \lambda_i^2 + (-6 - 9\beta) \lambda_i^3, \\ \eta_{200}^{(1)}(\lambda_j; \lambda_i; \beta) &= -36\lambda_j^2 + (36 - 18\beta) \lambda_j \lambda_i + (-3\beta^2 + 9\beta - 6) \lambda_i^2, \\ \eta_{200}^{(2)}(\lambda_j; \lambda_i; \beta) &= -72\lambda_j + (36 - 18\beta) \lambda_i, \\ \eta_{200}^{(3)}(\lambda_j; \lambda_i; \beta) &= -72 < 0.\end{aligned}$$

Now

$$\begin{aligned}\eta_{200}(\lambda_i; \lambda_i; \beta) &= (-3\beta^2 - 9\beta - 6) \lambda_i^3 < 0 \text{ for all } \beta > -1, \\ \eta_{200}^{(1)}(\lambda_i; \lambda_i; \beta) &= (-3\beta^2 - 9\beta - 6) \lambda_i^2 < 0 \text{ for all } \beta > -1, \\ \eta_{200}^{(2)}(\lambda_i; \lambda_i; \beta) &= (-36 - 18\beta) \lambda_i \leq 0 \text{ for all } \beta \geq -2.\end{aligned}$$

Hence, $\eta_{20}(\lambda_j; \lambda_j; \lambda_i; \beta)$ is negative for all $\lambda_j \geq \lambda_i$ and $\beta \geq -\frac{2}{3}$. This implies that $\eta_2(0; \lambda_k; \lambda_j; \lambda_i; \beta)$ is negative for all $\lambda_k \geq \lambda_j$ and $\beta \geq -\frac{2}{3}$.

Differentiating $\eta_{21}(\lambda_k; \lambda_j; \lambda_i; \beta)$ with respect to λ_k we obtain

$$\begin{aligned}
\eta_{21}(\lambda_k; \lambda_j; \lambda_i; \beta) &= \left(\frac{-54 - 9\beta}{2} \right) \sum_{\substack{(n_1, n_2, n_3) \\ \in \text{perm}(0,2,3)}} \lambda_i^{n_1} \lambda_j^{n_2} \lambda_k^{n_3} \\
&+ (36 - 18\beta) \sum_{\substack{(n_1, n_2, n_3) \\ \in \text{perm}(1,1,3)}} \lambda_i^{n_1} \lambda_j^{n_2} \lambda_k^{n_3} \\
&+ (-6\beta^2 + 9\beta + 6) \sum_{\substack{(n_1, n_2, n_3) \\ \in \text{perm}(1,2,2)}} (\lambda + \lambda_i)^{n_1} (\lambda + \lambda_j)^{n_2} (\lambda + \lambda_k)^{n_3}, \\
\eta_{21}^{(1)}(\lambda_k; \lambda_j; \lambda_i; \beta) &= \left(\frac{-162 - 27\beta}{2} \right) (\lambda_k^2 \lambda_j^2 + \lambda_k^2 \lambda_i^2) + (108 - 54\beta) \lambda_k^2 \lambda_j \lambda_i \\
&+ (-54 - 9\beta) (\lambda_k \lambda_j^3 + \lambda_k \lambda_i^3) + (-12\beta^2 + 18\beta + 12) (\lambda_k \lambda_j^2 \lambda_i + \lambda_k \lambda_j \lambda_i^2) \\
&+ (36 - 18\beta) (\lambda_j^3 \lambda_i + \lambda_j \lambda_i^3) + (-6\beta^2 + 9\beta + 6) \lambda_j^2 \lambda_i^2, \\
\eta_{21}^{(2)}(\lambda_k; \lambda_j; \lambda_i; \beta) &= (-162 - 27\beta) (\lambda_k \lambda_j^2 + \lambda_k \lambda_i^2) + (216 - 108\beta) \lambda_k \lambda_j \lambda_i \\
&+ (-54 - 9\beta) (\lambda_j^3 + \lambda_i^3) + (-12\beta^2 + 18\beta + 12) (\lambda_j^2 \lambda_i + \lambda_j \lambda_i^2), \\
\eta_{21}^{(3)}(\lambda_k; \lambda_j; \lambda_i; \beta) &= (-162 - 27\beta) (\lambda_j^2 + \lambda_i^2) + (216 - 108\beta) \lambda_j \lambda_i.
\end{aligned}$$

Define $\eta_{213}(\lambda_j; \lambda_i; \beta) = \eta_{21}^{(3)}(\lambda_k; \lambda_j; \lambda_i; \beta)$. We wish to show that η_{213} is non-positive for all $\lambda_j \geq \lambda_i$. Differentiating η_{213} with respect to λ_j gives

$$\begin{aligned}
\eta_{213}(\lambda_j; \lambda_i; \beta) &= (-162 - 27\beta) (\lambda_j^2 + \lambda_i^2) + (216 - 108\beta) \lambda_j \lambda_i, \\
\eta_{213}^{(1)}(\lambda_j; \lambda_i; \beta) &= (-324 - 54\beta) \lambda_j + (216 - 108\beta) \lambda_i, \\
\eta_{213}^{(2)}(\lambda_j; \lambda_i; \beta) &= (-324 - 54\beta) \leq 0 \text{ for all } \beta \geq -6.
\end{aligned}$$

Now

$$\begin{aligned}
\eta_{213}(\lambda_i; \lambda_i; \beta) &= (-108 - 162\beta) \lambda_i^2 \leq 0 \text{ for all } \beta \geq -\frac{2}{3}, \\
\eta_{213}^{(1)}(\lambda_i; \lambda_i; \beta) &= (-108 - 162\beta) \lambda_i \leq 0 \text{ for all } \beta \geq -\frac{2}{3}.
\end{aligned}$$

Hence, $\eta_{21}^{(3)}(\lambda_j; \lambda_j; \lambda_i; \beta)$ is non-positive for all $\lambda_k \geq \lambda_j \geq \lambda_i$ and $\beta \geq -\frac{2}{3}$.

Define $\eta_{212}(\lambda_j; \lambda_i; \beta) = \eta_{21}^{(2)}(\lambda_j; \lambda_j; \lambda_i; \beta)$. We wish to show that η_{212} is non-positive for all $\lambda_j \geq \lambda_i$. Differentiating η_{212} with respect to λ_j gives

$$\begin{aligned}
\eta_{212}(\lambda_j; \lambda_i; \beta) &= (-216 - 36\beta) \lambda_j^3 + (-12\beta^2 - 90\beta + 228) \lambda_j^2 \lambda_i \\
&+ (-12\beta^2 - 9\beta - 150) \lambda_j \lambda_i^2 + (-54 - 9\beta) \lambda_i^3, \\
\eta_{212}^{(1)}(\lambda_j; \lambda_i; \beta) &= (-648 - 108\beta) \lambda_j^2 + (-24\beta^2 - 180\beta + 456) \lambda_j \lambda_i + (-12\beta^2 - 9\beta - 150) \lambda_i^2, \\
\eta_{212}^{(2)}(\lambda_j; \lambda_i; \beta) &= (-1296 - 216\beta) \lambda_j + (-24\beta^2 - 180\beta + 456) \lambda_i, \\
\eta_{212}^{(3)}(\lambda_j; \lambda_i; \beta) &= -1296 - 216\beta \leq 0 \text{ for all } \beta \geq -6.
\end{aligned}$$

Now

$$\begin{aligned}\eta_{212}(\lambda_i; \lambda_i; \beta) &= (-24\beta^2 - 144\beta - 192) \lambda_i^3 \leq 0 \text{ for all } \beta \geq -2, \\ \eta_{212}^{(1)}(\lambda_i; \lambda_i; \beta) &= (-36\beta^2 - 297\beta - 342) \lambda_i^2 \leq 0 \text{ for all } \beta \geq -1.3835, \\ \eta_{212}^{(2)}(\lambda_i; \lambda_i; \beta) &= (-24\beta^2 - 396\beta - 840) \lambda_i \leq 0 \text{ for all } \beta \geq -2.5.\end{aligned}$$

Hence, $\eta_{21}^{(2)}(\lambda_j; \lambda_j; \lambda_i; \beta)$ is non-positive for all $\lambda_j \geq \lambda_i$ and $\beta \geq -\frac{2}{3}$.

Define $\eta_{211}(\lambda_j; \lambda_i; \beta) = \eta_{21}^{(1)}(\lambda_j; \lambda_j; \lambda_i; \beta)/\lambda_j$. We wish to show that η_{211} is non-positive for all $\lambda_j \geq \lambda_i$. Differentiating η_{211} with respect to λ_j gives

$$\begin{aligned}\eta_{211}(\lambda_j; \lambda_i; \beta) &= \left(\frac{-270 - 45\beta}{2}\right) \lambda_j^3 + (-12\beta^2 - 54\beta + 156) \lambda_j^2 \lambda_i \\ &\quad + (-18\beta^2 + \frac{27}{2}\beta - 63) \lambda_j \lambda_i^2 + (-27\beta - 18) \lambda_i^3, \\ \eta_{211}^{(1)}(\lambda_j; \lambda_i; \beta) &= \left(\frac{-810 - 135\beta}{2}\right) \lambda_j^2 + (-24\beta^2 - 108\beta + 312) \lambda_j \lambda_i \\ &\quad + \left(-18\beta^2 + \frac{27}{2}\beta - 63\right) \lambda_i^2, \\ \eta_{211}^{(2)}(\lambda_j; \lambda_i; \beta) &= (-810 - 135\beta) \lambda_j + (-24\beta^2 - 108\beta + 312) \lambda_i, \\ \eta_{211}^{(3)}(\lambda_j; \lambda_i; \beta) &= -810 - 135\beta \leq 0 \text{ for all } \beta \geq -6.\end{aligned}$$

Now

$$\begin{aligned}\eta_{211}(\lambda_i; \lambda_i; \beta) &= (-30\beta^2 - 90\beta - 60) \lambda_i^3 \leq 0 \text{ for all } \beta \geq -1, \\ \eta_{211}^{(1)}(\lambda_i; \lambda_i; \beta) &= (-42\beta^2 - 162\beta - 156) \lambda_i^2 \leq 0 \text{ for all } \beta \geq -\frac{13}{7}, \\ \eta_{211}^{(2)}(\lambda_i; \lambda_i; \beta) &= (-24\beta^2 - 243\beta - 498) \lambda_i \leq 0 \text{ for all } \beta \geq -2.8537.\end{aligned}$$

Hence, $\eta_{21}^{(1)}(\lambda_j; \lambda_j; \lambda_i; \beta)$ is non-positive for all $\lambda_j \geq \lambda_i$ and $\beta \geq -\frac{2}{3}$.

Define $\eta_{210}(\lambda_j; \lambda_i; \beta) = \eta_{21}(\lambda_j; \lambda_j; \lambda_i; \beta)/\lambda_j^2$. We wish to show that η_{210} is negative for all $\lambda_j \geq \lambda_i$. Differentiating η_{210} with respect to λ_j gives

$$\begin{aligned}\eta_{210}(\lambda_j; \lambda_i; \beta) &= (-54 - 9\beta) \lambda_j^3 + (-6\beta^2 - 27\beta + 78) \lambda_j^2 \lambda_i \\ &\quad + (-12\beta^2 + 9\beta - 42) \lambda_j \lambda_i^2 + (-18 - 27\beta) \lambda_i^3, \\ \eta_{210}^{(1)}(\lambda_j; \lambda_i; \beta) &= (-162 - 27\beta) \lambda_j^2 + (-12\beta^2 - 54\beta + 156) \lambda_j \lambda_i \\ &\quad + (-12\beta^2 + 9\beta - 42) \lambda_i^2, \\ \eta_{210}^{(2)}(\lambda_j; \lambda_i; \beta) &= (-324 - 54\beta) \lambda_j + (-12\beta^2 - 54\beta + 156) \lambda_i, \\ \eta_{210}^{(3)}(\lambda_j; \lambda_i; \beta) &= -324 - 54\beta \leq 0 \text{ for all } \beta \geq -6.\end{aligned}$$

Now

$$\begin{aligned}\eta_{210}(\lambda_i; \lambda_i; \beta) &= (-18\beta^2 - 54\beta - 36) \lambda_i^3 < 0 \text{ for all } \beta > -1, \\ \eta_{210}^{(1)}(\lambda_i; \lambda_i; \beta) &= (-24\beta^2 - 72\beta - 48) \lambda_i^2 < 0 \text{ for all } \beta > -1, \\ \eta_{210}^{(2)}(\lambda_i; \lambda_i; \beta) &= (-12\beta^2 - 108\beta - 168) \lambda_i \leq 0 \text{ for all } \beta \geq -2.\end{aligned}$$

Hence, $\eta_{21}(\lambda_j; \lambda_j; \lambda_i; \beta)$ is negative for all $\lambda_j \geq \lambda_i$ and $\beta \geq -\frac{2}{3}$. This implies that $\eta_2^{(1)}(0; \lambda_k; \lambda_j; \lambda_i; \beta)$ is negative for all $\lambda_k \geq \lambda_j \geq \lambda_i$ and $\beta \geq -\frac{2}{3}$.

Differentiating $\eta_{22}(\lambda_k; \lambda_j; \lambda_i; \beta)$ with respect to λ_k we obtain

$$\begin{aligned} \eta_{22}(\lambda_k; \lambda_j; \lambda_i; \beta) &= (-18 - 27\beta) \sum_{\substack{(n_1, n_2, n_3) \\ \in \text{perm}(0,1,3)}} \lambda_i^{n_1} \lambda_j^{n_2} \lambda_k^{n_3} \\ &\quad + (-6\beta^2 - 18\beta - 156) (\lambda_k^2 \lambda_j^2 + \lambda_k^2 \lambda_i^2 + \lambda_j^2 \lambda_i^2) \\ &\quad + (-24\beta^2 - 18\beta + 132) (\lambda_k^2 \lambda_j \lambda_i + \lambda_k \lambda_j^2 \lambda_i + \lambda_k \lambda_j \lambda_i^2), \\ \eta_{22}^{(1)}(\lambda_k; \lambda_j; \lambda_i; \beta) &= (-54 - 81\beta) (\lambda_k^2 \lambda_j + \lambda_k^2 \lambda_i) + (-12\beta^2 - 312 - 36\beta) (\lambda_k \lambda_j^2 + \lambda_k \lambda_i^2) \\ &\quad + (-36\beta - 48\beta^2 + 264) \lambda_k \lambda_j \lambda_i + (-27\beta - 18) (\lambda_j^3 + \lambda_i^3) \\ &\quad + (-24\beta^2 + 132 - 18\beta) (\lambda_j^2 \lambda_i + \lambda_j \lambda_i^2), \\ \eta_{22}^{(2)}(\lambda_k; \lambda_j; \lambda_i; \beta) &= (-108 - 162\beta) (\lambda_k \lambda_j + \lambda_k \lambda_i) + (-12\beta^2 - 312 - 36\beta) (\lambda_j^2 + \lambda_i^2) \\ &\quad + (-36\beta - 48\beta^2 + 264) \lambda_j \lambda_i, \\ \eta_{22}^{(3)}(\lambda_k; \lambda_j; \lambda_i; \beta) &= (-108 - 162\beta) (\lambda_j + \lambda_i) \leq 0 \text{ for all } \beta \geq -\frac{2}{3}. \end{aligned}$$

Define $\eta_{222}(\lambda_j; \lambda_i; \beta) = \eta_{22}^{(2)}(\lambda_j; \lambda_j; \lambda_i; \beta)$. We wish to show that η_{222} is non-positive for all $\lambda_j \geq \lambda_i$. Differentiating η_{222} with respect to λ_j gives

$$\begin{aligned} \eta_{222}(\lambda_j; \lambda_i; \beta) &= (-420 - 198\beta - 12\beta^2) \lambda_j^2 + (-48\beta^2 - 198\beta + 156) \lambda_j \lambda_i \\ &\quad + (-12\beta^2 - 36\beta - 312) \lambda_i^2, \\ \eta_{222}^{(1)}(\lambda_j; \lambda_i; \beta) &= (-396\beta - 840 - 24\beta^2) \lambda_j + (-48\beta^2 - 198\beta + 156) \lambda_i, \\ \eta_{222}^{(2)}(\lambda_j; \lambda_i; \beta) &= -396\beta - 840 - 24\beta^2 \leq 0 \text{ for all } \beta \geq -2.5. \end{aligned}$$

Now

$$\begin{aligned} \eta_{222}(\lambda_i; \lambda_i; \beta) &= (-576 - 432\beta - 72\beta^2) \lambda_i^2 \leq 0 \text{ for all } \beta \geq -2, \\ \eta_{222}^{(1)}(\lambda_i; \lambda_i; \beta) &= (-594\beta - 684 - 72\beta^2) \lambda_i \leq 0 \text{ for all } \beta \geq -1.3835. \end{aligned}$$

Hence, $\eta_{22}^{(2)}(\lambda_j; \lambda_j; \lambda_i; \beta)$ is non-positive for all $\lambda_j \geq \lambda_i$ and $\beta \geq -\frac{2}{3}$.

Define $\eta_{221}(\lambda_j; \lambda_i; \beta) = \eta_{22}^{(1)}(\lambda_j; \lambda_j; \lambda_i; \beta)$. We wish to show that η_{221} is non-positive for all $\lambda_j \geq \lambda_i$. Differentiating η_{221} with respect to λ_j gives

$$\begin{aligned} \eta_{221}(\lambda_j; \lambda_i; \beta) &= (-144\beta - 12\beta^2 - 384) \lambda_j^3 + (-72\beta^2 + 342 - 135\beta) \lambda_j^2 \lambda_i \\ &\quad + (-180 - 36\beta^2 - 54\beta) \lambda_j \lambda_i^2 + (-27\beta - 18) \lambda_i^3, \\ \eta_{221}^{(1)}(\lambda_j; \lambda_i; \beta) &= (-36\beta^2 - 1152 - 432\beta) \lambda_j^2 + (-270\beta + 684 - 144\beta^2) \lambda_j \lambda_i \\ &\quad + (-180 - 36\beta^2 - 54\beta) \lambda_i^2, \\ \eta_{221}^{(2)}(\lambda_j; \lambda_i; \beta) &= (-864\beta - 72\beta^2 - 2304) \lambda_j + (-270\beta + 684 - 144\beta^2) \lambda_i, \\ \eta_{221}^{(3)}(\lambda_j; \lambda_i; \beta) &= -864\beta - 72\beta^2 - 2304 \leq 0 \text{ for all } \beta \geq -4. \end{aligned}$$

Now

$$\begin{aligned}\eta_{221}(\lambda_i; \lambda_i; \beta) &= (-360\beta - 120\beta^2 - 240) \lambda_i^3 \leq 0 \text{ for all } \beta \geq -1, \\ \eta_{221}^{(1)}(\lambda_i; \lambda_i; \beta) &= (-216\beta^2 - 648 - 756\beta) \lambda_i^2 \leq 0 \text{ for all } \beta \geq -1.5, \\ \eta_{221}^{(2)}(\lambda_i; \lambda_i; \beta) &= (-1134\beta - 216\beta^2 - 1620) \lambda_i \leq 0 \text{ for all } \beta.\end{aligned}$$

Hence, $\eta_{22}^{(1)}(\lambda_j; \lambda_j; \lambda_i; \beta)$ is non-positive for all $\lambda_j \geq \lambda_i$ and $\beta \geq -\frac{2}{3}$.

Define $\eta_{220}(\lambda_j; \lambda_i; \beta) = \eta_{22}(\lambda_j; \lambda_j; \lambda_i; \beta)/\lambda_j$. We wish to show that η_{220} is negative for all $\lambda_j \geq \lambda_i$. Differentiating η_{220} with respect to λ_j gives

$$\begin{aligned}\eta_{220}(\lambda_j; \lambda_i; \beta) &= (-6\beta^2 - 72\beta - 192) \lambda_j^3 + (228 - 90\beta - 48\beta^2) \lambda_j^2 \lambda_i \\ &\quad + (-180 - 36\beta^2 - 54\beta) \lambda_j \lambda_i^2 + (-36 - 54\beta) \lambda_i^3, \\ \eta_{220}^{(1)}(\lambda_j; \lambda_i; \beta) &= (-216\beta - 576 - 18\beta^2) \lambda_j^2 + (-96\beta^2 - 180\beta + 456) \lambda_j \lambda_i \\ &\quad + (-180 - 36\beta^2 - 54\beta) \lambda_i^2, \\ \eta_{220}^{(2)}(\lambda_j; \lambda_i; \beta) &= (-36\beta^2 - 1152 - 432\beta) \lambda_j + (-96\beta^2 - 180\beta + 456) \lambda_i, \\ \eta_{220}^{(3)}(\lambda_j; \lambda_i; \beta) &= -36\beta^2 - 1152 - 432\beta \leq 0 \text{ for all } \beta \geq -4.\end{aligned}$$

Now

$$\begin{aligned}\eta_{220}(\lambda_i; \lambda_i; \beta) &= (-90\beta^2 - 270\beta - 180) \lambda_i^3 < 0 \text{ for all } \beta > -1, \\ \eta_{220}^{(1)}(\lambda_i; \lambda_i; \beta) &= (-450\beta - 300 - 150\beta^2) \lambda_i^2 < 0 \text{ for all } \beta > -1, \\ \eta_{220}^{(2)}(\lambda_i; \lambda_i; \beta) &= (-132\beta^2 - 696 - 612\beta) \lambda_i \leq 0 \text{ for all } \beta \geq -2.\end{aligned}$$

Hence, $\eta_{22}(\lambda_j; \lambda_j; \lambda_i; \beta)$ is negative for all $\lambda_j \geq \lambda_i$ and $\beta \geq -\frac{2}{3}$. This implies that $\eta_2^{(2)}(0; \lambda_k; \lambda_j; \lambda_i; \beta)$ is negative for all $\lambda_k \geq \lambda_j \geq \lambda_i$ and $\beta \geq -\frac{2}{3}$.

Differentiating $\eta_{23}(\lambda_k; \lambda_j; \lambda_i; \beta)$ with respect to λ_k we obtain

$$\begin{aligned}\eta_{23}(\lambda_k; \lambda_j; \lambda_i; \beta) &= (-36 - 54\beta) (\lambda_k^3 + \lambda_j^3 + \lambda_i^3) + (-144\beta^2 - 108\beta + 792) \lambda_k \lambda_j \lambda_i \\ &\quad + (-36\beta^2 - 135\beta - 234) \sum_{\substack{(n_1, n_2, n_3) \\ \in \text{perm}(0,1,2)}} \lambda_k^{n_1} \lambda_j^{n_2} \lambda_i^{n_3}, \\ \eta_{23}^{(1)}(\lambda_k; \lambda_j; \lambda_i; \beta) &= (-108 - 162\beta) \lambda_k^2 + (-468 - 72\beta^2 - 270\beta) (\lambda_k \lambda_j + \lambda_k \lambda_i) \\ &\quad + (-36\beta^2 - 135\beta - 234) (\lambda_j^2 + \lambda_i^2) + (-144\beta^2 - 108\beta + 792) \lambda_j \lambda_i, \\ \eta_{23}^{(2)}(\lambda_k; \lambda_j; \lambda_i; \beta) &= (-216 - 324\beta) \lambda_k + (-468 - 72\beta^2 - 270\beta) (\lambda_i + \lambda_j) \\ &\leq 0 \text{ for all } \beta \geq -\frac{2}{3}.\end{aligned}$$

Define $\eta_{231}(\lambda_j; \lambda_i; \beta) = \eta_{23}^{(1)}(\lambda_j; \lambda_j; \lambda_i; \beta)$. We wish to show that η_{231} is non-positive for all $\lambda_j \geq \lambda_i$. Differentiating η_{231} with respect to λ_j gives

$$\begin{aligned}\eta_{231}(\lambda_j; \lambda_i; \beta) &= (-810 - 567\beta - 108\beta^2) \lambda_j^2 + (-216\beta^2 + 324 - 378\beta) \lambda_j \lambda_i \\ &\quad + (-36\beta^2 - 135\beta - 234) \lambda_i^2, \\ \eta_{231}^{(1)}(\lambda_j; \lambda_i; \beta) &= (-1134\beta - 216\beta^2 - 1620) (\lambda_j + \lambda_i) \leq 0 \text{ for all } \beta.\end{aligned}$$

Now

$$\eta_{231}(\lambda_i; \lambda_i; \beta) = (-720 - 1080\beta - 360\beta^2) \lambda_i^2 \leq 0 \text{ for all } \beta \geq -1.$$

Hence, $\eta_{23}^{(1)}(\lambda_j; \lambda_j; \lambda_i; \beta)$ is non-positive for all $\lambda_j \geq \lambda_i$ and $\beta \geq -\frac{2}{3}$.

Define $\eta_{230}(\lambda_j; \lambda_i; \beta) = \eta_{23}(\lambda_j; \lambda_j; \lambda_i; \beta)$. We wish to show that η_{230} is negative for all $\lambda_j \geq \lambda_i$. Differentiating η_{231} with respect to λ_j gives

$$\begin{aligned} \eta_{230}(\lambda_j; \lambda_i; \beta) &= (-378\beta - 72\beta^2 - 540) \lambda_j^3 + (-216\beta^2 + 324 - 378\beta) \lambda_j^2 \lambda_i \\ &\quad + (-72\beta^2 - 270\beta - 468) \lambda_j \lambda_i^2 + (-36 - 54\beta) \lambda_i^3, \\ \eta_{230}^{(1)}(\lambda_j; \lambda_i; \beta) &= (-1134\beta - 216\beta^2 - 1620) \lambda_j^2 + (-756\beta + 648 - 432\beta^2) \lambda_j \lambda_i \\ &\quad + (-72\beta^2 - 270\beta - 468) \lambda_i^2, \\ \eta_{230}^{(2)}(\lambda_j; \lambda_i; \beta) &= (-3240 - 2268\beta - 432\beta^2) \lambda_j + (-756\beta + 648 - 432\beta^2) \lambda_i, \\ \eta_{230}^{(3)}(\lambda_j; \lambda_i; \beta) &= (-3240 - 2268\beta - 432\beta^2) \leq 0 \text{ for all } \beta. \end{aligned}$$

Now

$$\begin{aligned} \eta_{230}(\lambda_i; \lambda_i; \beta) &= (-720 - 1080\beta - 360\beta^2) \lambda_i^3 < 0 \text{ for all } \beta > -1, \\ \eta_{230}^{(1)}(\lambda_i; \lambda_i; \beta) &= (-2160\beta - 720\beta^2 - 1440) \lambda_i^2 < 0 \text{ for all } \beta > -1, \\ \eta_{230}^{(2)}(\lambda_i; \lambda_i; \beta) &= (-2592 - 3024\beta - 864\beta^2) \lambda_i \leq 0 \text{ for all } \beta \geq -1.5. \end{aligned}$$

Hence, $\eta_{23}(\lambda_j; \lambda_j; \lambda_i; \beta)$ is negative for all $\lambda_j \geq \lambda_i$ and $\beta \geq -\frac{2}{3}$. This implies that $\eta_2^{(3)}(0; \lambda_k; \lambda_j; \lambda_i; \beta)$ is negative for all $\lambda_k \geq \lambda_j \geq \lambda_i$ and $\beta \geq -\frac{2}{3}$.

Differentiating $\eta_{24}(\lambda_k; \lambda_j; \lambda_i; \beta)$ with respect to λ_k we obtain

$$\begin{aligned} \eta_{24}(\lambda_k; \lambda_j; \lambda_i; \beta) &= (-72\beta^2 - 432\beta - 576) (\lambda_k^2 + \lambda_j^2 + \lambda_i^2) \\ &\quad + (-288\beta^2 - 648\beta - 144) (\lambda_k \lambda_j + \lambda_k \lambda_i + \lambda_j \lambda_i), \\ \eta_{24}^{(1)}(\lambda_k; \lambda_j; \lambda_i; \beta) &= (-864\beta - 144\beta^2 - 1152) \lambda_k + (-288\beta^2 - 648\beta - 144) \lambda_j \\ &\quad + (-144 - 288\beta^2 - 648\beta) \lambda_i, \\ \eta_{24}^{(2)}(\lambda_k; \lambda_j; \lambda_i; \beta) &= -864\beta - 144\beta^2 - 1152 \leq 0 \text{ for all } \beta \geq -2. \end{aligned}$$

Define $\eta_{241}(\lambda_j; \lambda_i; \beta) = \eta_{24}^{(1)}(\lambda_j; \lambda_j; \lambda_i; \beta)$. We wish to show that η_{241} is non-positive for all $\lambda_j \geq \lambda_i$. Differentiating η_{241} with respect to λ_j gives

$$\begin{aligned} \eta_{241}(\lambda_j; \lambda_i; \beta) &= (-1296 - 1512\beta - 432\beta^2) \lambda_j + (-288\beta^2 - 648\beta - 144) \lambda_i, \\ \eta_{241}^{(1)}(\lambda_j; \lambda_i; \beta) &= -1296 - 1512\beta - 432\beta^2 \leq 0 \text{ for all } \beta \geq -1.5. \end{aligned}$$

Now

$$\eta_{241}(\lambda_i; \lambda_i; \beta) = (-2160\beta - 720\beta^2 - 1440) \lambda_i \leq 0 \text{ for all } \beta \geq -1.$$

Hence, $\eta_{24}^{(1)}(\lambda_j; \lambda_j; \lambda_i; \beta)$ is non-positive for all $\lambda_j \geq \lambda_i$ and $\beta \geq -\frac{2}{3}$.

Define $\eta_{240}(\lambda_j; \lambda_i; \beta) = \eta_{24}(\lambda_j; \lambda_j; \lambda_i; \beta)$. We wish to show that η_{240} is negative for all $\lambda_j \geq \lambda_i$. Differentiating η_{240} with respect to λ_j gives

$$\begin{aligned}\eta_{240}(\lambda_j; \lambda_i; \beta) &= (-1296 - 1512\beta - 432\beta^2) \lambda_j^2 + (-1296\beta - 576\beta^2 - 288) \lambda_j \lambda_i \\ &\quad + (-576 - 432\beta - 72\beta^2) \lambda_i^2, \\ \eta_{240}^{(1)}(\lambda_j; \lambda_i; \beta) &= (-2592 - 3024\beta - 864\beta^2) \lambda_j + (-1296\beta - 576\beta^2 - 288) \lambda_i, \\ \eta_{240}^{(2)}(\lambda_j; \lambda_i; \beta) &= -2592 - 3024\beta - 864\beta^2 \leq 0 \text{ for all } \beta \geq -1.5.\end{aligned}$$

Now

$$\begin{aligned}\eta_{240}(\lambda_i; \lambda_i; \beta) &= (-2160 - 3240\beta - 1080\beta^2) \lambda_i^2 < 0 \text{ for all } \beta > -1, \\ \eta_{240}^{(1)}(\lambda_i; \lambda_i; \beta) &= (-2880 - 4320\beta - 1440\beta^2) \lambda_i < 0 \text{ for all } \beta > -1.\end{aligned}$$

Hence, $\eta_{24}(\lambda_j; \lambda_j; \lambda_i; \beta)$ is negative for all $\lambda_j \geq \lambda_i$ and $\beta \geq -\frac{2}{3}$. This implies that $\eta_2^{(4)}(0; \lambda_k; \lambda_j; \lambda_i; \beta)$ is negative for all $\lambda_k \geq \lambda_j \geq \lambda_i$ and $\beta \geq -\frac{2}{3}$.

Differentiating $\eta_{25}(\lambda_k; \lambda_j; \lambda_i; \beta)$ with respect to λ_k we obtain

$$\begin{aligned}\eta_{25}(\lambda_k; \lambda_j; \lambda_i; \beta) &= (-1440 - 2160\beta - 720\beta^2) \lambda_k + (-720\beta^2 - 1440 - 2160\beta) \lambda_j \\ &\quad + (-1440 - 720\beta^2 - 2160\beta) \lambda_i, \\ \eta_{25}^{(1)}(\lambda_k; \lambda_j; \lambda_i; \beta) &= -1440 - 2160\beta - 720\beta^2 \leq 0 \text{ for all } \beta \geq -1.\end{aligned}$$

Define $\eta_{250}(\lambda_j; \lambda_i; \beta) = \eta_{25}(\lambda_j; \lambda_j; \lambda_i; \beta)$. We wish to show that η_{250} is negative for all $\lambda_j \geq \lambda_i$. Differentiating η_{250} with respect to λ_j gives

$$\begin{aligned}\eta_{250}(\lambda_j; \lambda_i; \beta) &= (-2880 - 4320\beta - 1440\beta^2) \lambda_j + (-720\beta^2 - 2160\beta - 1440) \lambda_i, \\ \eta_{250}^{(1)}(\lambda_j; \lambda_i; \beta) &= -2880 - 4320\beta - 1440\beta^2 \leq 0 \text{ for all } \beta \geq -1.\end{aligned}$$

Now

$$\eta_{250}(\lambda_i; \lambda_i; \beta) = (-4320 - 6480\beta - 2160\beta^2) \lambda_i < 0 \text{ for all } \beta > -1.$$

Hence, $\eta_{25}(\lambda_j; \lambda_j; \lambda_i; \beta)$ is negative for all $\lambda_j \geq \lambda_i$ and $\beta \geq -\frac{2}{3}$. This implies that $\eta_2^{(5)}(0; \lambda_k; \lambda_j; \lambda_i; \beta)$ is negative for all $\lambda_k \geq \lambda_j \geq \lambda_i$ and $\beta \geq -\frac{2}{3}$.

This completes the proof.

□

Appendix B

Lemma B.5 Let

$$\begin{aligned}\zeta_1(\lambda; \lambda_k; \lambda_i; \beta) &= (18\beta^2 + 42\beta + 24) (\lambda + \lambda_i)^4 + 60 (\lambda + \lambda_k)^4 \\ &\quad + (4\beta^3 - 12\beta^2 + 8\beta) (\lambda + \lambda_i)^3 (\lambda + \lambda_k) \\ &\quad - 48 (2 - \beta) (\lambda + \lambda_i) (\lambda + \lambda_k)^3 \\ &\quad + (18\beta^2 - 54\beta + 36) (\lambda + \lambda_i)^2 (\lambda + \lambda_k)^2\end{aligned}$$

and $\lambda_k \geq \lambda_i$. If $\lambda_i \leq 0$, the expression $\zeta_1(\lambda; \lambda_k; \lambda_i; \beta)$ is non-negative for all $\lambda \in [-\lambda_i, \infty)$ and $\beta \geq -1$; if $\lambda_i > 0$, the expression $\zeta_1(\lambda; \lambda_k; \lambda_i; \beta)$ is positive for all $\lambda \in [0, \infty)$ and $\beta > -1$.

Proof. Differentiating $\zeta_1(\lambda; \lambda_k; \lambda_i; \beta)$ with respect to λ we obtain

$$\begin{aligned}\zeta_1(\lambda; \lambda_k; \lambda_i; \beta) &= (18\beta^2 + 42\beta + 24) (\lambda + \lambda_i)^4 + 60 (\lambda + \lambda_k)^4 \\ &\quad + (4\beta^3 - 12\beta^2 + 8\beta) (\lambda + \lambda_i)^3 (\lambda + \lambda_k) \\ &\quad - 48 (2 - \beta) (\lambda + \lambda_i) (\lambda + \lambda_k)^3 \\ &\quad + (18\beta^2 - 54\beta + 36) (\lambda + \lambda_i)^2 (\lambda + \lambda_k)^2,\end{aligned}\tag{B.1}$$

$$\begin{aligned}\zeta_1^{(1)}(\lambda; \lambda_k; \lambda_i; \beta) &= (144 + 48\beta) (\lambda + \lambda_k)^3 \\ &\quad + (-216 + 36\beta + 36\beta^2) (\lambda + \lambda_k)^2 (\lambda + \lambda_i) \\ &\quad + (72 + 12\beta^3 - 84\beta) (\lambda + \lambda_k) (\lambda + \lambda_i)^2 \\ &\quad + (60\beta^2 + 176\beta + 96 + 4\beta^3) (\lambda + \lambda_i)^3,\end{aligned}\tag{B.2}$$

$$\begin{aligned}\zeta_1^{(2)}(\lambda; \lambda_k; \lambda_i; \beta) &= (216 + 180\beta + 36\beta^2) (\lambda + \lambda_k)^2 \\ &\quad + (24\beta^3 + 72\beta^2 - 96\beta - 288) (\lambda + \lambda_k) (\lambda + \lambda_i) \\ &\quad + (180\beta^2 + 444\beta + 360 + 24\beta^3) (\lambda + \lambda_i)^2,\end{aligned}\tag{B.3}$$

$$\begin{aligned}\zeta_1^{(3)}(\lambda; \lambda_k; \lambda_i; \beta) &= (24\beta^3 + 144\beta^2 + 264\beta + 144) (\lambda + \lambda_k) \\ &\quad + (432\beta^2 + 792\beta + 432 + 72\beta^3) (\lambda + \lambda_i) \\ &\geq 0 \text{ for all } \beta \geq -1.\end{aligned}$$

Consider the case $\lambda_i \leq 0$. Substituting $\lambda = -\lambda_i$ into (B.1)–(B.3) we obtain

$$\begin{aligned}\zeta_1(-\lambda_i; \lambda_k; \lambda_i; \beta) &= 60 (\lambda_k - \lambda_i)^4 \geq 0 \text{ for all } \beta, \\ \zeta_1^{(1)}(-\lambda_i; \lambda_k; \lambda_i; \beta) &= (144 + 48\beta) (\lambda_k - \lambda_i)^3 \geq 0 \text{ for all } \beta \geq -3, \\ \zeta_1^{(2)}(-\lambda_i; \lambda_k; \lambda_i; \beta) &= (216 + 180\beta + 36\beta^2) (\lambda_k - \lambda_i)^2 \geq 0 \text{ for all } \beta \geq -2.\end{aligned}$$

Hence, $\zeta_1(\lambda; \lambda_k; \lambda_i; \beta)$ is non-negative for all $\lambda \geq -\lambda_i$ and $\beta > -1$.

Consider the case $\lambda_i > 0$. Substituting $\lambda = 0$ into (B.1)–(B.3) we obtain

$$\begin{aligned}\zeta_1(0; \lambda_k; \lambda_i; \beta) &= (18\beta^2 + 42\beta + 24) \lambda_i^4 + 60\lambda_k^4 \\ &\quad + (4\beta^3 - 12\beta^2 + 8\beta) \lambda_i^3 \lambda_k - 48(2 - \beta) \lambda_i \lambda_k^3 \\ &\quad + (18\beta^2 - 54\beta + 36) \lambda_i^2 \lambda_k^2, \\ \zeta_1^{(1)}(0; \lambda_k; \lambda_i; \beta) &= (144 + 48\beta) \lambda_k^3 + (-216 + 36\beta + 36\beta^2) \lambda_k^2 \lambda_i \\ &\quad + (72 + 12\beta^3 - 84\beta) \lambda_k \lambda_i^2 + (60\beta^2 + 176\beta + 96 + 4\beta^3) \lambda_i^3, \\ \zeta_1^{(2)}(0; \lambda_k; \lambda_i; \beta) &= (216 + 180\beta + 36\beta^2) \lambda_k^2 + (24\beta^3 + 72\beta^2 - 96\beta - 288) \lambda_k \lambda_i \\ &\quad + (180\beta^2 + 444\beta + 360 + 24\beta^3) \lambda_i^2.\end{aligned}$$

We will show that the above equations are all positive for all $\lambda_k \geq \lambda_i$ and $\beta > -1$. This will then imply that $\zeta_1(\lambda; \lambda_k; \lambda_i; \beta)$ is positive for all $\lambda \geq 0$ and $\beta > -1$.

Define $\zeta_{10}(\lambda_k; \lambda_i; \beta) = \zeta_1(0; \lambda_k; \lambda_i; \beta)$. Differentiating $\zeta_{10}(\lambda_k; \lambda_i; \beta)$ with respect to λ_k we obtain

$$\begin{aligned}\zeta_{10}(\lambda_k; \lambda_i; \beta) &= (18\beta^2 + 42\beta + 24) \lambda_i^4 + 60\lambda_k^4 \\ &\quad + (4\beta^3 - 12\beta^2 + 8\beta) \lambda_i^3 \lambda_k - 48(2 - \beta) \lambda_i \lambda_k^3 \\ &\quad + (18\beta^2 - 54\beta + 36) \lambda_i^2 \lambda_k^2,\end{aligned}\tag{B.4}$$

$$\begin{aligned}\zeta_{10}^{(1)}(\lambda_k; \lambda_i; \beta) &= 240\lambda_k^3 + (-288 + 144\beta) \lambda_k^2 \lambda_i \\ &\quad + (72 - 108\beta + 36\beta^2) \lambda_k \lambda_i^2 + (4\beta^3 - 12\beta^2 + 8\beta) \lambda_i^3,\end{aligned}\tag{B.5}$$

$$\begin{aligned}\zeta_{10}^{(2)}(\lambda_k; \lambda_i; \beta) &= 720\lambda_k^2 + (288\beta - 576) \lambda_k \lambda_i \\ &\quad + (72 - 108\beta + 36\beta^2) \lambda_i^2,\end{aligned}\tag{B.6}$$

$$\zeta_{10}^{(3)}(\lambda_k; \lambda_i; \beta) = 1440\lambda_k + (288\beta - 576) \lambda_i,\tag{B.7}$$

$$\zeta_{10}^{(4)}(\lambda_k; \lambda_i; \beta) = 1440.$$

Substituting $\lambda_k = \lambda_i$ into (B.4)–(B.7) we obtain

$$\begin{aligned}\zeta_{10}(\lambda_i; \lambda_i; \beta) &= (24 + 44\beta + 24\beta^2 + 4\beta^3) \lambda_i^4 > 0 \text{ for all } \beta > -1, \\ \zeta_{10}^{(1)}(\lambda_i; \lambda_i; \beta) &= (24 + 44\beta + 24\beta^2 + 4\beta^3) \lambda_i^3 > 0 \text{ for all } \beta > -1, \\ \zeta_{10}^{(2)}(\lambda_i; \lambda_i; \beta) &= (216 + 180\beta + 36\beta^2) \lambda_i^2 > 0 \text{ for all } \beta > -2, \\ \zeta_{10}^{(3)}(\lambda_i; \lambda_i; \beta) &= (864 + 288\beta) \lambda_i > 0 \text{ for all } \beta > -3.\end{aligned}$$

Hence, $\zeta_1(0; \lambda_k; \lambda_i; \beta)$ is positive for all $\lambda_k \geq \lambda_i$ and $\beta > -1$.

Define $\zeta_{11}(\lambda_k; \lambda_i; \beta) = \zeta_1^{(1)}(0; \lambda_k; \lambda_i; \beta)$. Differentiating $\zeta_{11}(\lambda_k; \lambda_i; \beta)$ with respect to λ_k

we obtain

$$\begin{aligned} \zeta_{11}(\lambda_k; \lambda_i; \beta) &= (144 + 48\beta) \lambda_k^3 + (-216 + 36\beta + 36\beta^2) \lambda_k^2 \lambda_i \\ &\quad + (72 + 12\beta^3 - 84\beta) \lambda_k \lambda_i^2 + (60\beta^2 + 176\beta + 96 + 4\beta^3) \lambda_i^3, \end{aligned} \quad (\text{B.8})$$

$$\begin{aligned} \zeta_{11}^{(1)}(\lambda_k; \lambda_i; \beta) &= (432 + 144\beta) \lambda_k^2 + (72\beta^2 - 432 + 72\beta) \lambda_k \lambda_i \\ &\quad + (72 + 12\beta^3 - 84\beta) \lambda_i^2, \end{aligned} \quad (\text{B.9})$$

$$\zeta_{11}^{(2)}(\lambda_k; \lambda_i; \beta) = (864 + 288\beta) \lambda_k + (72\beta^2 - 432 + 72\beta) \lambda_i, \quad (\text{B.10})$$

$$\zeta_{11}^{(3)}(\lambda_k; \lambda_i; \beta) = 864 + 288\beta \geq 0 \text{ for all } \beta \geq -3.$$

Substituting $\lambda_k = \lambda_i$ into (B.8)–(B.10) we obtain

$$\begin{aligned} \zeta_{11}(\lambda_i; \lambda_i; \beta) &= (96 + 176\beta + 96\beta^2 + 16\beta^3) \lambda_i^3 > 0 \text{ for all } \beta > -1, \\ \zeta_{11}^{(1)}(\lambda_i; \lambda_i; \beta) &= (72 + 132\beta + 72\beta^2 + 12\beta^3) \lambda_i^2 > 0 \text{ for all } \beta > -1, \\ \zeta_{11}^{(2)}(\lambda_i; \lambda_i; \beta) &= (432 + 360\beta + 72\beta^2) \lambda_i > 0 \text{ for all } \beta > -2. \end{aligned}$$

Hence, $\zeta_1^{(1)}(0; \lambda_k; \lambda_i; \beta)$ is positive for all $\lambda_k \geq \lambda_i$ and $\beta > -1$.

Define $\zeta_{12}(\lambda_k; \lambda_i; \beta) = \zeta_1^{(2)}(0; \lambda_k; \lambda_i; \beta)$. Differentiating $\zeta_{12}(\lambda_k; \lambda_i; \beta)$ with respect to λ_k we obtain

$$\begin{aligned} \zeta_{12}(\lambda_k; \lambda_i; \beta) &= (216 + 180\beta + 36\beta^2) \lambda_k^2 \\ &\quad + (24\beta^3 + 72\beta^2 - 96\beta - 288) \lambda_k \lambda_i \\ &\quad + (180\beta^2 + 444\beta + 360 + 24\beta^3) \lambda_i^2, \end{aligned} \quad (\text{B.11})$$

$$\zeta_{12}^{(1)}(\lambda_k; \lambda_i; \beta) = (432 + 360\beta + 72\beta^2) \lambda_k + (24\beta^3 + 72\beta^2 - 96\beta - 288) \lambda_i \quad (\text{B.12})$$

$$\zeta_{12}^{(2)}(\lambda_k; \lambda_i; \beta) = 432 + 360\beta + 72\beta^2 \geq 0 \text{ for all } \beta \geq -2.$$

Substituting $\lambda_k = \lambda_i$ into (B.11) and (B.12) we obtain

$$\begin{aligned} \zeta_{12}(\lambda_i; \lambda_i; \beta) &= (288 + 528\beta + 288\beta^2 + 48\beta^3) \lambda_i^3 > 0 \text{ for all } \beta > -1, \\ \zeta_{12}^{(1)}(\lambda_i; \lambda_i; \beta) &= (24\beta^3 + 144\beta^2 + 264\beta + 144) \lambda_i^2 > 0 \text{ for all } \beta > -1. \end{aligned}$$

Hence, $\zeta_1^{(2)}(0; \lambda_k; \lambda_i; \beta)$ is positive for all $\lambda_k > \lambda_i$ and $\beta \geq -1$. This implies that $\zeta_1(\lambda; \lambda_k; \lambda_i; \beta)$ is positive for all $\lambda \geq 0$, $\lambda_k \geq \lambda_i$ and $\beta > -1$.

This completes the proof.

□

Lemma B.6 Let

$$\tilde{s}(a, b) = \sum_{\substack{(n_1, n_2) \\ \in \text{perm}(a, b)}} (\lambda + \lambda_i)^{n_1} (\lambda + \lambda_j)^{n_2}$$

and

$$\begin{aligned} \zeta_2(\lambda; \lambda_k; \lambda_j; \lambda_i; \beta) &= (75\beta + 30) \tilde{s}(4, 4) \\ &+ 90 (\lambda + \lambda_k)^4 \tilde{s}(0, 4) \\ &+ (18\beta^2 - 60\beta + 48) (\lambda + \lambda_k) \tilde{s}(3, 4) \\ &- 48 (2 - \beta) (\lambda + \lambda_k)^3 \tilde{s}(1, 4) \\ &- 24 (2 - \beta) (\lambda + \lambda_k)^4 \tilde{s}(1, 3) \\ &- 27 (2 - \beta) (\lambda + \lambda_k)^4 \tilde{s}(2, 2) \\ &+ (9\beta^2 - 36) (\lambda + \lambda_k)^2 \tilde{s}(2, 4) \\ &+ 18 (\beta^2 - 6\beta + 8) (\lambda + \lambda_k)^3 \tilde{s}(2, 3) \\ &+ (6\beta^3 - 54\beta^2 + 156\beta - 144) (\lambda + \lambda_k)^2 \tilde{s}(3, 3) \end{aligned}$$

and $\lambda_k \geq \max(\lambda_i, \lambda_j)$. If $\min(\lambda_i, \lambda_j) \leq 0$, the expression $\zeta_2(\lambda; \lambda_k; \lambda_j; \lambda_i; \beta)$ is non-negative for all $\lambda \in [\max(-\lambda_i, -\lambda_j), \infty)$ and $\beta \geq -\frac{2}{5}$; if $\min(\lambda_i, \lambda_j) > 0$, the expression $\zeta_2(\lambda; \lambda_k; \lambda_j; \lambda_i; \beta)$ is positive for all $\lambda \in [0, \infty)$ and $\beta \geq -\frac{2}{5}$.

Proof.

Without loss of generality assume that $\lambda_i \leq \lambda_j$.

Differentiating $\zeta_2(\lambda; \lambda_k; \lambda_j; \lambda_i; \beta)$ with respect to λ we obtain

$$\begin{aligned} \zeta_2(\lambda; \lambda_k; \lambda_j; \lambda_i; \beta) &= (75\beta + 30) \tilde{s}(4, 4) \\ &+ 90 (\lambda + \lambda_k)^4 \tilde{s}(0, 4) \\ &+ (18\beta^2 - 60\beta + 48) (\lambda + \lambda_k) \tilde{s}(3, 4) \\ &- 48 (2 - \beta) (\lambda + \lambda_k)^3 \tilde{s}(1, 4) \\ &- 24 (2 - \beta) (\lambda + \lambda_k)^4 \tilde{s}(1, 3) \\ &- 27 (2 - \beta) (\lambda + \lambda_k)^4 \tilde{s}(2, 2) \\ &+ (9\beta^2 - 36) (\lambda + \lambda_k)^2 \tilde{s}(2, 4) \\ &+ 18 (\beta^2 - 6\beta + 8) (\lambda + \lambda_k)^3 \tilde{s}(2, 3) \\ &+ (6\beta^3 - 54\beta^2 + 156\beta - 144) (\lambda + \lambda_k)^2 \tilde{s}(3, 3), \end{aligned} \tag{B.13}$$

$$\begin{aligned}
\zeta_2^{(1)}(\lambda; \lambda_k; \lambda_j; \lambda_i; \beta) &= (312 + 24\beta)(\lambda + \lambda_k)^4 \tilde{s}(1, 3) \\
&+ 126(\beta - 2)(\lambda + \lambda_k)^4 \tilde{s}(1, 2) \\
&+ 24(2\beta + 11)(\lambda + \lambda_k)^3 \tilde{s}(0, 4) \\
&+ 36(\beta^2 + \beta - 8)(\lambda + \lambda_k)^3 \tilde{s}(1, 3) \\
&+ 108(\beta^2 - 5\beta + 6)(\lambda + \lambda_k)^3 \tilde{s}(2, 2) \\
&+ 18(\beta^2 + 8\beta - 20)(\lambda + \lambda_k)^2 \tilde{s}(1, 4) \\
&+ 18(\beta^3 - 4\beta^2 + 8\beta - 8)(\lambda + \lambda_k)^2 \tilde{s}(2, 3) \\
&+ 36(2\beta^2 - 5\beta + 2)(\lambda + \lambda_k) \tilde{s}(2, 4) \\
&+ 12(\beta^3 + 3\beta^2 - 14\beta + 8)(\lambda + \lambda_k) \tilde{s}(3, 3) \\
&+ 6(3\beta^2 + 40\beta + 28)(\lambda + \lambda_k) \tilde{s}(3, 3),
\end{aligned} \tag{B.14}$$

$$\begin{aligned}
\zeta_2^{(2)}(\lambda; \lambda_k; \lambda_j; \lambda_i; \beta) &= (198\beta + 684)(\lambda + \lambda_k)^4 \tilde{s}(1, 2) \\
&+ 504(\beta - 2)(\lambda + \lambda_k)^4 \tilde{s}(1, 1) \\
&+ (2016 + 36\beta^2 + 360\beta)(\lambda + \lambda_k)^3 \tilde{s}(1, 3) \\
&+ (-576 + 324\beta^2 - 360\beta)(\lambda + \lambda_k)^3 \tilde{s}(1, 2) \\
&+ (432 + 18\beta^2 + 288\beta)(\lambda + \lambda_k)^2 \tilde{s}(0, 4) \\
&+ (36\beta^2 + 1080\beta + 36\beta^3 - 2592)(\lambda + \lambda_k)^2 \tilde{s}(1, 3) \\
&+ (-756\beta - 108\beta^2 + 1080 + 108\beta^3)(\lambda + \lambda_k)^2 \tilde{s}(2, 2) \\
&+ (-72\beta + 180\beta^2 - 576)(\lambda + \lambda_k) \tilde{s}(1, 4) \\
&+ (72\beta^3 + 252\beta^2 + 288 - 936\beta)(\lambda + \lambda_k) \tilde{s}(2, 3) \\
&+ (126\beta^2 + 576 + 540\beta) \tilde{s}(2, 4) \\
&+ (180\beta^2 + 1752\beta + 12\beta^3 + 1440) \tilde{s}(3, 3),
\end{aligned} \tag{B.15}$$

$$\begin{aligned}
\zeta_2^{(3)}(\lambda; \lambda_k; \lambda_j; \lambda_i; \beta) &= (900\beta + 360)(\lambda + \lambda_k)^4 ((\lambda + \lambda_j) + (\lambda + \lambda_i)) \\
&+ (432\beta^2 + 8208 + 1512\beta)(\lambda + \lambda_k)^3 \tilde{s}(0, 2) \\
&+ (-6336 + 576\beta + 1296\beta^2)(\lambda + \lambda_k)^3 \tilde{s}(1, 1) \\
&+ (3312\beta + 5184 + 36\beta^3 + 216\beta^2)(\lambda + \lambda_k)^2 \tilde{s}(0, 3) \\
&+ (-7344 + 864\beta^2 + 324\beta^3 + 648\beta)(\lambda + \lambda_k)^2 \tilde{s}(1, 2) \\
&+ (216\beta^2 + 288 + 504\beta)(\lambda + \lambda_k) \tilde{s}(0, 4) \\
&+ (1296\beta^2 + 216\beta^3 - 6912)(\lambda + \lambda_k) \tilde{s}(1, 3) \\
&+ (648\beta^3 - 7128\beta + 1296\beta^2 + 3888)(\lambda + \lambda_k) \tilde{s}(2, 2) \\
&+ (576 + 1008\beta + 432\beta^2) \tilde{s}(1, 4) \\
&+ (1296\beta^2 + 6480\beta + 6912 + 108\beta^3) \tilde{s}(2, 3),
\end{aligned} \tag{B.16}$$

$$\zeta_2^{(4)}(\lambda; \lambda_k; \lambda_j; \lambda_i; \beta) = (1800\beta + 720)(\lambda + \lambda_k)^4 \quad (\text{B.17})$$

$$\begin{aligned} & + (7200\beta + 2160\beta^2 + 11520)(\lambda + \lambda_k)^3 \tilde{s}(0, 1) \\ & + (432\beta^3 + 2808\beta^2 + 15120\beta + 32832)(\lambda + \lambda_k)^2 \tilde{s}(0, 2) \\ & + (-48384 + 4320\beta + 7344\beta^2 + 1296\beta^3)(\lambda + \lambda_k) \tilde{s}(1, 1) \\ & + (2592\beta^2 + 4608 + 8640\beta + 288\beta^3)(\lambda + \lambda_k) \tilde{s}(0, 3) \\ & + (8208\beta^2 + 2592\beta^3 - 12960\beta - 27648)(\lambda + \lambda_k) \tilde{s}(1, 2), \end{aligned} \quad (\text{B.18})$$

$$\begin{aligned} \zeta_2^{(5)}(\lambda; \lambda_k; \lambda_j; \lambda_i; \beta) & = (25920 + 21600\beta + 4320\beta^2)(\lambda + \lambda_k)^3 \\ & + (51840 + 56160\beta + 19440\beta^2 + 2160\beta^3)(\lambda + \lambda_k)^2 \tilde{s}(0, 1) \\ & + (4320\beta^3 + 51840 + 43200\beta + 21600\beta^2)(\lambda + \lambda_k) \tilde{s}(0, 2) \\ & + (-43200\beta + 12960\beta^3 - 207360 + 47520\beta^2)(\lambda + \lambda_k) \tilde{s}(1, 1) \\ & + (720\beta^3 + 10800\beta^2 + 17280 + 31680\beta) \tilde{s}(0, 3) \\ & + (90720 + 6480\beta^3 + 101520\beta + 43200\beta^2) \tilde{s}(1, 2), \end{aligned} \quad (\text{B.19})$$

$$\begin{aligned} \zeta_2^{(6)}(\lambda; \lambda_k; \lambda_j; \lambda_i; \beta) & = (4320\beta^3 + 177120\beta + 181440 + 51840\beta^2)(\lambda + \lambda_k)^2 \\ & + (155520\beta + 25920\beta^3 + 129600\beta^2)(\lambda + \lambda_k) \tilde{s}(0, 1) \\ & + (239760\beta + 194400 + 12960\beta^3 + 97200\beta^2) \tilde{s}(0, 2) \\ & + (220320\beta^2 + 38880\beta^3 + 362880\beta + 155520) \tilde{s}(1, 1), \end{aligned} \quad (\text{B.19})$$

$$\begin{aligned} \zeta_2^{(7)}(\lambda; \lambda_k; \lambda_j; \lambda_i; \beta) & = (665280\beta + 362880\beta^2 + 60480\beta^3 + 362880)(\lambda + \lambda_k) \\ & + (544320\beta^2 + 997920\beta + 544320 + 90720\beta^3) \tilde{s}(0, 1) \\ & \geq 0 \text{ for all } \beta \geq -1. \end{aligned}$$

Consider the case $\lambda_i \leq 0$. Substituting in $\lambda = -\lambda_i$ into (B.13)–(B.19) we obtain

$$\begin{aligned} \zeta_2(-\lambda_i; \lambda_k; \lambda_j; \lambda_i; \beta) & = 90(\lambda_j - \lambda_i)^4 (\lambda_k - \lambda_i)^4 \\ & \geq 0, \\ \zeta_2^{(1)}(-\lambda_i; \lambda_k; \lambda_j; \lambda_i; \beta) & = (312 + 24\beta)(\lambda_k - \lambda_i)^4 (\lambda_j - \lambda_i)^3 \\ & \quad + 24(2\beta + 11)(\lambda_k - \lambda_i)^3 (\lambda_j - \lambda_i)^4 \\ & \geq 0 \text{ for all } \beta \geq -5.5, \\ \zeta_2^{(2)}(-\lambda_i; \lambda_k; \lambda_j; \lambda_i; \beta) & = (198\beta + 684)(\lambda_k - \lambda_i)^4 (\lambda_j - \lambda_i)^2 \\ & \quad + (2016 + 36\beta^2 + 360\beta)(\lambda_k - \lambda_i)^3 (\lambda_j - \lambda_i)^3 \\ & \quad + (432 + 18\beta^2 + 288\beta)(\lambda_k - \lambda_i)^2 (\lambda_j - \lambda_i)^4 \\ & \geq 0 \text{ for all } \beta \geq -1.6754, \\ \zeta_2^{(3)}(-\lambda_i; \lambda_k; \lambda_j; \lambda_i; \beta) & = (900\beta + 360)(\lambda_k - \lambda_i)^4 (\lambda_j - \lambda_i) \\ & \quad + (432\beta^2 + 8208 + 1512\beta)(\lambda_k - \lambda_i)^3 (\lambda_j - \lambda_i)^2 \\ & \quad + (3312\beta + 5184 + 36\beta^3 + 216\beta^2)(\lambda_k - \lambda_i)^2 (\lambda_j - \lambda_i)^3 \\ & \quad + (216\beta^2 + 288 + 504\beta)(\lambda_k - \lambda_i)(\lambda_j - \lambda_i)^4 \\ & \geq 0 \text{ for all } \beta \geq -0.4, \end{aligned}$$

$$\begin{aligned}
\zeta_2^{(4)}(\lambda; \lambda_k; \lambda_j; \lambda_i; \beta) &= (1800\beta + 720)(\lambda_k - \lambda_i)^4 \\
&\quad + (7200\beta + 2160\beta^2 + 11520)(\lambda_k - \lambda_i)^3(\lambda_j - \lambda_i) \\
&\quad + (432\beta^3 + 2808\beta^2 + 15120\beta + 32832)(\lambda_k - \lambda_i)^2(\lambda_j - \lambda_i)^2 \\
&\quad + (2592\beta^2 + 4608 + 8640\beta + 288\beta^3)(\lambda_k - \lambda_i)(\lambda_j - \lambda_i)^3 \\
&\geq 0 \text{ for all } \beta \geq -0.4, \\
\zeta_2^{(5)}(-\lambda_i; \lambda_k; \lambda_j; \lambda_i; \beta) &= (25920 + 21600\beta + 4320\beta^2)(\lambda_k - \lambda_i)^3 \\
&\quad + (51840 + 56160\beta + 19440\beta^2 + 2160\beta^3)(\lambda_k - \lambda_i)^2(\lambda_j - \lambda_i) \\
&\quad + (4320\beta^3 + 51840 + 43200\beta + 21600\beta^2)(\lambda_k - \lambda_i)(\lambda_j - \lambda_i)^2 \\
&\quad + (720\beta^3 + 10800\beta^2 + 17280 + 31680\beta)(\lambda_j - \lambda_i)^3 \\
&\geq 0 \text{ for all } \beta \geq -0.7085, \\
\zeta_2^{(6)}(-\lambda_i; \lambda_k; \lambda_j; \lambda_i; \beta) &= (4320\beta^3 + 177120\beta + 181440 + 51840\beta^2)(\lambda_k - \lambda_i)^2 \\
&\quad + (155520\beta + 25920\beta^3 + 129600\beta^2)(\lambda_k - \lambda_i)(\lambda_j - \lambda_i) \\
&\quad + (239760\beta + 194400 + 12960\beta^3 + 97200\beta^2)(\lambda_j - \lambda_i)^2.
\end{aligned}$$

Define $\hat{\zeta}_2(\lambda_k; \lambda_j; \lambda_i; \beta) = \zeta_2^{(6)}(-\lambda_i; \lambda_k; \lambda_j; \lambda_i; \beta)$. We wish to show that $\hat{\zeta}_2(\lambda_k; \lambda_j; \lambda_i; \beta)$ is non-negative for all $\beta \geq -\frac{2}{5}$. Differentiating $\hat{\zeta}_2(\lambda_k; \lambda_j; \lambda_i; \beta)$ with respect to λ_k we obtain

$$\begin{aligned}
\hat{\zeta}_2(\lambda_k; \lambda_j; \lambda_i; \beta) &= (4320\beta^3 + 177120\beta + 181440 + 51840\beta^2)(\lambda_k - \lambda_i)^2 \quad (\text{B.20}) \\
&\quad + (155520\beta + 25920\beta^3 + 129600\beta^2)(\lambda_k - \lambda_i)(\lambda_j - \lambda_i) \\
&\quad + (239760\beta + 194400 + 12960\beta^3 + 97200\beta^2)(\lambda_j - \lambda_i)^2,
\end{aligned}$$

$$\begin{aligned}
\hat{\zeta}_2^{(1)}(\lambda_k; \lambda_j; \lambda_i; \beta) &= 2(4320\beta^3 + 177120\beta + 181440 + 51840\beta^2)(\lambda_k - \lambda_i) \quad (\text{B.21}) \\
&\quad + (155520\beta + 25920\beta^3 + 129600\beta^2)(\lambda_j - \lambda_i)
\end{aligned}$$

$$\begin{aligned}
\hat{\zeta}_2^{(2)}(\lambda_k; \lambda_j; \lambda_i; \beta) &= 2(4320\beta^3 + 177120\beta + 181440 + 51840\beta^2) \\
&\geq 0 \text{ for all } \beta \geq -2.
\end{aligned}$$

Substituting $\lambda_k = \lambda_j$ into (B.20) and (B.21) we obtain

$$\begin{aligned}
\hat{\zeta}_2(\lambda_k; \lambda_j; \lambda_i; \beta) &= (375840 + 43200\beta^3 + 278640\beta^2 + 572400\beta)(\lambda_j - \lambda_i)^2 \\
&\geq 0 \text{ for all } \beta \geq -1.45, \\
\hat{\zeta}_2^{(1)}(\lambda_k; \lambda_j; \lambda_i; \beta) &= (34560\beta^3 + 509760\beta + 233280\beta^2 + 362880)(\lambda_k - \lambda_i) \\
&\geq 0 \text{ for all } \beta \geq -1.75.
\end{aligned}$$

Hence, $\zeta_2^{(6)}(-\lambda_i; \lambda_k; \lambda_j; \lambda_i; \beta)$ is non-negative for all $\beta \geq -\frac{2}{5}$ when $\lambda_i \leq 0$. From this we deduce that $\zeta_2(\lambda; \lambda_k; \lambda_j; \lambda_i; \beta)$ is non-negative for all $\lambda \geq -\lambda_i$.

Consider the case $\lambda_i > 0$. Let us define the functions $\zeta_{20}(\lambda_k; \lambda_j; \lambda_i; \beta)$, $\zeta_{21}(\lambda_k; \lambda_j; \lambda_i; \beta)$, $\zeta_{22}(\lambda_k; \lambda_j; \lambda_i; \beta)$, $\zeta_{23}(\lambda_k; \lambda_j; \lambda_i; \beta)$, $\zeta_{24}(\lambda_k; \lambda_j; \lambda_i; \beta)$, $\zeta_{25}(\lambda_k; \lambda_j; \lambda_i; \beta)$, and $\zeta_{26}(\lambda_k; \lambda_j; \lambda_i; \beta)$

as follows:

$$\begin{aligned}
\zeta_{20}(\lambda_k; \lambda_j; \lambda_i; \beta) &= \zeta_2(0; \lambda_k; \lambda_j; \lambda_i; \beta), \\
\zeta_{21}(\lambda_k; \lambda_j; \lambda_i; \beta) &= \zeta_2^{(1)}(0; \lambda_k; \lambda_j; \lambda_i; \beta), \\
\zeta_{22}(\lambda_k; \lambda_j; \lambda_i; \beta) &= \zeta_2^{(2)}(0; \lambda_k; \lambda_j; \lambda_i; \beta), \\
\zeta_{23}(\lambda_k; \lambda_j; \lambda_i; \beta) &= \zeta_2^{(3)}(0; \lambda_k; \lambda_j; \lambda_i; \beta), \\
\zeta_{24}(\lambda_k; \lambda_j; \lambda_i; \beta) &= \zeta_2^{(4)}(0; \lambda_k; \lambda_j; \lambda_i; \beta), \\
\zeta_{25}(\lambda_k; \lambda_j; \lambda_i; \beta) &= \zeta_2^{(5)}(0; \lambda_k; \lambda_j; \lambda_i; \beta), \\
\zeta_{26}(\lambda_k; \lambda_j; \lambda_i; \beta) &= \zeta_2^{(6)}(0; \lambda_k; \lambda_j; \lambda_i; \beta).
\end{aligned}$$

We will show that the above equations are all positive for all $\lambda_k \geq \lambda_j$ and $\beta \geq -\frac{2}{5}$ and, hence, prove that $\zeta_2(\lambda; \lambda_k; \lambda_j; \lambda_i; \beta)$ is positive for all $\lambda \geq 0$.

Differentiating $\zeta_{20}(\lambda_k; \lambda_j; \lambda_i; \beta)$ with respect to λ_k we obtain

$$\begin{aligned}
\zeta_{20}(\lambda_k; \lambda_j; \lambda_i; \beta) &= (75\beta + 30) \lambda_i^4 \lambda_j^4 \\
&\quad + 90\lambda_i^4 \lambda_k^4 + 90\lambda_j^4 \lambda_k^4 \\
&\quad + (18\beta^2 - 60\beta + 48) (\lambda_i^3 \lambda_j^4 \lambda_k + \lambda_i^4 \lambda_j^3 \lambda_k) \\
&\quad - 48(2 - \beta) (\lambda_i \lambda_j^4 \lambda_k^3 + \lambda_i^4 \lambda_j \lambda_k^3) \\
&\quad - 24(2 - \beta) (\lambda_i \lambda_j^3 \lambda_k^4 + \lambda_i^3 \lambda_j \lambda_k^4) \\
&\quad - 27(2 - \beta) \lambda_i^2 \lambda_j^2 \lambda_k^4 \\
&\quad + (9\beta^2 - 36) (\lambda_i^2 \lambda_j^4 \lambda_k^2 + \lambda_i^4 \lambda_j^2 \lambda_k^2) \\
&\quad + 18(\beta^2 - 6\beta + 8) (\lambda_i^2 \lambda_j^3 \lambda_k^3 + \lambda_i^3 \lambda_j^2 \lambda_k^3) \\
&\quad + (6\beta^3 - 54\beta^2 + 156\beta - 144) \lambda_i^3 \lambda_j^3 \lambda_k^2, \\
\zeta_{20}^{(1)}(\lambda_k; \lambda_j; \lambda_i; \beta) &= 360\lambda_k^3 (\lambda_i^4 + \lambda_j^4) \\
&\quad + (96\beta - 192) \lambda_k^3 (\lambda_i \lambda_j^3 + \lambda_i^3 \lambda_j) \\
&\quad + (108\beta - 216) \lambda_k^3 \lambda_i^2 \lambda_j^2 \\
&\quad + (54\beta^2 - 324\beta + 432) (\lambda_i^3 \lambda_j^2 + \lambda_i^2 \lambda_j^3) \lambda_k^2 \\
&\quad + (144\beta - 288) \lambda_k^2 (\lambda_i \lambda_j^4 + \lambda_i^4 \lambda_j) \\
&\quad + (18\beta^2 - 72) \lambda_k (\lambda_i^4 \lambda_j^2 + \lambda_i^2 \lambda_j^4) \\
&\quad + (12\beta^3 - 108\beta^2 + 312\beta - 288) \lambda_k \lambda_i^3 \lambda_j^3 \\
&\quad + (18\beta^2 - 60\beta + 48) (\lambda_i^3 \lambda_j^4 + \lambda_i^4 \lambda_j^3), \\
\zeta_{20}^{(2)}(\lambda_k; \lambda_j; \lambda_i; \beta) &= (288\beta - 576) \lambda_k^2 (\lambda_i^3 \lambda_j + \lambda_i \lambda_j^3) \\
&\quad + (324\beta - 648) \lambda_k^2 \lambda_i^2 \lambda_j^2 \\
&\quad + 1080\lambda_k^2 (\lambda_i^4 + \lambda_j^4) \\
&\quad + (108\beta^2 - 648\beta + 864) \lambda_k (\lambda_i^3 \lambda_j^2 + \lambda_i^2 \lambda_j^3) \\
&\quad + (288\beta - 576) \lambda_k (\lambda_i^4 \lambda_j + \lambda_i \lambda_j^4) \\
&\quad + (18\beta^2 - 72) (\lambda_i^4 \lambda_j^2 + \lambda_i^2 \lambda_j^4) \\
&\quad + (12\beta^3 - 108\beta^2 + 312\beta - 288) \lambda_i^3 \lambda_j^3,
\end{aligned}$$

$$\begin{aligned}
\zeta_{20}^{(3)}(\lambda_k; \lambda_j; \lambda_i; \beta) &= (576\beta - 1152) \lambda_k (\lambda_i \lambda_j^3 + \lambda_i^3 \lambda_j) \\
&\quad + (648\beta - 1296) \lambda_k \lambda_i^2 \lambda_j^2 \\
&\quad + 2160 \lambda_k (\lambda_i^4 + \lambda_j^4) \\
&\quad + (108\beta^2 - 648\beta + 864) (\lambda_i^3 \lambda_j^2 + \lambda_i^2 \lambda_j^3) \\
&\quad + (288\beta - 576) (\lambda_i^4 \lambda_j + \lambda_i \lambda_j^4), \\
\zeta_{20}^{(4)}(\lambda_k; \lambda_j; \lambda_i; \beta) &= (576\beta - 1152) \lambda_k (\lambda_i \lambda_j^3 + \lambda_i^3 \lambda_j) \\
&\quad + (648\beta - 1296) \lambda_i^2 \lambda_j^2 \\
&\quad + 2160 (\lambda_i^4 + \lambda_j^4).
\end{aligned}$$

Define $\zeta_{204}(\lambda_j; \lambda_i; \beta) = \zeta_{20}^{(4)}(\lambda_k; \lambda_j; \lambda_i; \beta)$. We wish to show that ζ_{204} is non-negative for all $\lambda_j \geq \lambda_i$. Differentiating ζ_{204} with respect to λ_j gives

$$\begin{aligned}
\zeta_{204}(\lambda_j; \lambda_i; \beta) &= (576\beta - 1152) \lambda_k (\lambda_i \lambda_j^3 + \lambda_i^3 \lambda_j) \\
&\quad (648\beta - 1296) \lambda_i^2 \lambda_j^2 \\
&\quad 2160 (\lambda_i^4 + \lambda_j^4), \\
\zeta_{204}^{(1)}(\lambda_j; \lambda_i; \beta) &= 8640 \lambda_j^3 + (-3456 \lambda_i + 1728 \lambda_i \beta) \lambda_j^2 + (1296 \lambda_i^2 \beta - 2592 \lambda_i^2) \lambda_j + (576\beta - 1152) \lambda_i^3, \\
\zeta_{204}^{(2)}(\lambda_j; \lambda_i; \beta) &= 25920 \lambda_j^2 + (-6912 \lambda_i + 3456 \lambda_i \beta) \lambda_j + 1296 \lambda_i^2 \beta - 2592 \lambda_i^2, \\
\zeta_{204}^{(3)}(\lambda_j; \lambda_i; \beta) &= 51840 \lambda_j - 6912 \lambda_i + 3456 \lambda_i \beta, \\
\zeta_{204}^{(4)}(\lambda_j; \lambda_i; \beta) &= 51840 > 0.
\end{aligned}$$

Now

$$\begin{aligned}
\zeta_{204}(\lambda_i; \lambda_i; \beta) &= (1800\beta + 720) \lambda_i^4 \geq 0 \text{ for all } \beta \geq -\frac{2}{5}, \\
\zeta_{204}^{(1)}(\lambda_i; \lambda_i; \beta) &= (1440 + 3600\beta) \lambda_i^3 \geq 0 \text{ for all } \beta \geq -\frac{2}{5}, \\
\zeta_{204}^{(2)}(\lambda_i; \lambda_i; \beta) &= (16416 + 4752\beta) \lambda_i^2 \geq 0 \text{ for all } \beta \geq -\frac{38}{11}, \\
\zeta_{204}^{(3)}(\lambda_i; \lambda_i; \beta) &= (44928 + 3456\beta) \lambda_i \geq 0 \text{ for all } \beta \geq -13.
\end{aligned}$$

Hence, $\zeta_{20}^{(4)}(\lambda_k; \lambda_j; \lambda_i; \beta)$ is non-negative for all $\lambda_j \geq \lambda_i$ and $\beta \geq -\frac{2}{5}$.

Define $\zeta_{203}(\lambda_j; \lambda_i; \beta) = \zeta_{20}^{(3)}(\lambda_j; \lambda_j; \lambda_i; \beta) / \lambda_j$. We wish to show that ζ_{203} is non-negative for all $\lambda_j \geq \lambda_i$. Differentiating ζ_{203} with respect to λ_j gives

$$\begin{aligned}
\zeta_{203}(\lambda_j; \lambda_i; \beta) &= 2160 \lambda_j^4 + (864\beta - 1728) \lambda_j^3 \lambda_i + (108\beta^2 - 432) \lambda_j^2 \lambda_i^2 \\
&\quad + (-72\beta + 108\beta^2 - 288) \lambda_j \lambda_i^3 + (1584 + 288\beta) \lambda_i^4, \\
\zeta_{203}^{(1)}(\lambda_j; \lambda_i; \beta) &= 8640 \lambda_j^3 + (2592\beta - 5184) \lambda_j^2 \lambda_i + (-864 + 216\beta^2) \lambda_j \lambda_i^2 \\
&\quad + (-72\beta + 108\beta^2 - 288) \lambda_i^3, \\
\zeta_{203}^{(2)}(\lambda_j; \lambda_i; \beta) &= 25920 \lambda_j^2 + (5184\beta - 10368) \lambda_j \lambda_i + (-864 + 216\beta^2) \lambda_i^2, \\
\zeta_{203}^{(3)}(\lambda_j; \lambda_i; \beta) &= 51840 \lambda_j + 5184 \lambda_i \beta - 10368 \lambda_i, \\
\zeta_{203}^{(4)}(\lambda_j; \lambda_i; \beta) &= 51840 > 0.
\end{aligned}$$

Now

$$\begin{aligned}
\zeta_{203}(\lambda_i; \lambda_i; \beta) &= (1296 + 1080\beta + 216\beta^2)\lambda_i^4 \geq 0 \text{ for all } \beta \geq -2, \\
\zeta_{203}^{(1)}(\lambda_i; \lambda_i; \beta) &= (2304 + 2520\beta + 324\beta^2)\lambda_i^3 \geq 0 \text{ for all } \beta \geq -1.05, \\
\zeta_{203}^{(2)}(\lambda_i; \lambda_i; \beta) &= (14688 + 5184\beta + 216\beta^2)\lambda_i^2 \geq 0 \text{ for all } \beta \geq -3.28, \\
\zeta_{203}^{(3)}(\lambda_i; \lambda_i; \beta) &= (41472 + 5184\beta)\lambda_i \geq 0 \text{ for all } \beta \geq -8.
\end{aligned}$$

Hence, $\zeta_{20}^{(3)}(\lambda_j; \lambda_j; \lambda_i; \beta)$ is non-negative for all $\lambda_j \geq \lambda_i$ and $\beta \geq -\frac{2}{5}$.

Define $\zeta_{202}(\lambda_j; \lambda_i; \beta) = \zeta_{20}^{(2)}(\lambda_j; \lambda_j; \lambda_i; \beta)/\lambda_j^2$. We wish to show that ζ_{202} is non-negative for all $\lambda_j \geq \lambda_i$. Differentiating ζ_{202} with respect to λ_j gives

$$\begin{aligned}
\zeta_{202}(\lambda_j; \lambda_i; \beta) &= 1080\lambda_j^4 + (576\beta - 1152)\lambda_j^3\lambda_i + (144 - 324\beta + 126\beta^2)\lambda_j^2\lambda_i^2 \\
&\quad + (12\beta^3 - 48\beta)\lambda_j\lambda_i^3 + (288\beta + 432 + 18\beta^2)\lambda_i^4, \\
\zeta_{202}^{(1)}(\lambda_j; \lambda_i; \beta) &= 4320\lambda_j^3 + (1728\beta - 3456)\lambda_j^2\lambda_i + (-648\beta + 252\beta^2 + 288)\lambda_j\lambda_i^2 \\
&\quad + (12\beta^3 - 48\beta)\lambda_i^3, \\
\zeta_{202}^{(2)}(\lambda_j; \lambda_i; \beta) &= 12960\lambda_j^2 + (-6912 + 3456\beta)\lambda_j\lambda_i + (-648\beta + 252\beta^2 + 288)\lambda_i^2, \\
\zeta_{202}^{(3)}(\lambda_j; \lambda_i; \beta) &= 25920\lambda_j - 6912\lambda_i + 3456\lambda_i\beta, \\
\zeta_{202}^{(4)}(\lambda_j; \lambda_i; \beta) &= 25920 > 0.
\end{aligned}$$

Now

$$\begin{aligned}
\zeta_{202}(\lambda_i; \lambda_i; \beta) &= (504 + 492\beta + 144\beta^2 + 12\beta^3)\lambda_i^4 \geq 0 \text{ for all } \beta \geq -2, \\
\zeta_{202}^{(1)}(\lambda_i; \lambda_i; \beta) &= (1152 + 1032\beta + 252\beta^2 + 12\beta^3)\lambda_i^3 \geq 0 \text{ for all } \beta \geq -2, \\
\zeta_{202}^{(2)}(\lambda_i; \lambda_i; \beta) &= (6336 + 2808\beta + 252\beta^2)\lambda_i^2 \geq 0 \text{ for all } \beta \geq -3.14, \\
\zeta_{202}^{(3)}(\lambda_i; \lambda_i; \beta) &= (19008 + 3456\beta)\lambda_i \geq 0 \text{ for all } \beta \geq -5.5.
\end{aligned}$$

Hence, $\zeta_{20}^{(2)}(\lambda_j; \lambda_j; \lambda_i; \beta)$ is non-negative for all $\lambda_j \geq \lambda_i$ and $\beta \geq -\frac{2}{5}$.

Define $\zeta_{201}(\lambda_j; \lambda_i; \beta) = \zeta_{20}^{(1)}(\lambda_j; \lambda_j; \lambda_i; \beta)/\lambda_j^3$. We wish to show that ζ_{201} is non-negative for all $\lambda_j \geq \lambda_i$. Differentiating ζ_{201} with respect to λ_j gives

$$\begin{aligned}
\zeta_{201}(\lambda_j; \lambda_i; \beta) &= 360\lambda_j^4 + (-480 + 240\beta)\lambda_j^3\lambda_i + (72\beta^2 - 216\beta + 144)\lambda_j^2\lambda_i^2 \\
&\quad + (-36\beta^2 + 24\beta + 12\beta^3)\lambda_j\lambda_i^3 + (84\beta + 36\beta^2 + 48)\lambda_i^4, \\
\zeta_{201}^{(1)}(\lambda_j; \lambda_i; \beta) &= 1440\lambda_j^3 + (-1440 + 720\beta)\lambda_j^2\lambda_i + (288 - 432\beta + 144\beta^2)\lambda_j\lambda_i^2 \\
&\quad + (-36\beta^2 + 24\beta + 12\beta^3)\lambda_i^3, \\
\zeta_{201}^{(2)}(\lambda_j; \lambda_i; \beta) &= 4320\lambda_j^2 + (-2880 + 1440\beta)\lambda_j\lambda_i + (288 - 432\beta + 144\beta^2)\lambda_i^2, \\
\zeta_{201}^{(3)}(\lambda_j; \lambda_i; \beta) &= 8640\lambda_j - 2880\lambda_i + 1440\lambda_i\beta, \\
\zeta_{201}^{(4)}(\lambda_j; \lambda_i; \beta) &= 8640 > 0.
\end{aligned}$$

Now

$$\begin{aligned}
\zeta_{201}(\lambda_i; \lambda_i; \beta) &= (72 + 132\beta + 72\beta^2 + 12\beta^3)\lambda_i^4 \geq 0 \text{ for all } \beta \geq -1, \\
\zeta_{201}^{(1)}(\lambda_i; \lambda_i; \beta) &= (288 + 312\beta + 108\beta^2 + 12\beta^3)\lambda_i^3 \geq 0 \text{ for all } \beta \geq -2, \\
\zeta_{201}^{(2)}(\lambda_i; \lambda_i; \beta) &= (1728 + 1008\beta + 144\beta^2)\lambda_i^2 \geq 0 \text{ for all } \beta \geq -3, \\
\zeta_{201}^{(3)}(\lambda_i; \lambda_i; \beta) &= (5760 + 1440\beta)\lambda_i \geq 0 \text{ for all } \beta \geq -4.
\end{aligned}$$

Hence, $\zeta_{20}^{(1)}(\lambda_j; \lambda_j; \lambda_i; \beta)$ is non-negative for all $\lambda_j \geq \lambda_i$ and $\beta \geq -\frac{2}{5}$.

Define $\zeta_{200}(\lambda_j; \lambda_i; \beta) = \zeta_{20}(\lambda_j; \lambda_j; \lambda_i; \beta)/\lambda_j^4$. We wish to show that ζ_{200} is positive for all $\lambda_j \geq \lambda_i$. Differentiating ζ_{200} with respect to λ_j gives

$$\begin{aligned}
\zeta_{200}(\lambda_j; \lambda_i; \beta) &= 90\lambda_j^4 + (-144\lambda_i + 72\lambda_i\beta)\lambda_j^3 + (54\lambda_i^2 + 27\lambda_i^2\beta^2 - 81\lambda_i^2\beta)\lambda_j^2 \\
&\quad + (-18\lambda_i^3\beta^2 + 6\lambda_i^3\beta^3 + 12\lambda_i^3\beta)\lambda_j + 36\lambda_i^4 + 27\lambda_i^4\beta^2 + 63\lambda_i^4\beta, \\
\zeta_{200}^{(1)}(\lambda_j; \lambda_i; \beta) &= 360\lambda_j^3 + (-432\lambda_i + 216\lambda_i\beta)\lambda_j^2 + (-162\lambda_i^2\beta + 54\lambda_i^2\beta^2 + 108\lambda_i^2)\lambda_j \\
&\quad - 18\lambda_i^3\beta^2 + 6\lambda_i^3\beta^3 + 12\lambda_i^3\beta, \\
\zeta_{200}^{(2)}(\lambda_j; \lambda_i; \beta) &= 1080\lambda_j^2 + (432\lambda_i\beta - 864\lambda_i)\lambda_j - 162\lambda_i^2\beta + 54\lambda_i^2\beta^2 + 108\lambda_i^2, \\
\zeta_{200}^{(3)}(\lambda_j; \lambda_i; \beta) &= 2160\lambda_j + 432\lambda_i\beta - 864\lambda_i, \\
\zeta_{200}^{(4)}(\lambda_j; \lambda_i; \beta) &= 2160 > 0.
\end{aligned}$$

Now

$$\begin{aligned}
\zeta_{200}(\lambda_i; \lambda_i; \beta) &= (36 + 66\beta + 36\beta^2 + 6\beta^3)\lambda_i^4 > 0 \text{ for all } \beta > -1, \\
\zeta_{200}^{(1)}(\lambda_i; \lambda_i; \beta) &= (36 + 66\beta + 36\beta^2 + 6\beta^3)\lambda_i^3 > 0 \text{ for all } \beta > -1, \\
\zeta_{200}^{(2)}(\lambda_i; \lambda_i; \beta) &= (324 + 270\beta + 54\beta^2)\lambda_i^2 > 0 \text{ for all } \beta > -2, \\
\zeta_{200}^{(3)}(\lambda_i; \lambda_i; \beta) &= (1296 + 432\beta)\lambda_i > 0 \text{ for all } \beta > -3.
\end{aligned}$$

Hence, $\zeta_{20}(\lambda_j; \lambda_j; \lambda_i; \beta)$ is positive for all $\lambda_j \geq \lambda_i$ and $\beta \geq -\frac{2}{5}$. This implies that $\zeta(0; \lambda_k; \lambda_j; \lambda_i; \beta)$ is positive for all $\lambda_k \geq \lambda_j$ and $\beta \geq -\frac{2}{5}$.

Differentiating $\zeta_{21}(\lambda_k; \lambda_j; \lambda_i; \beta)$ with respect to λ_k we obtain

$$\begin{aligned}
\zeta_{21}(\lambda_k; \lambda_j; \lambda_i; \beta) &= (312 + 24\beta) \lambda_k^4 (\lambda_j^3 + \lambda_i^3) + 126 (\beta - 2) \lambda_k^4 (\lambda_j^2 \lambda_i + \lambda_j \lambda_i^2) \\
&\quad + 24 (2\beta + 11) \lambda_k^3 (\lambda_j^4 + \lambda_i^4) + 36 (\beta^2 + \beta - 8) \lambda_k^3 (\lambda_j^3 \lambda_i + \lambda_j \lambda_i^3) \\
&\quad + 108 (\beta^2 - 5\beta + 6) \lambda_k^3 \lambda_j^2 \lambda_i^2 + 18 (\beta^2 + 8\beta - 20) \lambda_k^2 (\lambda_j^4 \lambda_i + \lambda_j \lambda_i^4) \\
&\quad + 18 (\beta^3 - 4\beta^2 + 8\beta - 8) \lambda_k^2 (\lambda_j^3 \lambda_i^2 + \lambda_j^2 \lambda_i^3) \\
&\quad + 36 (2\beta^2 - 5\beta + 2) \lambda_k (\lambda_j^4 \lambda_i^2 + \lambda_j^2 \lambda_i^4) \\
&\quad + 12 (\beta^3 + 3\beta^2 - 14\beta + 8) \lambda_k \lambda_j^3 \lambda_i^3 + 6 (3\beta^2 + 40\beta + 28) \lambda_k \lambda_j^3 \lambda_i^3, \\
\zeta_{21}^{(1)}(\lambda_k; \lambda_j; \lambda_i; \beta) &= (96\beta + 1248) \lambda_k^3 (\lambda_j^3 + \lambda_i^3) + (504\beta - 1008) \lambda_k^3 (\lambda_j^2 \lambda_i + \lambda_j \lambda_i^2) \\
&\quad + (48\beta + 264) \lambda_k^2 (\lambda_j^4 + \lambda_i^4) + (-288\lambda_i + 36\lambda_i \beta^2 + 72\lambda_i \beta) \lambda_k^2 (\lambda_j^3 \lambda_i + \lambda_j \lambda_i^3) \\
&\quad + (648 - 540\beta + 108\beta^2) \lambda_k^2 \lambda_j^2 \lambda_i^2 \\
&\quad + (36\lambda_i \beta^2 + 288\lambda_i \beta - 720\lambda_i) \lambda_k (\lambda_j^4 \lambda_i + \lambda_j \lambda_i^4) \\
&\quad + (36\beta^3 + 288\beta - 144\beta^2 - 288) \lambda_k (\lambda_j^3 \lambda_i^2 + \lambda_j^2 \lambda_i^3) \\
&\quad + (72\beta^2 + 72 - 180\beta) (\lambda_j^4 \lambda_i^2 + \lambda_j^2 \lambda_i^4) + (36\beta^2 - 168\beta + 96 + 12\beta^3) \lambda_j^3 \lambda_i^3, \\
\zeta_{21}^{(2)}(\lambda_k; \lambda_j; \lambda_i; \beta) &= (3744 + 288\beta) \lambda_k^2 (\lambda_j^3 + \lambda_i^3) + (1512\beta - 3024) \lambda_k^2 (\lambda_j^2 \lambda_i + \lambda_j \lambda_i^2) \\
&\quad + (288\beta + 1584) \lambda_k (\lambda_j^4 + \lambda_i^4) + (216\beta^2 - 1728 + 432\beta) \lambda_k (\lambda_j^3 \lambda_i + \lambda_j \lambda_i^3) \\
&\quad + (-3240\beta + 648\beta^2 + 3888) \lambda_k \lambda_j^2 \lambda_i^2 + (36\beta^2 + 288\beta - 720) (\lambda_j^4 \lambda_i + \lambda_j \lambda_i^4) \\
&\quad + (36\beta^3 + 288\beta - 144\beta^2 - 288) (\lambda_j^3 \lambda_i^2 + \lambda_j^2 \lambda_i^3), \\
\zeta_{21}^{(3)}(\lambda_k; \lambda_j; \lambda_i; \beta) &= (7488 + 576\beta) \lambda_k (\lambda_j^3 + \lambda_i^3) + (3024\beta - 6048) \lambda_k (\lambda_j^2 \lambda_i + \lambda_j \lambda_i^2) \\
&\quad + (288\beta + 1584) (\lambda_j^4 + \lambda_i^4) + (216\beta^2 - 1728 + 432\beta) (\lambda_j^3 \lambda_i + \lambda_j \lambda_i^3) \\
&\quad + (-3240\beta + 648\beta^2 + 3888) \lambda_j^2 \lambda_i^2, \\
\zeta_{21}^{(4)}(\lambda_k; \lambda_j; \lambda_i; \beta) &= (7488 + 576\beta) (\lambda_j^3 + \lambda_i^3) + (3024\beta - 6048) (\lambda_j^2 \lambda_i + \lambda_j \lambda_i^2).
\end{aligned}$$

Define $\zeta_{214}(\lambda_j; \lambda_i; \beta) = \zeta_{21}^{(4)}(\lambda_k; \lambda_j; \lambda_i; \beta)$. We wish to show that ζ_{214} is non-negative for all $\lambda_j \geq \lambda_i$. Differentiating ζ_{214} with respect to λ_j gives

$$\begin{aligned}
\zeta_{214}(\lambda_j; \lambda_i; \beta) &= (7488 + 576\beta) (\lambda_j^3 + \lambda_i^3) + (3024\beta - 6048) (\lambda_j^2 \lambda_i + \lambda_j \lambda_i^2), \\
\zeta_{214}^{(1)}(\lambda_j; \lambda_i; \beta) &= (1728\beta + 22464) \lambda_j^2 + (6048\lambda_i \beta - 12096\lambda_i) \lambda_j - 6048\lambda_i^2 + 3024\lambda_i^2 \beta, \\
\zeta_{214}^{(2)}(\lambda_j; \lambda_i; \beta) &= (3456\beta + 44928) \lambda_j + 6048\lambda_i \beta - 12096\lambda_i, \\
\zeta_{214}^{(3)}(\lambda_j; \lambda_i; \beta) &= 3456\beta + 44928 \geq 0 \text{ for all } \beta \geq -13.
\end{aligned}$$

Now

$$\begin{aligned}
\zeta_{214}(\lambda_i; \lambda_i; \beta) &= (7200\beta + 2880) \lambda_i^3 \geq 0 \text{ for all } \beta \geq -\frac{2}{5}, \\
\zeta_{214}^{(1)}(\lambda_i; \lambda_i; \beta) &= (10800\beta + 4320) \lambda_i^2 \geq 0 \text{ for all } \beta \geq -\frac{2}{5}, \\
\zeta_{214}^{(2)}(\lambda_i; \lambda_i; \beta) &= (9504\beta + 32832) \lambda_i \geq 0 \text{ for all } \beta \geq -\frac{38}{11}.
\end{aligned}$$

Hence, $\zeta_{21}^{(4)}(\lambda_k; \lambda_j; \lambda_i; \beta)$ is non-negative for all $\lambda_j \geq \lambda_i$ and $\beta \geq -\frac{2}{5}$.

Define $\zeta_{213}(\lambda_j; \lambda_i; \beta) = \zeta_{21}^{(3)}(\lambda_j; \lambda_j; \lambda_i; \beta)$. We wish to show that ζ_{213} is non-negative for all $\lambda_j \geq \lambda_i$. Differentiating ζ_{213} with respect to λ_j gives

$$\begin{aligned}\zeta_{213}(\lambda_j; \lambda_i; \beta) &= (864\beta + 9072)\lambda_j^4 + (216\lambda_i\beta^2 + 3456\lambda_i\beta - 7776\lambda_i)\lambda_j^3 \\ &\quad + (-216\lambda_i^2\beta - 2160\lambda_i^2 + 648\lambda_i^2\beta^2)\lambda_j^2 \\ &\quad + (5760\lambda_i^3 + 1008\lambda_i^3\beta + 216\lambda_i^3\beta^2)\lambda_j + 1584\lambda_i^4 + 288\lambda_i^4\beta, \\ \zeta_{213}^{(1)}(\lambda_j; \lambda_i; \beta) &= (3456\beta + 36288)\lambda_j^3 + (-23328\lambda_i + 10368\lambda_i\beta + 648\lambda_i\beta^2)\lambda_j^2 \\ &\quad + (1296\lambda_i^2\beta^2 - 432\lambda_i^2\beta - 4320\lambda_i^2)\lambda_j + 5760\lambda_i^3 + 1008\lambda_i^3\beta + 216\lambda_i^3\beta^2, \\ \zeta_{213}^{(2)}(\lambda_j; \lambda_i; \beta) &= (10368\beta + 108864)\lambda_j^2 + (1296\lambda_i\beta^2 - 46656\lambda_i + 20736\lambda_i\beta)\lambda_j \\ &\quad + 1296\lambda_i^2\beta^2 - 432\lambda_i^2\beta - 4320\lambda_i^2, \\ \zeta_{213}^{(3)}(\lambda_j; \lambda_i; \beta) &= (217728 + 20736\beta)\lambda_j + 1296\lambda_i\beta^2 - 46656\lambda_i + 20736\lambda_i\beta, \\ \zeta_{213}^{(4)}(\lambda_j; \lambda_i; \beta) &= 217728 + 20736\beta \geq 0 \text{ for all } \beta \geq -10.5.\end{aligned}$$

Now

$$\begin{aligned}\zeta_{213}(\lambda_i; \lambda_i; \beta) &= (5400\beta + 6480 + 1080\beta^2)\lambda_i^4 \geq 0 \text{ for all } \beta \geq -2, \\ \zeta_{213}^{(1)}(\lambda_i; \lambda_i; \beta) &= (14400\beta + 14400 + 2160\beta^2)\lambda_i^3 \geq 0 \text{ for all } \beta \geq -1.22, \\ \zeta_{213}^{(2)}(\lambda_i; \lambda_i; \beta) &= (30672\beta + 57888 + 2592\beta^2)\lambda_i^2 \geq 0 \text{ for all } \beta \geq -2.35, \\ \zeta_{213}^{(3)}(\lambda_i; \lambda_i; \beta) &= (171072 + 41472\beta + 1296\beta^2)\lambda_i \geq 0 \text{ for all } \beta \geq -4.86.\end{aligned}$$

Hence, $\zeta_{21}^{(3)}(\lambda_j; \lambda_j; \lambda_i; \beta)$ is non-negative for all $\lambda_j \geq \lambda_i$ and $\beta \geq -\frac{2}{5}$.

Define $\zeta_{212}(\lambda_j; \lambda_i; \beta) = \zeta_{21}^{(2)}(\lambda_j; \lambda_j; \lambda_i; \beta)/\lambda_j$. We wish to show that ζ_{212} is non-negative for all $\lambda_j \geq \lambda_i$. Differentiating ζ_{212} with respect to λ_j gives

$$\begin{aligned}\zeta_{212}(\lambda_j; \lambda_i; \beta) &= (5328 + 576\beta)\lambda_j^4 + (2232\lambda_i\beta + 252\lambda_i\beta^2 - 5472\lambda_i)\lambda_j^3 \\ &\quad + (504\lambda_i^2\beta^2 + 36\lambda_i^2\beta^3 - 1440\lambda_i^2\beta + 576\lambda_i^2)\lambda_j^2 \\ &\quad + (1008\lambda_i^3\beta + 1728\lambda_i^3 + 36\lambda_i^3\beta^3 + 72\lambda_i^3\beta^2)\lambda_j \\ &\quad + 576\lambda_i^4\beta + 36\lambda_i^4\beta^2 + 864\lambda_i^4, \\ \zeta_{212}^{(1)}(\lambda_j; \lambda_i; \beta) &= (2304\beta + 21312)\lambda_j^3 + (-16416\lambda_i + 6696\lambda_i\beta + 756\lambda_i\beta^2)\lambda_j^2 \\ &\quad + (1152\lambda_i^2 + 1008\lambda_i^2\beta^2 + 72\lambda_i^2\beta^3 - 2880\lambda_i^2\beta)\lambda_j \\ &\quad + 1008\lambda_i^3\beta + 1728\lambda_i^3 + 36\lambda_i^3\beta^3 + 72\lambda_i^3\beta^2, \\ \zeta_{212}^{(2)}(\lambda_j; \lambda_i; \beta) &= (63936 + 6912\beta)\lambda_j^2 + (1512\lambda_i\beta^2 - 32832\lambda_i + 13392\lambda_i\beta)\lambda_j \\ &\quad + 1152\lambda_i^2 + 1008\lambda_i^2\beta^2 + 72\lambda_i^2\beta^3 - 2880\lambda_i^2\beta, \\ \zeta_{212}^{(3)}(\lambda_j; \lambda_i; \beta) &= (13824\beta + 127872)\lambda_j + 1512\lambda_i\beta^2 - 32832\lambda_i + 13392\lambda_i\beta, \\ \zeta_{212}^{(4)}(\lambda_j; \lambda_i; \beta) &= (13824\beta + 127872) \geq 0 \text{ for all } \beta \geq -9.25.\end{aligned}$$

Now

$$\begin{aligned}
\zeta_{212}(\lambda_i; \lambda_i; \beta) &= (3024 + 2952\beta + 864\beta^2 + 72\beta^3)\lambda_i^4 \geq 0 \text{ for all } \beta \geq -2, \\
\zeta_{212}^{(1)}(\lambda_i; \lambda_i; \beta) &= (7128\beta + 7776 + 1836\beta^2 + 108\beta^3)\lambda_i^3 \geq 0 \text{ for all } \beta \geq -2, \\
\zeta_{212}^{(2)}(\lambda_i; \lambda_i; \beta) &= (32256 + 17424\beta + 2520\beta^2 + 72\beta^3)\lambda_i^2 \geq 0 \text{ for all } \beta \geq -3.18, \\
\zeta_{212}^{(3)}(\lambda_i; \lambda_i; \beta) &= (27216\beta + 95040 + 1512\beta^2)\lambda_i \geq 0 \text{ for all } \beta \geq -4.74.
\end{aligned}$$

Hence, $\zeta_{21}^{(2)}(\lambda_j; \lambda_j; \lambda_i; \beta)$ is non-negative for all $\lambda_j \geq \lambda_i$ and $\beta \geq -\frac{2}{5}$.

Define $\zeta_{211}(\lambda_j; \lambda_i; \beta) = \zeta_{21}^{(1)}(\lambda_j; \lambda_j; \lambda_i; \beta)/\lambda_j^2$. We wish to show that ζ_{211} is non-negative for all $\lambda_j \geq \lambda_i$. Differentiating ζ_{211} with respect to λ_j gives

$$\begin{aligned}
\zeta_{211}(\lambda_j; \lambda_i; \beta) &= (2040 + 240\beta)\lambda_j^4 + (144\lambda_i\beta^2 - 2592\lambda_i + 1008\lambda_i\beta)\lambda_j^3 \\
&\quad + (720\lambda_i^2 - 1008\lambda_i^2\beta + 36\lambda_i^2\beta^3 + 252\lambda_i^2\beta^2)\lambda_j^2 \\
&\quad + (192\lambda_i^3 + 48\lambda_i^3\beta^3 + 432\lambda_i^3\beta)\lambda_j + 144\lambda_i^4 + 252\lambda_i^4\beta + 108\lambda_i^4\beta^2, \\
\zeta_{211}^{(1)}(\lambda_j; \lambda_i; \beta) &= (8160 + 960\beta)\lambda_j^3 + (3024\lambda_i\beta + 432\lambda_i\beta^2 - 7776\lambda_i)\lambda_j^2 \\
&\quad + (504\lambda_i^2\beta^2 + 1440\lambda_i^2 - 2016\lambda_i^2\beta + 72\lambda_i^2\beta^3)\lambda_j \\
&\quad + 192\lambda_i^3 + 48\lambda_i^3\beta^3 + 432\lambda_i^3\beta, \\
\zeta_{211}^{(2)}(\lambda_j; \lambda_i; \beta) &= (2880\beta + 24480)\lambda_j^2 + (-15552\lambda_i + 6048\lambda_i\beta + 864\lambda_i\beta^2)\lambda_j \\
&\quad + 504\lambda_i^2\beta^2 + 1440\lambda_i^2 - 2016\lambda_i^2\beta + 72\lambda_i^2\beta^3, \\
\zeta_{211}^{(3)}(\lambda_j; \lambda_i; \beta) &= (48960 + 5760\beta)\lambda_j - 15552\lambda_i + 6048\lambda_i\beta + 864\lambda_i\beta^2, \\
\zeta_{211}^{(4)}(\lambda_j; \lambda_i; \beta) &= 48960 + 5760\beta \geq 0 \text{ for all } \beta \geq -8.5.
\end{aligned}$$

Now

$$\begin{aligned}
\zeta_{211}(\lambda_i; \lambda_i; \beta) &= (504 + 924\beta + 504\beta^2 + 84\beta^3)\lambda_i^4 \geq 0 \text{ for all } \beta \geq -1, \\
\zeta_{211}^{(1)}(\lambda_i; \lambda_i; \beta) &= (2016 + 2400\beta + 936\beta^2 + 120\beta^3)\lambda_i^3 \geq 0 \text{ for all } \beta \geq -2, \\
\zeta_{211}^{(2)}(\lambda_i; \lambda_i; \beta) &= (6912\beta + 10368 + 1368\beta^2 + 72\beta^3)\lambda_i^2 \geq 0 \text{ for all } \beta \geq -3, \\
\zeta_{211}^{(3)}(\lambda_i; \lambda_i; \beta) &= (33408 + 11808\beta + 864\beta^2)\lambda_i \geq 0 \text{ for all } \beta \geq -4.
\end{aligned}$$

Hence, $\zeta_{21}^{(1)}(\lambda_j; \lambda_j; \lambda_i; \beta)$ is non-negative for all $\lambda_j \geq \lambda_i$ and $\beta \geq -\frac{2}{5}$.

Define $\zeta_{210}(\lambda_j; \lambda_i; \beta) = \zeta_{21}(\lambda_j; \lambda_j; \lambda_i; \beta)/\lambda_j^3$. We wish to show that ζ_{210} is positive for all

$\lambda_j \geq \lambda_i$. Differentiating ζ_{210} with respect to λ_j gives

$$\begin{aligned}
\zeta_{210}(\lambda_j; \lambda_i; \beta) &= (576 + 72\beta)\lambda_j^4 + (-900\lambda_i + 54\lambda_i\beta^2 + 342\lambda_i\beta)\lambda_j^3 \\
&\quad + (-450\lambda_i^2\beta + 18\lambda_i^2\beta^3 + 324\lambda_i^2 + 108\lambda_i^2\beta^2)\lambda_j^2 \\
&\quad + (30\lambda_i^3\beta^3 + 312\lambda_i^3\beta + 18\lambda_i^3\beta^2 + 144\lambda_i^3)\lambda_j \\
&\quad + 108\lambda_i^4\beta^2 + 252\lambda_i^4\beta + 144\lambda_i^4, \\
\zeta_{210}^{(1)}(\lambda_j; \lambda_i; \beta) &= (288\beta + 2304)\lambda_j^3 + (1026\lambda_i\beta - 2700\lambda_i + 162\lambda_i\beta^2)\lambda_j^2 \\
&\quad + (216\lambda_i^2\beta^2 - 900\lambda_i^2\beta + 36\lambda_i^2\beta^3 + 648\lambda_i^2)\lambda_j \\
&\quad + 30\lambda_i^3\beta^3 + 312\lambda_i^3\beta + 18\lambda_i^3\beta^2 + 144\lambda_i^3, \\
\zeta_{210}^{(2)}(\lambda_j; \lambda_i; \beta) &= (6912 + 864\beta)\lambda_j^2 + (324\lambda_i\beta^2 + 2052\lambda_i\beta - 5400\lambda_i)\lambda_j \\
&\quad + 216\lambda_i^2\beta^2 - 900\lambda_i^2\beta + 36\lambda_i^2\beta^3 + 648\lambda_i^2, \\
\zeta_{210}^{(3)}(\lambda_j; \lambda_i; \beta) &= (1728\beta + 13824)\lambda_j + 324\lambda_i\beta^2 + 2052\lambda_i\beta - 5400\lambda_i, \\
\zeta_{210}^{(4)}(\lambda_j; \lambda_i; \beta) &= 1728\beta + 13824 \geq 0 \text{ for all } \beta \geq -8.
\end{aligned}$$

Now

$$\begin{aligned}
\zeta_{210}(\lambda_i; \lambda_i; \beta) &= (288 + 528\beta + 288\beta^2 + 48\beta^3)\lambda_i^4 > 0 \text{ for all } \beta > -1, \\
\zeta_{210}^{(1)}(\lambda_i; \lambda_i; \beta) &= (726\beta + 396 + 396\beta^2 + 66\beta^3)\lambda_i^3 > 0 \text{ for all } \beta > -1, \\
\zeta_{210}^{(2)}(\lambda_i; \lambda_i; \beta) &= (2160 + 2016\beta + 540\beta^2 + 36\beta^3)\lambda_i^2 > 0 \text{ for all } \beta > -2, \\
\zeta_{210}^{(3)}(\lambda_i; \lambda_i; \beta) &= (3780\beta + 8424 + 324\beta^2)\lambda_i > 0 \text{ for all } \beta > -3.
\end{aligned}$$

Hence, $\zeta_{21}(\lambda_j; \lambda_j; \lambda_i; \beta)$ is positive for all $\lambda_j \geq \lambda_i$ and $\beta \geq -\frac{2}{5}$. This implies that $\zeta^{(1)}(0; \lambda_k; \lambda_j; \lambda_i; \beta)$ is positive for all $\lambda_k \geq \lambda_j$ and $\beta \geq -\frac{2}{5}$.

Differentiating $\zeta_{22}(\lambda_k; \lambda_j; \lambda_i; \beta)$ with respect to λ_k we obtain

$$\begin{aligned}
\zeta_{22}(\lambda_k; \lambda_j; \lambda_i; \beta) &= (198\beta + 684)\lambda_k^4(\lambda_j^2 + \lambda_i^2) \\
&\quad + 504(\beta - 2)\lambda_k^4\lambda_j\lambda_i \\
&\quad + (2016 + 36\beta^2 + 360\beta)\lambda_k^3(\lambda_j^3 + \lambda_i^3) \\
&\quad + (-576 + 324\beta^2 - 360\beta)\lambda_k^3(\lambda_j^2\lambda_i + \lambda_j\lambda_i^2) \\
&\quad + (432 + 18\beta^2 + 288\beta)\lambda_k^2(\lambda_j^4 + \lambda_i^4) \\
&\quad + (36\beta^2 + 1080\beta + 36\beta^3 - 2592)(\lambda_k^2\lambda_j^3\lambda_i + \lambda_k^2\lambda_j\lambda_i^3) \\
&\quad + (-756\beta - 108\beta^2 + 1080 + 108\beta^3)\lambda_k^2\lambda_j^2\lambda_i^2 \\
&\quad + (-72\beta + 180\beta^2 - 576)\lambda_k(\lambda_j^4\lambda_i + \lambda_j\lambda_i^4) \\
&\quad + (72\beta^3 + 252\beta^2 + 288 - 936\beta)(\lambda_k\lambda_j^3\lambda_i^2 + \lambda_k\lambda_j\lambda_i^3) \\
&\quad + (126\beta^2 + 576 + 540\beta)(\lambda_j^4\lambda_i^2 + \lambda_j^2\lambda_i^4) \\
&\quad + (180\beta^2 + 1752\beta + 12\beta^3 + 1440)\lambda_j^3\lambda_i^3,
\end{aligned}$$

$$\begin{aligned}
\zeta_{22}^{(1)}(\lambda_k; \lambda_j; \lambda_i; \beta) &= (792\beta + 2736)\lambda_k^3 (\lambda_j^2 + \lambda_i^2) + (-4032 + 2016\beta)\lambda_k^3 \lambda_j \lambda_i \\
&\quad + (108\beta^2 + 6048 + 1080\beta)\lambda_k^2 (\lambda_j^3 + \lambda_i^3) \\
&\quad + (972\beta^2 - 1080\beta - 1728)\lambda_k^2 (\lambda_j^2 \lambda_i + \lambda_j \lambda_i^2) \\
&\quad + (864 + 36\beta^2 + 576\beta)\lambda_k (\lambda_j^4 + \lambda_i^4) \\
&\quad + (-5184 + 2160\beta + 72\beta^2 + 72\beta^3)\lambda_k (\lambda_j^3 \lambda_i + \lambda_j \lambda_i^3) \\
&\quad + (2160 - 1512\beta - 216\beta^2 + 216\beta^3)\lambda_k \lambda_j^2 \lambda_i^2 \\
&\quad + (-72\beta + 180\beta^2 - 576) (\lambda_j^4 \lambda_i + \lambda_j \lambda_i^4) \\
&\quad + (72\beta^3 + 252\beta^2 + 288 - 936\beta) (\lambda_j^3 \lambda_i^2 + \lambda_j^2 \lambda_i^3), \\
\zeta_{22}^{(2)}(\lambda_k; \lambda_j; \lambda_i; \beta) &= (2376\beta + 8208)\lambda_k^2 (\lambda_j^2 + \lambda_i^2) + (-12096 + 6048\beta)\lambda_k^2 \lambda_j \lambda_i \\
&\quad + (216\beta^2 + 12096 + 2160\beta)\lambda_k (\lambda_j^3 + \lambda_i^3) \\
&\quad + (-3456 - 2160\beta + 1944\beta^2)\lambda_k (\lambda_j^2 \lambda_i + \lambda_j \lambda_i^2) \\
&\quad + (864 + 36\beta^2 + 576\beta) (\lambda_j^4 + \lambda_i^4) \\
&\quad + (-5184 + 2160\beta + 72\beta^2 + 72\beta^3) (\lambda_j^3 \lambda_i + \lambda_j \lambda_i^3) \\
&\quad + (2160 - 1512\beta - 216\beta^2 + 216\beta^3)\lambda_j^2 \lambda_i^2, \\
\zeta_{22}^{(3)}(\lambda_k; \lambda_j; \lambda_i; \beta) &= (16416 + 4752\beta)\lambda_k (\lambda_j^2 + \lambda_i^2) + (-24192 + 12096\beta)\lambda_k \lambda_j \lambda_i \\
&\quad + (216\beta^2 + 12096 + 2160\beta) (\lambda_j^3 + \lambda_i^3) \\
&\quad + (-3456 - 2160\beta + 1944\beta^2) (\lambda_i \lambda_j^2 + \lambda_i^2 \lambda_j), \\
\zeta_{22}^{(4)}(\lambda_k; \lambda_j; \lambda_i; \beta) &= (16416 + 4752\beta) (\lambda_j^2 + \lambda_i^2) + (-24192 + 12096\beta)\lambda_i \lambda_j.
\end{aligned}$$

Define $\zeta_{224}(\lambda_j; \lambda_i; \beta) = \zeta_{22}^{(4)}(\lambda_k; \lambda_j; \lambda_i; \beta)$. We wish to show that ζ_{224} is non-negative for all $\lambda_j \geq \lambda_i$. Differentiating ζ_{224} with respect to λ_j gives

$$\begin{aligned}
\zeta_{224}(\lambda_j; \lambda_i; \beta) &= (16416 + 4752\beta) (\lambda_j^2 + \lambda_i^2) + (-24192 + 12096\beta)\lambda_i \lambda_j, \\
\zeta_{224}^{(1)}(\lambda_j; \lambda_i; \beta) &= (32832 + 9504\beta)\lambda_j + (-24192 + 12096\beta)\lambda_i, \\
\zeta_{224}^{(2)}(\lambda_j; \lambda_i; \beta) &= 32832 + 9504\beta \geq 0 \text{ for all } \beta \geq -\frac{38}{11}.
\end{aligned}$$

Now

$$\begin{aligned}
\zeta_{224}(\lambda_i; \lambda_i; \beta) &= (8640 + 21600\beta)\lambda_i^2 \geq 0 \text{ for all } \beta \geq -\frac{2}{5}, \\
\zeta_{224}^{(1)}(\lambda_i; \lambda_i; \beta) &= (8640 + 21600\beta)\lambda_i \geq 0 \text{ for all } \beta \geq -\frac{2}{5}.
\end{aligned}$$

Hence, $\zeta_{22}^{(4)}(\lambda_k; \lambda_j; \lambda_i; \beta)$ is non-negative for all $\lambda_j \geq \lambda_i$ and $\beta \geq -\frac{2}{5}$.

Define $\zeta_{223}(\lambda_j; \lambda_i; \beta) = \zeta_{22}^{(3)}(\lambda_j; \lambda_j; \lambda_i; \beta)$. We wish to show that ζ_{223} is non-negative for

all $\lambda_j \geq \lambda_i$. Differentiating ζ_{223} with respect to λ_j gives

$$\begin{aligned}\zeta_{223}(\lambda_j; \lambda_i; \beta) &= (28512 + 216\beta^2 + 6912\beta)\lambda_j^3 + (9936\beta + 1944\beta^2 - 27648)\lambda_i\lambda_j^2 \\ &\quad + (12960 + 2592\beta + 1944\beta^2)\lambda_i^2\lambda_j + (216\beta^2 + 12096 + 2160\beta)\lambda_i^3, \\ \zeta_{223}^{(1)}(\lambda_j; \lambda_i; \beta) &= (85536 + 648\beta^2 + 20736\beta)\lambda_j^2 + (19872\beta + 3888\beta^2 - 55296)\lambda_i\lambda_j \\ &\quad + (12960 + 2592\beta + 1944\beta^2)\lambda_i^2, \\ \zeta_{223}^{(2)}(\lambda_j; \lambda_i; \beta) &= (171072 + 41472\beta + 1296\beta^2)\lambda_j + (19872\beta + 3888\beta^2 - 55296)\lambda_i, \\ \zeta_{223}^{(3)}(\lambda_j; \lambda_i; \beta) &= 171072 + 41472\beta + 1296\beta^2 \geq 0 \text{ for all } \beta \geq -4.86.\end{aligned}$$

Now

$$\begin{aligned}\zeta_{223}(\lambda_i; \lambda_i; \beta) &= (25920 + 4320\beta^2 + 21600\beta)\lambda_i^3 \geq 0 \text{ for all } \beta \geq -2, \\ \zeta_{223}^{(1)}(\lambda_i; \lambda_i; \beta) &= (43200 + 6480\beta^2 + 43200\beta)\lambda_i^2 \geq 0 \text{ for all } \beta \geq -1.22, \\ \zeta_{223}^{(2)}(\lambda_i; \lambda_i; \beta) &= (115776 + 61344\beta + 5184\beta^2)\lambda_i \geq 0 \text{ for all } \beta \geq -2.35.\end{aligned}$$

Hence, $\zeta_{22}^{(3)}(\lambda_j; \lambda_j; \lambda_i; \beta)$ is non-negative for all $\lambda_j \geq \lambda_i$ and $\beta \geq -\frac{2}{5}$.

Define $\zeta_{222}(\lambda_j; \lambda_i; \beta) = \zeta_{22}^{(2)}(\lambda_j; \lambda_j; \lambda_i; \beta)$. We wish to show that ζ_{222} is non-negative for all $\lambda_j \geq \lambda_i$. Differentiating ζ_{222} with respect to λ_j gives

$$\begin{aligned}\zeta_{222}(\lambda_j; \lambda_i; \beta) &= (5112\beta + 252\beta^2 + 21168)\lambda_j^4 + (6048\beta + 72\beta^3 - 20736 + 2016\beta^2)\lambda_i\lambda_j^3 \\ &\quad + (216\beta^3 + 1728\beta^2 - 1296\beta + 6912)\lambda_i^2\lambda_j^2 \\ &\quad + (6912 + 72\beta^3 + 4320\beta + 288\beta^2)\lambda_i^3\lambda_j \\ &\quad + (864 + 36\beta^2 + 576\beta)\lambda_i^4, \\ \zeta_{222}^{(1)}(\lambda_j; \lambda_i; \beta) &= (20448\beta + 1008\beta^2 + 84672)\lambda_j^3 + (18144\beta + 216\beta^3 - 62208 + 6048\beta^2)\lambda_i\lambda_j^2 \\ &\quad + (432\beta^3 + 3456\beta^2 - 2592\beta + 13824)\lambda_i^2\lambda_j + (6912 + 72\beta^3 + 4320\beta + 288\beta^2)\lambda_i^3, \\ \zeta_{222}^{(2)}(\lambda_j; \lambda_i; \beta) &= (61344\beta + 3024\beta^2 + 254016)\lambda_j^2 + (36288\beta + 432\beta^3 - 124416 + 12096\beta^2)\lambda_i\lambda_j \\ &\quad + (432\beta^3 + 3456\beta^2 - 2592\beta + 13824)\lambda_i^2, \\ \zeta_{222}^{(3)}(\lambda_j; \lambda_i; \beta) &= (122688\beta + 6048\beta^2 + 508032)\lambda_j + (36288\beta + 432\beta^3 - 124416 + 12096\beta^2)\lambda_i, \\ \zeta_{222}^{(4)}(\lambda_j; \lambda_i; \beta) &= 122688\beta + 6048\beta^2 + 508032 \geq 0 \text{ for all } \beta \geq -5.79.\end{aligned}$$

Now

$$\begin{aligned}\zeta_{222}(\lambda_i; \lambda_i; \beta) &= (15120 + 4320\beta^2 + 14760\beta + 360\beta^3)\lambda_i^4 \geq 0 \text{ for all } \beta \geq -2, \\ \zeta_{222}^{(1)}(\lambda_i; \lambda_i; \beta) &= (40320\beta + 10800\beta^2 + 43200 + 720\beta^3)\lambda_i^3 \geq 0 \text{ for all } \beta \geq -2, \\ \zeta_{222}^{(2)}(\lambda_i; \lambda_i; \beta) &= (95040\beta + 18576\beta^2 + 143424 + 864\beta^3)\lambda_i^2 \geq 0 \text{ for all } \beta \geq -14.8, \\ \zeta_{222}^{(3)}(\lambda_i; \lambda_i; \beta) &= (158976\beta + 18144\beta^2 + 383616 + 432\beta^3)\lambda_i \geq 0 \text{ for all } \beta \geq -4.34.\end{aligned}$$

Hence, $\zeta_{22}^{(2)}(\lambda_j; \lambda_j; \lambda_i; \beta)$ is non-negative for all $\lambda_j \geq \lambda_i$ and $\beta \geq -\frac{2}{5}$.

Define $\zeta_{221}(\lambda_j; \lambda_i; \beta) = \zeta_{22}^{(1)}(\lambda_j; \lambda_j; \lambda_i; \beta)/\lambda_j$. We wish to show that ζ_{221} is non-negative for all $\lambda_j \geq \lambda_i$. Differentiating ζ_{221} with respect to λ_j gives

$$\begin{aligned}\zeta_{221}(\lambda_j; \lambda_i; \beta) &= (9648 + 144\beta^2 + 2448\beta)\lambda_j^4 + (3024\beta + 1224\beta^2 + 72\beta^3 - 11520)\lambda_i\lambda_j^3 \\ &\quad + (3456 - 2736\beta + 1008\beta^2 + 288\beta^3)\lambda_i^2\lambda_j^2 \\ &\quad + (2304\beta + 144\beta^3 + 1152 + 432\beta^2)\lambda_i^3\lambda_j + (216\beta^2 + 504\beta + 288)\lambda_i^4, \\ \zeta_{221}^{(1)}(\lambda_j; \lambda_i; \beta) &= (9792\beta + 576\beta^2 + 38592)\lambda_j^3 + (9072\beta - 34560 + 216\beta^3 + 3672\beta^2)\lambda_i\lambda_j^2 \\ &\quad + (2016\beta^2 + 576\beta^3 - 5472\beta + 6912)\lambda_i^2\lambda_j \\ &\quad + (2304\beta + 144\beta^3 + 1152 + 432\beta^2)\lambda_i^3, \\ \zeta_{221}^{(2)}(\lambda_j; \lambda_i; \beta) &= (29376\beta + 1728\beta^2 + 115776)\lambda_j^2 + (18144\beta - 69120 + 432\beta^3 + 7344\beta^2)\lambda_i\lambda_j \\ &\quad + (2016\beta^2 + 576\beta^3 - 5472\beta + 6912)\lambda_i^2, \\ \zeta_{221}^{(3)}(\lambda_j; \lambda_i; \beta) &= (58752\beta + 3456\beta^2 + 231552)\lambda_j + (18144\beta - 69120 + 432\beta^3 + 7344\beta^2)\lambda_i, \\ \zeta_{221}^{(4)}(\lambda_j; \lambda_i; \beta) &= 58752\beta + 3456\beta^2 + 231552 \geq 0 \text{ for all } \beta \geq -6.20.\end{aligned}$$

Now

$$\begin{aligned}\zeta_{221}(\lambda_i; \lambda_i; \beta) &= (3024\beta^2 + 5544\beta + 3024 + 504\beta^3)\lambda_i^4 \geq 0 \text{ for all } \beta \geq -1, \\ \zeta_{221}^{(1)}(\lambda_i; \lambda_i; \beta) &= (15696\beta + 6696\beta^2 + 12096 + 936\beta^3)\lambda_i^3 \geq 0 \text{ for all } \beta \geq -2, \\ \zeta_{221}^{(2)}(\lambda_i; \lambda_i; \beta) &= (42048\beta + 11088\beta^2 + 53568 + 1008\beta^3)\lambda_i^2 \geq 0 \text{ for all } \beta \geq -3, \\ \zeta_{221}^{(3)}(\lambda_i; \lambda_i; \beta) &= (76896\beta + 10800\beta^2 + 162432 + 432\beta^3)\lambda_i \geq 0 \text{ for all } \beta \geq -4.\end{aligned}$$

Hence, $\zeta_{22}^{(1)}(\lambda_j; \lambda_j; \lambda_i; \beta)$ is non-negative for all $\lambda_j \geq \lambda_i$ and $\beta \geq -\frac{2}{5}$.

Define $\zeta_{220}(\lambda_j; \lambda_i; \beta) = \zeta_{22}(\lambda_j; \lambda_j; \lambda_i; \beta)/\lambda_j^2$. We wish to show that ζ_{220} is non-negative for all $\lambda_j \geq \lambda_i$. Differentiating ζ_{220} with respect to λ_j gives

$$\begin{aligned}\zeta_{220}(\lambda_j; \lambda_i; \beta) &= (3132 + 54\beta^2 + 846\beta)\lambda_j^4 + (1152\beta - 4752 + 36\beta^3 + 540\beta^2)\lambda_i\lambda_j^3 \\ &\quad + (594\beta^2 + 180\beta^3 - 1314\beta + 2052)\lambda_i^2\lambda_j^2 \\ &\quad + (2256\beta + 504\beta^2 + 120\beta^3 + 1152)\lambda_i^3\lambda_j \\ &\quad + (324\beta^2 + 756\beta + 432)\lambda_i^4, \\ \zeta_{220}^{(1)}(\lambda_j; \lambda_i; \beta) &= (3384\beta + 216\beta^2 + 12528)\lambda_j^3 + (3456\beta - 14256 + 108\beta^3 + 1620\beta^2)\lambda_i\lambda_j^2 \\ &\quad + (1188\beta^2 + 360\beta^3 - 2628\beta + 4104)\lambda_i^2\lambda_j \\ &\quad + (2256\beta + 504\beta^2 + 120\beta^3 + 1152)\lambda_i^3, \\ \zeta_{220}^{(2)}(\lambda_j; \lambda_i; \beta) &= (10152\beta + 648\beta^2 + 37584)\lambda_j^2 + (6912\beta - 28512 + 216\beta^3 + 3240\beta^2)\lambda_i\lambda_j \\ &\quad + (1188\beta^2 + 360\beta^3 - 2628\beta + 4104)\lambda_i^2, \\ \zeta_{220}^{(3)}(\lambda_j; \lambda_i; \beta) &= (20304\beta + 1296\beta^2 + 75168)\lambda_j + (6912\beta - 28512 + 216\beta^3 + 3240\beta^2)\lambda_i, \\ \zeta_{220}^{(4)}(\lambda_j; \lambda_i; \beta) &= 20304\beta + 1296\beta^2 + 75168 \geq 0 \text{ for all } \beta \geq -6.\end{aligned}$$

Now

$$\begin{aligned}
\zeta_{220}(\lambda_i; \lambda_i; \beta) &= (2016\beta^2 + 3696\beta + 2016 + 336\beta^3)\lambda_i^4 > 0 \text{ for all } \beta > -1, \\
\zeta_{220}^{(1)}(\lambda_i; \lambda_i; \beta) &= (6468\beta + 3528\beta^2 + 3528 + 588\beta^3)\lambda_i^3 > 0 \text{ for all } \beta > -1, \\
\zeta_{220}^{(2)}(\lambda_i; \lambda_i; \beta) &= (14436\beta + 5076\beta^2 + 13176 + 576\beta^3)\lambda_i^2 > 0 \text{ for all } \beta > -2, \\
\zeta_{220}^{(3)}(\lambda_i; \lambda_i; \beta) &= (27216\beta + 4536\beta^2 + 46656 + 216\beta^3)\lambda_i > 0 \text{ for all } \beta > -3.
\end{aligned}$$

Hence, $\zeta_{22}(\lambda_j; \lambda_j; \lambda_i; \beta)$ is positive for all $\lambda_j \geq \lambda_i$ and $\beta \geq -\frac{2}{5}$. This implies that $\zeta^{(2)}(0; \lambda_k; \lambda_j; \lambda_i; \beta)$ is positive for all $\lambda_k \geq \lambda_j$ and $\beta \geq -\frac{2}{5}$.

Differentiating $\zeta_{23}(\lambda_k; \lambda_j; \lambda_i; \beta)$ with respect to λ_k we obtain

$$\begin{aligned}
\zeta_{23}(\lambda_k; \lambda_j; \lambda_i; \beta) &= (900\beta + 360)\lambda_k^4(\lambda_j + \lambda_i) \\
&+ (432\beta^2 + 8208 + 1512\beta)\lambda_k^3(\lambda_j^2 + \lambda_i^2) \\
&+ (-6336 + 576\beta + 1296\beta^2)\lambda_k^3\lambda_j\lambda_i \\
&+ (3312\beta + 5184 + 36\beta^3 + 216\beta^2)\lambda_k^2\lambda_j^3 \\
&+ (3312\beta + 5184 + 36\beta^3 + 216\beta^2)\lambda_k^2\lambda_i^3 \\
&+ (-7344 + 864\beta^2 + 324\beta^3 + 648\beta)\lambda_k^2\lambda_j^2\lambda_i \\
&+ (-7344 + 864\beta^2 + 324\beta^3 + 648\beta)\lambda_k^2\lambda_j\lambda_i^2 \\
&+ (216\beta^2 + 288 + 504\beta)\lambda_k(\lambda_j^4 + \lambda_i^4) \\
&+ (1296\beta^2 + 216\beta^3 - 6912)\lambda_k(\lambda_j^3\lambda_i + \lambda_j\lambda_i^3) \\
&+ (648\beta^3 - 7128\beta + 1296\beta^2 + 3888)\lambda_k\lambda_j^2\lambda_i^2 \\
&+ (576 + 1008\beta + 432\beta^2)(\lambda_j^4\lambda_i + \lambda_j\lambda_i^4) \\
&+ (1296\beta^2 + 6480\beta + 6912 + 108\beta^3)\lambda_j^3\lambda_i^2 \\
&+ (1296\beta^2 + 6480\beta + 6912 + 108\beta^3)\lambda_j^2\lambda_i^3,
\end{aligned}$$

$$\begin{aligned}
\zeta_{23}^{(1)}(\lambda_k; \lambda_j; \lambda_i; \beta) &= (3600\beta + 1440)\lambda_k^3(\lambda_j + \lambda_i) + (24624 + 1296\beta^2 + 4536\beta)\lambda_k^2(\lambda_j^2 + \lambda_i^2) \\
&\quad + (-19008 + 1728\beta + 3888\beta^2)\lambda_k^2\lambda_j\lambda_i \\
&\quad + (1728\beta^2 - 14688 + 648\beta^3 + 1296\beta)\lambda_k(\lambda_j^2\lambda_i + \lambda_j\lambda_i^2) \\
&\quad + (432\beta^2 + 10368 + 72\beta^3 + 6624\beta)\lambda_k(\lambda_j^3 + \lambda_i^3) \\
&\quad + (216\beta^2 + 504\beta + 288)(\lambda_j^4 + \lambda_i^4) \\
&\quad + (1296\beta^2 + 216\beta^3 - 6912)(\lambda_j^3\lambda_i + \lambda_j\lambda_i^3) \\
&\quad + (648\beta^3 - 7128\beta + 1296\beta^2 + 3888)\lambda_j^2\lambda_i^2, \\
\zeta_{23}^{(2)}(\lambda_k; \lambda_j; \lambda_i; \beta) &= (10800\beta + 4320)\lambda_k^2(\lambda_j + \lambda_i) + (7776\beta^2 - 38016 + 3456\beta)\lambda_k\lambda_i\lambda_j \\
&\quad + (9072\beta + 49248 + 2592\beta^2)\lambda_k(\lambda_j^2 + \lambda_i^2) \\
&\quad + (1728\beta^2 - 14688 + 648\beta^3 + 1296\beta)(\lambda_j^2\lambda_i + \lambda_j\lambda_i^2) \\
&\quad + (432\beta^2 + 10368 + 72\beta^3 + 6624\beta)(\lambda_j^3 + \lambda_i^3), \\
\zeta_{23}^{(3)}(\lambda_k; \lambda_j; \lambda_i; \beta) &= (21600\beta + 8640)\lambda_k(\lambda_j + \lambda_i) + (7776\beta^2 - 38016 + 3456\beta)\lambda_i\lambda_j \\
&\quad + (9072\beta + 49248 + 2592\beta^2)(\lambda_j^2 + \lambda_i^2), \\
\zeta_{23}^{(4)}(\lambda_k; \lambda_j; \lambda_i; \beta) &= (21600\beta + 8640)(\lambda_j + \lambda_i) \geq 0 \text{ for all } \beta \geq -\frac{2}{5}.
\end{aligned}$$

Define $\zeta_{233}(\lambda_j; \lambda_i; \beta) = \zeta_{23}^{(3)}(\lambda_j; \lambda_j; \lambda_i; \beta)$. We wish to show that ζ_{233} is non-negative for all $\lambda_j \geq \lambda_i$. Differentiating ζ_{233} with respect to λ_j gives

$$\begin{aligned}
\zeta_{233}(\lambda_j; \lambda_i; \beta) &= (30672\beta + 57888 + 2592\beta^2)\lambda_j^2 + (25056\beta - 29376 + 7776\beta^2)\lambda_i\lambda_j \\
&\quad + (9072\beta + 49248 + 2592\beta^2)\lambda_i^2, \\
\zeta_{233}^{(1)}(\lambda_j; \lambda_i; \beta) &= (61344\beta + 115776 + 5184\beta^2)\lambda_j + (25056\beta - 29376 + 7776\beta^2)\lambda_i, \\
\zeta_{233}^{(2)}(\lambda_j; \lambda_i; \beta) &= 61344\beta + 115776 + 5184\beta^2 \geq 0 \text{ for all } \beta \geq -2.35.
\end{aligned}$$

Now

$$\begin{aligned}
\zeta_{233}(\lambda_i; \lambda_i; \beta) &= (64800\beta + 77760 + 12960\beta^2)\lambda_i^2 \geq 0 \text{ for all } \beta \geq -2, \\
\zeta_{233}^{(1)}(\lambda_i; \lambda_i; \beta) &= (86400\beta + 86400 + 12960\beta^2)\lambda_i \geq 0 \text{ for all } \beta \geq -1.22.
\end{aligned}$$

Hence, $\zeta_{23}^{(3)}(\lambda_j; \lambda_j; \lambda_i; \beta)$ is non-negative for all $\lambda_j \geq \lambda_i$ and $\beta \geq -\frac{2}{5}$.

Define $\zeta_{232}(\lambda_j; \lambda_i; \beta) = \zeta_{23}^{(2)}(\lambda_j; \lambda_j; \lambda_i; \beta)$. We wish to show that ζ_{232} is non-negative for

all $\lambda_j \geq \lambda_i$. Differentiating ζ_{232} with respect to λ_j gives

$$\begin{aligned}
\zeta_{232}(\lambda_j; \lambda_i; \beta) &= (63936 + 3024\beta^2 + 26496\beta + 72\beta^3)\lambda_j^3 \\
&\quad + (648\beta^3 + 15552\beta + 9504\beta^2 - 48384)\lambda_i\lambda_j^2 \\
&\quad + (4320\beta^2 + 10368\beta + 648\beta^3 + 34560)\lambda_i^2\lambda_j \\
&\quad + (432\beta^2 + 10368 + 72\beta^3 + 6624\beta)\lambda_i^3, \\
\zeta_{232}^{(1)}(\lambda_j; \lambda_i; \beta) &= (191808 + 9072\beta^2 + 79488\beta + 216\beta^3)\lambda_j^2 \\
&\quad + (1296\beta^3 + 31104\beta + 19008\beta^2 - 96768)\lambda_i\lambda_j \\
&\quad + (4320\beta^2 + 10368\beta + 648\beta^3 + 34560)\lambda_i^2, \\
\zeta_{232}^{(2)}(\lambda_j; \lambda_i; \beta) &= (158976\beta + 18144\beta^2 + 383616 + 432\beta^3)\lambda_j \\
&\quad + (1296\beta^3 + 31104\beta + 19008\beta^2 - 96768)\lambda_i, \\
\zeta_{232}^{(3)}(\lambda_j; \lambda_i; \beta) &= 158976\beta + 18144\beta^2 + 383616 + 432\beta^3 \geq 0 \text{ for all } \beta \geq -4.35.
\end{aligned}$$

Now

$$\begin{aligned}
\zeta_{232}(\lambda_i; \lambda_i; \beta) &= \geq 0 \text{ for all } \beta \geq -2, \\
\zeta_{232}^{(1)}(\lambda_i; \lambda_i; \beta) &= \geq 0 \text{ for all } \beta \geq -2, \\
\zeta_{232}^{(2)}(\lambda_i; \lambda_i; \beta) &= \geq 0 \text{ for all } \beta \geq -3.18.
\end{aligned}$$

Hence, $\zeta_{23}^{(2)}(\lambda_j; \lambda_j; \lambda_i; \beta)$ is non-negative for all $\lambda_j \geq \lambda_i$ and $\beta \geq -\frac{2}{5}$.

Define $\zeta_{231}(\lambda_j; \lambda_i; \beta) = \zeta_{23}^{(1)}(\lambda_j; \lambda_j; \lambda_i; \beta)$. We wish to show that ζ_{231} is non-negative for all $\lambda_j \geq \lambda_i$. Differentiating ζ_{231} with respect to λ_j gives

$$\begin{aligned}
\zeta_{231}(\lambda_j; \lambda_i; \beta) &= (72\beta^3 + 15264\beta + 1944\beta^2 + 36720)\lambda_j^4 \\
&\quad + (6624\beta - 39168 + 864\beta^3 + 6912\beta^2)\lambda_i\lambda_j^3 \\
&\quad + (4320\beta^2 + 1296\beta^3 - 1296\beta + 13824)\lambda_i^2\lambda_j^2 \\
&\quad + (6624\beta + 1728\beta^2 + 288\beta^3 + 3456)\lambda_i^3\lambda_j + (216\beta^2 + 504\beta + 288)\lambda_i^4, \\
\zeta_{231}^{(1)}(\lambda_j; \lambda_i; \beta) &= (288\beta^3 + 61056\beta + 7776\beta^2 + 146880)\lambda_j^3 \\
&\quad + (19872\beta - 117504 + 2592\beta^3 + 20736\beta^2)\lambda_i\lambda_j^2 \\
&\quad + (8640\beta^2 + 2592\beta^3 - 2592\beta + 27648)\lambda_i^2\lambda_j \\
&\quad + (6624\beta + 1728\beta^2 + 288\beta^3 + 3456)\lambda_i^3, \\
\zeta_{231}^{(2)}(\lambda_j; \lambda_i; \beta) &= (864\beta^3 + 183168\beta + 23328\beta^2 + 440640)\lambda_j^2 \\
&\quad + (39744\beta - 235008 + 5184\beta^3 + 41472\beta^2)\lambda_i\lambda_j \\
&\quad + (8640\beta^2 + 2592\beta^3 - 2592\beta + 27648)\lambda_i^2, \\
\zeta_{231}^{(3)}(\lambda_j; \lambda_i; \beta) &= (881280 + 46656\beta^2 + 366336\beta + 1728\beta^3)\lambda_j \\
&\quad + (39744\beta - 235008 + 5184\beta^3 + 41472\beta^2)\lambda_i, \\
\zeta_{231}^{(4)}(\lambda_j; \lambda_i; \beta) &= 881280 + 46656\beta^2 + 366336\beta + 1728\beta^3 \geq 0 \text{ for all } \beta \geq -5.
\end{aligned}$$

Now

$$\begin{aligned}\zeta_{231}(\lambda_i; \lambda_i; \beta) &= (15120\beta^2 + 27720\beta + 15120 + 2520\beta^3)\lambda_i^4 \geq 0 \text{ for all } \beta \geq -1, \\ \zeta_{231}^{(1)}(\lambda_i; \lambda_i; \beta) &= (5760\beta^3 + 84960\beta + 38880\beta^2 + 60480)\lambda_i^3 \geq 0 \text{ for all } \beta \geq -2, \\ \zeta_{231}^{(2)}(\lambda_i; \lambda_i; \beta) &= (8640\beta^3 + 220320\beta + 73440\beta^2 + 233280)\lambda_i^2 \geq 0 \text{ for all } \beta \geq -3, \\ \zeta_{231}^{(3)}(\lambda_i; \lambda_i; \beta) &= (646272 + 88128\beta^2 + 406080\beta + 6912\beta^3)\lambda_i \geq 0 \text{ for all } \beta \geq -3.70.\end{aligned}$$

Hence, $\zeta_{23}^{(1)}(\lambda_j; \lambda_j; \lambda_i; \beta)$ is non-negative for all $\lambda_j \geq \lambda_i$ and $\beta \geq -\frac{2}{5}$.

Define $\zeta_{230}(\lambda_j; \lambda_i; \beta) = \zeta_{23}(\lambda_j; \lambda_j; \lambda_i; \beta)/\lambda_j$. We wish to show that ζ_{230} is non-negative for all $\lambda_j \geq \lambda_i$. Differentiating ζ_{230} with respect to λ_j gives

$$\begin{aligned}\zeta_{230}(\lambda_j; \lambda_i; \beta) &= (36\beta^3 + 864\beta^2 + 6228\beta + 14040)\lambda_j^4 \\ &\quad + (540\beta^3 - 19656 + 3888\beta^2 + 3132\beta)\lambda_i\lambda_j^3 \\ &\quad + (11664 + 1080\beta^3 + 3888\beta^2 + 1512\beta)\lambda_i^2\lambda_j^2 \\ &\quad + (360\beta^3 + 9792\beta + 2808\beta^2 + 5184)\lambda_i^3\lambda_j \\ &\quad + (864 + 648\beta^2 + 1512\beta)\lambda_i^4,\end{aligned}$$

$$\begin{aligned}\zeta_{230}^{(1)}(\lambda_j; \lambda_i; \beta) &= (56160 + 24912\beta + 3456\beta^2 + 144\beta^3)\lambda_j^3 \\ &\quad + (1620\beta^3 + 9396\beta + 11664\beta^2 - 58968)\lambda_i\lambda_j^2 \\ &\quad + (23328 + 7776\beta^2 + 2160\beta^3 + 3024\beta)\lambda_i^2\lambda_j \\ &\quad + (360\beta^3 + 9792\beta + 2808\beta^2 + 5184)\lambda_i^3, \\ \zeta_{230}^{(2)}(\lambda_j; \lambda_i; \beta) &= (168480 + 74736\beta + 10368\beta^2 + 432\beta^3)\lambda_j^2 \\ &\quad + (3240\beta^3 + 18792\beta + 23328\beta^2 - 117936)\lambda_i\lambda_j \\ &\quad + (23328 + 7776\beta^2 + 2160\beta^3 + 3024\beta)\lambda_i^2, \\ \zeta_{230}^{(3)}(\lambda_j; \lambda_i; \beta) &= (336960 + 149472\beta + 20736\beta^2 + 864\beta^3)\lambda_j \\ &\quad + (3240\beta^3 + 18792\beta + 23328\beta^2 - 117936)\lambda_i, \\ \zeta_{230}^{(4)}(\lambda_j; \lambda_i; \beta) &= 336960 + 149472\beta + 20736\beta^2 + 864\beta^3 \geq 0 \text{ for all } \beta \geq -5.\end{aligned}$$

Now

$$\begin{aligned}\zeta_{230}(\lambda_i; \lambda_i; \beta) &= (12096 + 12096\beta^2 + 22176\beta + 2016\beta^3)\lambda_i^4 > 0 \text{ for all } \beta > -1, \\ \zeta_{230}^{(1)}(\lambda_i; \lambda_i; \beta) &= (25704 + 47124\beta + 25704\beta^2 + 4284\beta^3)\lambda_i^3 > 0 \text{ for all } \beta > -1, \\ \zeta_{230}^{(2)}(\lambda_i; \lambda_i; \beta) &= (73872 + 96552\beta + 41472\beta^2 + 5832\beta^3)\lambda_i^2 > 0 \text{ for all } \beta > -2, \\ \zeta_{230}^{(3)}(\lambda_i; \lambda_i; \beta) &= (219024 + 168264\beta + 44064\beta^2 + 4104\beta^3)\lambda_i > 0 \text{ for all } \beta > -3.\end{aligned}$$

Hence, $\zeta_{23}(\lambda_j; \lambda_j; \lambda_i; \beta)$ is positive for all $\lambda_j \geq \lambda_i$ and $\beta \geq -\frac{2}{5}$. This implies that $\zeta^{(1)}(0; \lambda_k; \lambda_j; \lambda_i; \beta)$ is positive for all $\lambda_k \geq \lambda_j$ and $\beta \geq -\frac{2}{5}$.

Differentiating $\zeta_{24}(\lambda_k; \lambda_j; \lambda_i; \beta)$ with respect to λ_k we obtain

$$\begin{aligned}
\zeta_{24}(\lambda_k; \lambda_j; \lambda_i; \beta) &= (1800\beta + 720)\lambda_k^4 + (7200\beta + 2160\beta^2 + 11520)\lambda_k^3 (\lambda_j + \lambda_i) \\
&\quad + (432\beta^3 + 2808\beta^2 + 15120\beta + 32832)\lambda_k^2 (\lambda_j^2 + \lambda_i^2) \\
&\quad + (-48384 + 4320\beta + 7344\beta^2 + 1296\beta^3)\lambda_k^2 \lambda_j \lambda_i \\
&\quad + (2592\beta^2 + 4608 + 8640\beta + 288\beta^3)\lambda_k (\lambda_j^3 + \lambda_i^3) \\
&\quad + (8208\beta^2 + 2592\beta^3 - 12960\beta - 27648)\lambda_k \lambda_j^2 \lambda_i \\
&\quad + (8208\beta^2 + 2592\beta^3 - 12960\beta - 27648)\lambda_k \lambda_j \lambda_i^2, \\
\zeta_{24}^{(1)}(\lambda_k; \lambda_j; \lambda_i; \beta) &= (2880 + 7200\beta)\lambda_k^3 + (21600\beta + 34560 + 6480\beta^2)\lambda_k^2 (\lambda_j + \lambda_i) \\
&\quad + (14688\beta^2 + 2592\beta^3 - 96768 + 8640\beta)\lambda_k \lambda_i \lambda_j \\
&\quad + (30240\beta + 864\beta^3 + 65664 + 5616\beta^2)\lambda_k (\lambda_j^2 + \lambda_i^2) \\
&\quad + (4608 + 2592\beta^2 + 8640\beta + 288\beta^3)(\lambda_j^3 + \lambda_i^3) \\
&\quad + (-27648 + 8208\beta^2 - 12960\beta + 2592\beta^3)(\lambda_i \lambda_j^2 + \lambda_i^2 \lambda_j), \\
\zeta_{24}^{(2)}(\lambda_k; \lambda_j; \lambda_i; \beta) &= (8640 + 21600\beta)\lambda_k^2 + (43200\beta + 69120 + 12960\beta^2)\lambda_k (\lambda_j + \lambda_i) \\
&\quad + (14688\beta^2 + 2592\beta^3 - 96768 + 8640\beta)\lambda_i \lambda_j \\
&\quad + (30240\beta + 864\beta^3 + 65664 + 5616\beta^2)(\lambda_j^2 + \lambda_i^2), \\
\zeta_{24}^{(3)}(\lambda_k; \lambda_j; \lambda_i; \beta) &= (17280 + 43200\beta)\lambda_k + (43200\beta + 69120 + 12960\beta^2)(\lambda_j + \lambda_i) \\
&\geq 0 \text{ for all } \beta \geq -\frac{2}{5}.
\end{aligned}$$

Define $\zeta_{242}(\lambda_j; \lambda_i; \beta) = \zeta_{24}^{(2)}(\lambda_j; \lambda_j; \lambda_i; \beta)$. We wish to show that ζ_{242} is non-negative for all $\lambda_j \geq \lambda_i$. Differentiating ζ_{242} with respect to λ_j gives

$$\begin{aligned}
\zeta_{242}(\lambda_j; \lambda_i; \beta) &= (143424 + 95040\beta + 18576\beta^2 + 864\beta^3)\lambda_j^2 \\
&\quad + (-27648 + 27648\beta^2 + 2592\beta^3 + 51840\beta)\lambda_i \lambda_j \\
&\quad + (30240\beta + 864\beta^3 + 65664 + 5616\beta^2)\lambda_i^2, \\
\zeta_{242}^{(1)}(\lambda_j; \lambda_i; \beta) &= (286848 + 190080\beta + 37152\beta^2 + 1728\beta^3)\lambda_j \\
&\quad + (-27648 + 27648\beta^2 + 2592\beta^3 + 51840\beta)\lambda_i, \\
\zeta_{242}^{(2)}(\lambda_j; \lambda_i; \beta) &= 286848 + 190080\beta + 37152\beta^2 + 1728\beta^3 \geq 0 \text{ for all } \beta \geq -14.8.
\end{aligned}$$

Now

$$\begin{aligned}
\zeta_{242}(\lambda_i; \lambda_i; \beta) &= (177120\beta + 4320\beta^3 + 181440 + 51840\beta^2)\lambda_i^2 \geq 0 \text{ for all } \beta \geq -2, \\
\zeta_{242}^{(1)}(\lambda_i; \lambda_i; \beta) &= (259200 + 241920\beta + 64800\beta^2 + 4320\beta^3)\lambda_i \geq 0 \text{ for all } \beta \geq -2.
\end{aligned}$$

Hence, $\zeta_{24}^{(2)}(\lambda_j; \lambda_j; \lambda_i; \beta)$ is non-negative for all $\lambda_j \geq \lambda_i$ and $\beta \geq -\frac{2}{5}$.

Define $\zeta_{241}(\lambda_j; \lambda_i; \beta) = \zeta_{24}^{(1)}(\lambda_j; \lambda_j; \lambda_i; \beta)$. We wish to show that ζ_{241} is non-negative for

all $\lambda_j \geq \lambda_i$. Differentiating ζ_{241} with respect to λ_j gives

$$\begin{aligned}\zeta_{241}(\lambda_j; \lambda_i; \beta) &= (107712 + 1152\beta^3 + 14688\beta^2 + 67680\beta)\lambda_j^3 \\ &\quad + (29376\beta^2 - 89856 + 5184\beta^3 + 17280\beta)\lambda_i\lambda_j^2 \\ &\quad + (3456\beta^3 + 38016 + 17280\beta + 13824\beta^2)\lambda_i^2\lambda_j \\ &\quad + (4608 + 2592\beta^2 + 8640\beta + 288\beta^3)\lambda_i^3, \\ \zeta_{241}^{(1)}(\lambda_j; \lambda_i; \beta) &= (323136 + 3456\beta^3 + 44064\beta^2 + 203040\beta)\lambda_j^2 \\ &\quad + (58752\beta^2 - 179712 + 10368\beta^3 + 34560\beta)\lambda_i\lambda_j \\ &\quad + (3456\beta^3 + 38016 + 17280\beta + 13824\beta^2)\lambda_i^2, \\ \zeta_{241}^{(2)}(\lambda_j; \lambda_i; \beta) &= (646272 + 88128\beta^2 + 406080\beta + 6912\beta^3)\lambda_j \\ &\quad + (58752\beta^2 - 179712 + 10368\beta^3 + 34560\beta)\lambda_i, \\ \zeta_{241}^{(3)}(\lambda_j; \lambda_i; \beta) &= 646272 + 88128\beta^2 + 406080\beta + 6912\beta^3 \geq 0 \text{ for all } \beta \geq -3.70.\end{aligned}$$

Now

$$\begin{aligned}\zeta_{241}(\lambda_i; \lambda_i; \beta) &= (60480 + 60480\beta^2 + 110880\beta + 10080\beta^3)\lambda_i^3 \geq 0 \text{ for all } \beta \geq -1, \\ \zeta_{241}^{(1)}(\lambda_i; \lambda_i; \beta) &= (181440 + 17280\beta^3 + 116640\beta^2 + 254880\beta)\lambda_i^2 \geq 0 \text{ for all } \beta \geq -2, \\ \zeta_{241}^{(2)}(\lambda_i; \lambda_i; \beta) &= (466560 + 146880\beta^2 + 440640\beta + 17280\beta^3)\lambda_i \geq 0 \text{ for all } \beta \geq -3.\end{aligned}$$

Hence, $\zeta_{24}^{(1)}(\lambda_j; \lambda_j; \lambda_i; \beta)$ is non-negative for all $\lambda_j \geq \lambda_i$ and $\beta \geq -\frac{2}{5}$.

Define $\zeta_{240}(\lambda_j; \lambda_i; \beta) = \zeta_{24}(\lambda_j; \lambda_j; \lambda_i; \beta)$. We wish to show that ζ_{240} is non-negative for all $\lambda_j \geq \lambda_i$. Differentiating ζ_{240} with respect to λ_j gives

$$\begin{aligned}\zeta_{240}(\lambda_j; \lambda_i; \beta) &= (720\beta^3 + 34272\beta + 50544 + 8208\beta^2)\lambda_j^4 \\ &\quad + (15552\beta + 23328\beta^2 + 4320\beta^3 - 55296)\lambda_i\lambda_j^3 \\ &\quad + (33912\beta + 20088\beta^2 + 4320\beta^3 + 50544)\lambda_i^2\lambda_j^2 \\ &\quad + (720\beta^3 + 13824 + 8208\beta^2 + 25632\beta)\lambda_i^3\lambda_j \\ &\quad + (648\beta^2 + 1512\beta + 864)\lambda_i^4, \\ \zeta_{240}^{(1)}(\lambda_j; \lambda_i; \beta) &= (2880\beta^3 + 137088\beta + 202176 + 32832\beta^2)\lambda_j^3 \\ &\quad + (46656\beta + 69984\beta^2 + 12960\beta^3 - 165888)\lambda_i\lambda_j^2 \\ &\quad + (67824\beta + 40176\beta^2 + 8640\beta^3 + 101088)\lambda_i^2\lambda_j \\ &\quad + (720\beta^3 + 13824 + 8208\beta^2 + 25632\beta)\lambda_i^3, \\ \zeta_{240}^{(2)}(\lambda_j; \lambda_i; \beta) &= (8640\beta^3 + 411264\beta + 606528 + 98496\beta^2)\lambda_j^2 \\ &\quad + (93312\beta + 139968\beta^2 + 25920\beta^3 - 331776)\lambda_i\lambda_j \\ &\quad + (67824\beta + 40176\beta^2 + 8640\beta^3 + 101088)\lambda_i^2, \\ \zeta_{240}^{(3)}(\lambda_j; \lambda_i; \beta) &= (17280\beta^3 + 822528\beta + 1213056 + 196992\beta^2)\lambda_j \\ &\quad + (93312\beta + 139968\beta^2 + 25920\beta^3 - 331776)\lambda_i, \\ \zeta_{240}^{(4)}(\lambda_j; \lambda_i; \beta) &= 17280\beta^3 + 822528\beta + 1213056 + 196992\beta^2 \geq 0 \text{ for all } \beta \geq -3.58.\end{aligned}$$

Now

$$\begin{aligned}\zeta_{240}(\lambda_i; \lambda_i; \beta) &= (60480 + 60480\beta^2 + 110880\beta + 10080\beta^3)\lambda_i^4 > 0 \text{ for all } \beta > -1, \\ \zeta_{240}^{(1)}(\lambda_i; \lambda_i; \beta) &= (25200\beta^3 + 277200\beta + 151200 + 151200\beta^2)\lambda_i^3 > 0 \text{ for all } \beta > -1, \\ \zeta_{240}^{(2)}(\lambda_i; \lambda_i; \beta) &= (43200\beta^3 + 572400\beta + 375840 + 278640\beta^2)\lambda_i^2 > 0 \text{ for all } \beta > -2, \\ \zeta_{240}^{(3)}(\lambda_i; \lambda_i; \beta) &= (43200\beta^3 + 915840\beta + 881280 + 336960\beta^2)\lambda_i > 0 \text{ for all } \beta > -3.\end{aligned}$$

Hence, $\zeta_{24}(\lambda_j; \lambda_j; \lambda_i; \beta)$ is positive for all $\lambda_j \geq \lambda_i$ and $\beta \geq -\frac{2}{5}$. This implies that $\zeta^{(1)}(0; \lambda_k; \lambda_j; \lambda_i; \beta)$ is positive for all $\lambda_k \geq \lambda_j$ and $\beta \geq -\frac{2}{5}$.

Differentiating $\zeta_{25}(\lambda_k; \lambda_j; \lambda_i; \beta)$ with respect to λ_k we obtain

$$\begin{aligned}\zeta_{25}(\lambda_k; \lambda_j; \lambda_i; \beta) &= (25920 + 21600\beta + 4320\beta^2)\lambda_k^3 \\ &\quad + (51840 + 56160\beta + 19440\beta^2 + 2160\beta^3)\lambda_k^2(\lambda_j + \lambda_i) \\ &\quad + (4320\beta^3 + 51840 + 43200\beta + 21600\beta^2)\lambda_k(\lambda_j^2 + \lambda_i^2) \\ &\quad + (-43200\beta + 12960\beta^3 - 207360 + 47520\beta^2)\lambda_k\lambda_j\lambda_i \\ &\quad + (720\beta^3 + 10800\beta^2 + 17280 + 31680\beta)(\lambda_j^3 + \lambda_i^3) \\ &\quad + (90720 + 6480\beta^3 + 101520\beta + 43200\beta^2)(\lambda_j^2\lambda_i + \lambda_j\lambda_i^2),\end{aligned}$$

$$\begin{aligned}\zeta_{25}^{(1)}(\lambda_k; \lambda_j; \lambda_i; \beta) &= (77760 + 64800\beta + 12960\beta^2)\lambda_k^2 \\ &\quad + (112320\beta + 103680 + 38880\beta^2 + 4320\beta^3)\lambda_k(\lambda_j + \lambda_i) \\ &\quad + (21600\beta^2 + 43200\beta + 51840 + 4320\beta^3)(\lambda_j^2 + \lambda_i^2) \\ &\quad + (12960\beta^3 - 43200\beta - 207360 + 47520\beta^2)\lambda_i\lambda_j,\end{aligned}$$

$$\begin{aligned}\zeta_{25}^{(2)}(\lambda_k; \lambda_j; \lambda_i; \beta) &= (155520 + 129600\beta + 25920\beta^2)\lambda_k \\ &\quad + (112320\beta + 103680 + 38880\beta^2 + 4320\beta^3)(\lambda_j + \lambda_i) \\ &\geq 0 \text{ for all } \beta \geq -2.\end{aligned}$$

Define $\zeta_{251}(\lambda_j; \lambda_i; \beta) = \zeta_{25}^{(1)}(\lambda_j; \lambda_j; \lambda_i; \beta)$. We wish to show that ζ_{251} is non-negative for all $\lambda_j \geq \lambda_i$. Differentiating ζ_{251} with respect to λ_j gives

$$\begin{aligned}\zeta_{251}(\lambda_j; \lambda_i; \beta) &= (8640\beta^3 + 220320\beta + 73440\beta^2 + 233280)\lambda_j^2 \\ &\quad + (69120\beta - 103680 + 86400\beta^2 + 17280\beta^3)\lambda_i\lambda_j \\ &\quad + (21600\beta^2 + 43200\beta + 51840 + 4320\beta^3)\lambda_i^2, \\ \zeta_{251}^{(1)}(\lambda_j; \lambda_i; \beta) &= (466560 + 146880\beta^2 + 440640\beta + 17280\beta^3)\lambda_j \\ &\quad + (69120\beta - 103680 + 86400\beta^2 + 17280\beta^3)\lambda_i, \\ \zeta_{251}^{(2)}(\lambda_j; \lambda_i; \beta) &= 466560 + 146880\beta^2 + 440640\beta + 17280\beta^3 \geq 0 \text{ for all } \beta \geq -3.\end{aligned}$$

Now

$$\begin{aligned}\zeta_{251}(\lambda_i; \lambda_i; \beta) &= (181440\beta^2 + 332640\beta + 181440 + 30240\beta^3)\lambda_i^2 \geq 0 \text{ for all } \beta \geq -1, \\ \zeta_{251}^{(1)}(\lambda_i; \lambda_i; \beta) &= (362880 + 233280\beta^2 + 509760\beta + 34560\beta^3)\lambda_i \geq 0 \text{ for all } \beta \geq -1.75.\end{aligned}$$

Hence, $\zeta_{25}^{(1)}(\lambda_j; \lambda_j; \lambda_i; \beta)$ is non-negative for all $\lambda_j \geq \lambda_i$ and $\beta \geq -\frac{2}{5}$.

Define $\zeta_{250}(\lambda_j; \lambda_i; \beta) = \zeta_{25}(\lambda_j; \lambda_j; \lambda_i; \beta)$. We wish to show that ζ_{250} is non-negative for all $\lambda_j \geq \lambda_i$. Differentiating ζ_{250} with respect to λ_j gives

$$\begin{aligned}\zeta_{250}(\lambda_j; \lambda_i; \beta) &= (7200\beta^3 + 146880 + 152640\beta + 56160\beta^2)\lambda_j^3 \\ &\quad + (114480\beta + 21600\beta^3 - 64800 + 110160\beta^2)\lambda_i\lambda_j^2 \\ &\quad + (10800\beta^3 + 64800\beta^2 + 144720\beta + 142560)\lambda_i^2\lambda_j \\ &\quad + (10800\beta^2 + 720\beta^3 + 31680\beta + 17280)\lambda_i^3, \\ \zeta_{250}^{(1)}(\lambda_j; \lambda_i; \beta) &= (21600\beta^3 + 440640 + 457920\beta + 168480\beta^2)\lambda_j^2 \\ &\quad + (228960\beta + 43200\beta^3 - 129600 + 220320\beta^2)\lambda_i\lambda_j \\ &\quad + (10800\beta^3 + 64800\beta^2 + 144720\beta + 142560)\lambda_i^2, \\ \zeta_{250}^{(2)}(\lambda_j; \lambda_i; \beta) &= (43200\beta^3 + 915840\beta + 881280 + 336960\beta^2)\lambda_j \\ &\quad + (228960\beta + 43200\beta^3 - 129600 + 220320\beta^2)\lambda_i, \\ \zeta_{250}^{(3)}(\lambda_j; \lambda_i; \beta) &= 43200\beta^3 + 915840\beta + 881280 + 336960\beta^2 \geq 0 \text{ for all } \beta \geq -5.\end{aligned}$$

Now

$$\begin{aligned}\zeta_{250}(\lambda_i; \lambda_i; \beta) &= (241920\beta^2 + 40320\beta^3 + 443520\beta + 241920)\lambda_i^3 > 0 \text{ for all } \beta > -1, \\ \zeta_{250}^{(1)}(\lambda_i; \lambda_i; \beta) &= (75600\beta^3 + 453600 + 831600\beta + 453600\beta^2)\lambda_i^2 > 0 \text{ for all } \beta > -1, \\ \zeta_{250}^{(2)}(\lambda_i; \lambda_i; \beta) &= (86400\beta^3 + 1144800\beta + 751680 + 557280\beta^2)\lambda_i > 0 \text{ for all } \beta > -1.45.\end{aligned}$$

Hence, $\zeta_{25}(\lambda_j; \lambda_j; \lambda_i; \beta)$ is positive for all $\lambda_j \geq \lambda_i$ and $\beta \geq -\frac{2}{5}$. This implies that $\zeta^{(1)}(0; \lambda_k; \lambda_j; \lambda_i; \beta)$ is positive for all $\lambda_k \geq \lambda_j$ and $\beta \geq -\frac{2}{5}$.

Differentiating $\zeta_{26}(\lambda_k; \lambda_j; \lambda_i; \beta)$ with respect to λ_k we obtain

$$\begin{aligned}\zeta_{26}(\lambda_k; \lambda_j; \lambda_i; \beta) &= (4320\beta^3 + 177120\beta + 181440 + 51840\beta^2)\lambda_k^2 \\ &\quad + (155520\beta + 25920\beta^3 + 129600\beta^2)\lambda_k\lambda_j \\ &\quad + (155520\beta + 25920\beta^3 + 129600\beta^2)\lambda_k\lambda_i \\ &\quad + (239760\beta + 194400 + 12960\beta^3 + 97200\beta^2)\lambda_j^2 \\ &\quad + (239760\beta + 194400 + 12960\beta^3 + 97200\beta^2)\lambda_i^2 \\ &\quad + (220320\beta^2 + 38880\beta^3 + 362880\beta + 155520)\lambda_j\lambda_i, \\ \zeta_{26}^{(1)}(\lambda_k; \lambda_j; \lambda_i; \beta) &= (354240\beta + 8640\beta^3 + 362880 + 103680\beta^2)\lambda_k \\ &\quad + (129600\beta^2 + 25920\beta^3 + 155520\beta)(\lambda_j + \lambda_i), \\ \zeta_{26}^{(2)}(\lambda_k; \lambda_j; \lambda_i; \beta) &= 354240\beta + 8640\beta^3 + 362880 + 103680\beta^2 \geq 0 \text{ for all } \beta \geq -2.\end{aligned}$$

Define $\zeta_{261}(\lambda_j; \lambda_i; \beta) = \zeta_{26}^{(1)}(\lambda_j; \lambda_j; \lambda_i; \beta)$. We wish to show that ζ_{261} is non-negative for all $\lambda_j \geq \lambda_i$. Differentiating ζ_{261} with respect to λ_j gives

$$\begin{aligned}\zeta_{261}(\lambda_j; \lambda_i; \beta) &= (362880 + 233280\beta^2 + 509760\beta + 34560\beta^3)\lambda_j \\ &\quad + (129600\beta^2 + 25920\beta^3 + 155520\beta)\lambda_i, \\ \zeta_{261}^{(1)}(\lambda_j; \lambda_i; \beta) &= 362880 + 233280\beta^2 + 509760\beta + 34560\beta^3 \geq 0 \text{ for all } \beta \geq -1.75.\end{aligned}$$

Now

$$\zeta_{261}(\lambda_i; \lambda_i; \beta) = (362880\beta^2 + 60480\beta^3 + 665280\beta + 362880)\lambda_i \geq 0 \text{ for all } \beta \geq -1.$$

Hence, $\zeta_{26}^{(1)}(\lambda_j; \lambda_j; \lambda_i; \beta)$ is non-negative for all $\lambda_j \geq \lambda_i$ and $\beta \geq -\frac{2}{5}$.

Define $\zeta_{260}(\lambda_j; \lambda_i; \beta) = \zeta_{26}(\lambda_j; \lambda_j; \lambda_i; \beta)$. We wish to show that ζ_{260} is non-negative for all $\lambda_j \geq \lambda_i$. Differentiating ζ_{260} with respect to λ_j gives

$$\begin{aligned} \zeta_{260}(\lambda_j; \lambda_i; \beta) &= (43200\beta^3 + 572400\beta + 375840 + 278640\beta^2)\lambda_j^2 \\ &\quad + (349920\beta^2 + 64800\beta^3 + 518400\beta + 155520)\lambda_i\lambda_j \\ &\quad + (239760\beta + 97200\beta^2 + 12960\beta^3 + 194400)\lambda_i^2, \\ \zeta_{260}^{(1)}(\lambda_j; \lambda_i; \beta) &= (86400\beta^3 + 1144800\beta + 751680 + 557280\beta^2)\lambda_j \\ &\quad + (349920\beta^2 + 64800\beta^3 + 518400\beta + 155520)\lambda_i \\ &\geq 0 \text{ for all } \beta \geq -\frac{2}{5}. \end{aligned}$$

Now

$$\zeta_{260}(\lambda_i; \lambda_i; \beta) = (1330560\beta + 725760\beta^2 + 120960\beta^3 + 725760)\lambda_i^2 > 0 \text{ for all } \beta > -1.$$

Hence, $\zeta_{26}(\lambda_j; \lambda_j; \lambda_i; \beta)$ is positive for all $\lambda_j \geq \lambda_i$ and $\beta \geq -\frac{2}{5}$. This implies that $\zeta^{(6)}(0; \lambda_k; \lambda_j; \lambda_i; \beta)$ is positive for all $\lambda_k \geq \lambda_j$ and $\beta \geq -\frac{2}{5}$. □

Lemma B.7 Let

$$\begin{aligned} \zeta_3(\lambda; \lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta) &= 60 \sum_{\substack{(n_1, n_2, n_3, n_4) \\ \in \text{perm}(0, 4, 4, 4)}} (\lambda + \lambda_i)^{n_1} (\lambda + \lambda_j)^{n_2} (\lambda + \lambda_k)^{n_3} (\lambda + \lambda_l)^{n_4} \\ &\quad + (16\beta - 32) \sum_{\substack{(n_1, n_2, n_3, n_4) \\ \in \text{perm}(1, 3, 4, 4)}} (\lambda + \lambda_i)^{n_1} (\lambda + \lambda_j)^{n_2} (\lambda + \lambda_k)^{n_3} (\lambda + \lambda_l)^{n_4} \\ &\quad + (18\beta - 36) \sum_{\substack{(n_1, n_2, n_3, n_4) \\ \in \text{perm}(2, 2, 4, 4)}} (\lambda + \lambda_i)^{n_1} (\lambda + \lambda_j)^{n_2} (\lambda + \lambda_k)^{n_3} (\lambda + \lambda_l)^{n_4} \\ &\quad + (6\beta^2 - 36\beta + 48) \sum_{\substack{(n_1, n_2, n_3, n_4) \\ \in \text{perm}(2, 3, 3, 4)}} (\lambda + \lambda_i)^{n_1} (\lambda + \lambda_j)^{n_2} (\lambda + \lambda_k)^{n_3} (\lambda + \lambda_l)^{n_4} \\ &\quad + (4\beta^3 - 48\beta^2 + 176\beta - 192) (\lambda + \lambda_k)^3 (\lambda + \lambda_i)^3 (\lambda + \lambda_j)^3 (\lambda + \lambda_l)^3 \end{aligned}$$

and $\lambda_k \geq \max(\lambda_i, \lambda_j, \lambda_l)$. The expression $\zeta_3(\lambda; \lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta)$ is non-negative for all $\lambda \in (\lambda_s, \infty)$ and $\beta \geq -\frac{2}{5}$.

Proof.

Without loss of generality assume that $\lambda_i \leq \lambda_j \leq \lambda_l$.

Define

$$s(a, b, c, d) = \sum_{\substack{(n_1, n_2, n_3, n_4) \\ \in \text{perm}(a, b, c, d)}} (\lambda + \lambda_i)^{n_1} (\lambda + \lambda_j)^{n_2} (\lambda + \lambda_k)^{n_3} (\lambda + \lambda_l)^{n_4}.$$

Differentiating $\zeta_3(\lambda; \lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta)$ with respect to λ we obtain

$$\begin{aligned} \zeta_3(\lambda; \lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta) &= 60s(0, 4, 4, 4) + (16\beta - 32)s(1, 3, 4, 4) & (B.22) \\ &+ (18\beta - 36)s(2, 2, 4, 4) + (6\beta^2 - 36\beta + 48)s(2, 3, 3, 4) \\ &+ (4\beta^3 - 48\beta^2 + 176\beta - 192)s(3, 3, 3, 3), \end{aligned}$$

$$\begin{aligned} \zeta_3^{(1)}(\lambda; \lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta) &= (56\beta + 12\beta^2 - 160)s(1, 3, 3, 4) & (B.23) \\ &+ (-144\beta + 36\beta^2 + 144)s(2, 2, 3, 4) \\ &+ (208 + 16\beta)s(0, 3, 4, 4) \\ &+ (84\beta - 168)s(1, 2, 4, 4) \\ &+ (-72\beta^2 + 96\beta + 12\beta^3)s(2, 3, 3, 3) \end{aligned}$$

$$\begin{aligned} \zeta_3^{(2)}(\lambda; \lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta) &= (-1296\beta + 324\beta^2 + 1296)s(2, 2, 2, 4) & (B.24) \\ &+ (336\beta - 672)s(1, 1, 4, 4) \\ &+ (-864 + 108\beta^2 + 216\beta)s(1, 2, 3, 4) \\ &+ (12\beta^2 + 1504 + 184\beta)s(0, 3, 3, 4) \\ &+ (132\beta + 456)s(0, 2, 4, 4) \\ &+ (24\beta^3 + 864\beta - 1920)s(1, 3, 3, 3) \\ &+ (1152 + 72\beta^3 - 144\beta^2 - 576\beta)s(2, 2, 3, 3), \end{aligned}$$

$$\begin{aligned} \zeta_3^{(3)}(\lambda; \lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta) &= (648\beta^3 + 15552 - 10368\beta)s(2, 2, 2, 3) & (B.25) \\ &+ (2208\beta + 432\beta^2 - 6144)s(1, 1, 3, 4) \\ &+ (1296\beta^2 - 1296\beta - 2592)s(1, 2, 2, 4) \\ &+ (576\beta^2 + 3168\beta + 216\beta^3 - 10368)s(1, 2, 3, 3) \\ &+ (144\beta^2 + 1296\beta + 5472)s(0, 2, 3, 4) \\ &+ (240 + 600\beta)s(0, 1, 4, 4) \\ &+ (144\beta^2 + 3072\beta + 16128 + 24\beta^3)s(0, 3, 3, 3), \end{aligned}$$

$$\begin{aligned}
\zeta_3^{(4)}(\lambda; \lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta) &= (1200\beta + 480)s(0, 0, 4, 4) & (B.26) \\
&+ (7200\beta + 720\beta^2 + 5760)s(0, 1, 3, 4) \\
&+ (30240 + 6480\beta + 2160\beta^2)s(0, 2, 2, 4) \\
&+ (-28800 + 6480\beta^2 + 1440\beta)s(1, 1, 2, 4) \\
&+ (2160\beta^2 + 288\beta^3 + 81792 + 22752\beta)s(0, 2, 3, 3) \\
&+ (30336\beta + 864\beta^3 + 5760\beta^2 - 90624)s(1, 1, 3, 3) \\
&+ (2592\beta^3 + 8640\beta^2 - 41472 - 6912\beta)s(1, 2, 2, 3) \\
&+ (7776\beta^3 + 186624 - 124416\beta)s(2, 2, 2, 2),
\end{aligned}$$

$$\begin{aligned}
\zeta_3^{(5)}(\lambda; \lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta) &= (8640\beta + 38880\beta^2 - 172800)s(1, 1, 1, 4) & (B.27) \\
&+ (12960\beta^2 + 48960 + 36000\beta)s(0, 1, 2, 4) \\
&+ (1440\beta^2 + 19200\beta + 13440)s(0, 0, 3, 4) \\
&+ (15840\beta^2 + 119040 + 133440\beta + 1440\beta^3)s(0, 1, 3, 3) \\
&+ (12960\beta^3 - 552960 + 69120\beta + 77760\beta^2)s(1, 1, 2, 3) \\
&+ (30240\beta^2 + 4320\beta^3 + 570240 + 155520\beta)s(0, 2, 2, 3) \\
&+ (-311040\beta + 38880\beta^3 + 77760\beta^2)s(1, 2, 2, 2),
\end{aligned}$$

$$\begin{aligned}
\zeta_3^{(6)}(\lambda; \lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta) &= (345600 + 420480\beta + 43200\beta^2 + 2880\beta^3)s(0, 0, 3, 3) & (B.28) \\
&+ (1140480 + 924480\beta + 237600\beta^2 + 25920\beta^3)s(0, 1, 2, 3) \\
&+ (152640\beta + 23040 + 90720\beta^2)s(0, 1, 1, 4) \\
&+ (77760\beta^3 - 4008960 + 449280\beta + 622080\beta^2)s(1, 1, 1, 3) \\
&+ (777600\beta^2 - 829440\beta + 233280\beta^3 - 3317760)s(1, 1, 2, 2) \\
&+ (129600\beta + 30240\beta^2 + 138240)s(0, 0, 2, 4) \\
&+ (349920\beta^2 + 77760\beta^3 + 5132160 + 1088640\beta)s(0, 2, 2, 2),
\end{aligned}$$

$$\begin{aligned}
\zeta_3^{(7)}(\lambda; \lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta) &= 20160(322560 + 564480\beta + 241920\beta^2)s(0, 0, 1, 4) & (B.29) \\
&+ (60480\beta^3 + 3870720 + 3628800\beta + 725760\beta^2)s(0, 0, 2, 3) \\
&+ (1935360\beta^2 + 181440\beta^3 + 645120 + 4757760\beta)s(0, 1, 1, 3) \\
&+ (2903040\beta^2 + 544320\beta^3 + 13789440 + 6894720\beta)s(0, 1, 2, 2) \\
&+ (-3628800\beta + 6531840\beta^2 - 31933440 + 1632960\beta^3)s(1, 1, 1, 2),
\end{aligned}$$

$$\begin{aligned}
\zeta_3^{(8)}(\lambda; \lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta) &= (725760\beta^2 + 967680 + 1693440\beta)s(0, 0, 0, 4) & (B.30) \\
&+ (10321920 + 483840\beta^3 + 19031040\beta + 6289920\beta^2)s(0, 0, 1, 3) \\
&+ (38223360\beta + 23950080\beta^2 + 25159680 + 4354560\beta^3)s(0, 1, 1, 2) \\
&+ (50803200 + 10160640\beta^2 + 35562240\beta + 1451520\beta^3)s(0, 0, 2, 2) \\
&+ (-29030400\beta + 52254720\beta^2 - 255467520 + 13063680\beta^3)s(1, 1, 1, 1),
\end{aligned}$$

$$\begin{aligned}
\zeta_3^{(9)}(\lambda; \lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta) &= 1451520(15\beta^2 + 44\beta + \beta^3 + 24)s(0, 0, 0, 3) & (B.31) \\
&+ 4354560(3\beta^3 + 42 + 47\beta + 20\beta^2)s(0, 0, 1, 2) \\
&+ 4354560(9\beta^3 + 45\beta^2 - 24 + 46\beta)s(0, 1, 1, 1),
\end{aligned}$$

$$\begin{aligned}
\zeta_3^{(10)}(\lambda; \lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta) &= 21772800(2\beta + 5)(\beta + 3)(\beta + 2)s(0, 0, 0, 2) \\
&\quad + 43545600(3\beta + 2)(\beta + 3)(\beta + 2)s(0, 0, 1, 1), \\
\zeta_3^{(11)}(\lambda; \lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta) &= 479001600(\beta + 3)(\beta + 2)(\beta + 1)s(0, 0, 0, 1) \\
&> 0 \text{ for all } \beta > -1.
\end{aligned} \tag{B.32}$$

Consider the case $\lambda_i \leq 0$. Substituting in $\lambda = -\lambda_i$ into (B.22)–(B.32) and defining

$$\hat{s}(a, b, c) = \sum_{\substack{(n_1, n_2, n_3) \\ \in \text{perm}(a, b, c)}} (\lambda_j - \lambda_i)^{n_1} (\lambda_k - \lambda_i)^{n_2} (\lambda_l - \lambda_i)^{n_3}$$

we obtain

$$\begin{aligned}
\zeta_3(-\lambda_i; \lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta) &= 60\hat{s}(4, 4, 4) \geq 0, \\
\zeta_3^{(1)}(-\lambda_i; \lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta) &= (208 + 16\beta)\hat{s}(3, 4, 4) \geq 0 \text{ for all } \beta > -13, \\
\zeta_3^{(2)}(-\lambda_i; \lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta) &= (12\beta^2 + 1504 + 184\beta)\hat{s}(3, 3, 4) \\
&\quad + (132\beta + 456)\hat{s}(2, 4, 4) \\
&\geq 0 \text{ for all } \beta > -\frac{38}{11}, \\
\zeta_3^{(3)}(-\lambda_i; \lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta) &= (144\beta^2 + 1296\beta + 5472)\hat{s}(2, 3, 4) \\
&\quad + (240 + 600\beta)\hat{s}(1, 4, 4) \\
&\quad + (144\beta^2 + 3072\beta + 16128 + 24\beta^3)\hat{s}(3, 3, 3) \\
&\geq 0 \text{ for all } \beta \geq -\frac{2}{5}, \\
\zeta_3^{(4)}(-\lambda_i; \lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta) &= (1200\beta + 480)\hat{s}(0, 4, 4) + (7200\beta + 720\beta^2 + 5760)\hat{s}(1, 3, 4) \\
&\quad + (30240 + 6480\beta + 2160\beta^2)\hat{s}(2, 2, 4) \\
&\quad + (2160\beta^2 + 288\beta^3 + 81792 + 22752\beta)\hat{s}(2, 3, 3) \\
&\geq 0 \text{ for all } \beta \geq -\frac{2}{5}, \\
\zeta_3^{(5)}(-\lambda_i; \lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta) &= (12960\beta^2 + 48960 + 36000\beta)\hat{s}(1, 2, 4) \\
&\quad + (1440\beta^2 + 19200\beta + 13440)\hat{s}(0, 3, 4) \\
&\quad + (15840\beta^2 + 119040 + 133440\beta + 1440\beta^3)\hat{s}(1, 3, 3) \\
&\quad + (30240\beta^2 + 4320\beta^3 + 570240 + 155520\beta)\hat{s}(2, 2, 3) \\
&\geq 0 \text{ for all } \beta \geq -0.74,
\end{aligned}$$

$$\begin{aligned}
\zeta_3^{(6)}(-\lambda_i; \lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta) &= 2880(120 + 146\beta + 15\beta^2 + \beta^3)\hat{s}(0, 3, 3) \\
&\quad + 4320(264 + 214\beta + 55\beta^2 + 6\beta^3)\hat{s}(1, 2, 3) \\
&\quad + 1440(106\beta + 16 + 63\beta^2)\hat{s}(1, 1, 4) \\
&\quad + 4320(30\beta + 7\beta^2 + 32)\hat{s}(0, 2, 4) \\
&\quad + 38880(9\beta^2 + 2\beta^3 + 132 + 28\beta)\hat{s}(2, 2, 2), \\
\zeta_3^{(7)}(-\lambda_i; \lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta) &= 80640(4 + 7\beta + 3\beta^2)\hat{s}(0, 1, 4) \\
&\quad + 60480(\beta^3 + 64 + 60\beta + 12\beta^2)\hat{s}(0, 2, 3) \\
&\quad + 20160(96\beta^2 + 9\beta^3 + 32 + 236\beta)\hat{s}(1, 1, 3) \\
&\quad + 181440(16\beta^2 + 3\beta^3 + 76 + 38\beta)\hat{s}(1, 2, 2), \\
\zeta_3^{(8)}(-\lambda_i; \lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta) &= (725760\beta^2 + 967680 + 1693440\beta)\hat{s}(0, 0, 4) \\
&\quad + (10321920 + 483840\beta^3 + 19031040\beta + 6289920\beta^2)\hat{s}(0, 1, 3) \\
&\quad + (38223360\beta + 23950080\beta^2 + 25159680 + 4354560\beta^3)\hat{s}(1, 1, 2) \\
&\quad + (50803200 + 10160640\beta^2 + 35562240\beta + 1451520\beta^3)\hat{s}(0, 2, 2) \\
&\geq 0 \text{ for all } \beta \geq -0.69, \\
\zeta_3^{(9)}(-\lambda_i; \lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta) &= 1451520(15\beta^2 + 44\beta + \beta^3 + 24)\hat{s}(0, 0, 3) \\
&\quad + 4354560(3\beta^3 + 42 + 47\beta + 20\beta^2)\hat{s}(0, 1, 2) \\
&\quad + 4354560(9\beta^3 + 45\beta^2 - 24 + 46\beta)\hat{s}(1, 1, 1), \\
\zeta_3^{(10)}(-\lambda_i; \lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta) &= 21772800(2\beta + 5)(\beta + 3)(\beta + 2)\hat{s}(0, 0, 2) \\
&\quad + 43545600(3\beta + 2)(\beta + 3)(\beta + 2)\hat{s}(0, 1, 1) \\
&\geq 0 \text{ for all } \beta \geq -\frac{2}{3}.
\end{aligned}$$

From Lemmas B.8—B.10 we deduce that $\zeta_3^{(6)}(-\lambda_i; \lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta)$, $\zeta_3^{(7)}(-\lambda_i; \lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta)$ and $\zeta_3^{(9)}(-\lambda_i; \lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta)$ are non-negative for all $\beta \geq -\frac{2}{5}$. Hence, $\zeta_3(\lambda; \lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta)$ is non-negative for all $\lambda \geq -\lambda_i$ and $\beta \geq -\frac{2}{5}$.

Consider the case $\lambda_i > 0$. From Lemmas B.11-B.21 we deduce that $\zeta_3(0; \lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta)$, $\zeta_3^{(1)}(0; \lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta)$, \dots , $\zeta_3^{(11)}(0; \lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta)$ are positive for all $\lambda_k \geq \lambda_l \geq \lambda_j \geq \lambda_i$ and $\beta \geq -\frac{2}{5}$. Hence, $\zeta_3(\lambda; \lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta)$ is non-negative for all $\lambda \geq 0$ and $\beta \geq -\frac{2}{5}$.

□

Lemma B.8 Let

$$\begin{aligned} \xi(\lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta) &= (106\beta + 16 + 63\beta^2)\hat{s}(1, 1, 4) \\ &\quad + 3(30\beta + 7\beta^2 + 32)\hat{s}(0, 2, 4), \end{aligned} \quad (\text{B.33})$$

where

$$\hat{s}(a, b, c) = \sum_{\substack{(n_1, n_2, n_3) \\ \in \text{perm}(a, b, c)}} (\lambda_j - \lambda_i)^{n_1} (\lambda_k - \lambda_i)^{n_2} (\lambda_l - \lambda_i)^{n_3}.$$

The expression $\xi(\lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta)$ is non-negative for all $\lambda_k \geq \lambda_l \geq \lambda_j \geq \lambda_i$, and $\beta \geq -\frac{2}{5}$.

Proof. Differentiating $\xi(\lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta)$ with respect to λ_k we obtain

$$\begin{aligned} \xi^{(1)}(\lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta) &= 2(90\beta + 96 + 21\beta^2)(\lambda_k - \lambda_i)(\lambda_l - \lambda_i)^4 \\ &\quad + 4(90\beta + 96 + 21\beta^2)(\lambda_k - \lambda_i)^3(\lambda_l - \lambda_i)^2 \\ &\quad + 4(90\beta + 96 + 21\beta^2)(\lambda_j - \lambda_i)^2(\lambda_k - \lambda_i)^3 \\ &\quad + 2(90\beta + 96 + 21\beta^2)(\lambda_j - \lambda_i)^4(\lambda_k - \lambda_i) \\ &\quad + (106\beta + 16 + 63\beta^2)(\lambda_j - \lambda_i)(\lambda_l - \lambda_i)^4 \\ &\quad + 4(106\beta + 16 + 63\beta^2)(\lambda_j - \lambda_i)(\lambda_k - \lambda_i)^3(\lambda_l - \lambda_i) \\ &\quad + (106\beta + 16 + 63\beta^2)(\lambda_j - \lambda_i)^4(\lambda_l - \lambda_i), \end{aligned} \quad (\text{B.34})$$

$$\begin{aligned} \xi^{(2)}(\lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta) &= 2(90\beta + 96 + 21\beta^2)(\lambda_l - \lambda_i)^4 \\ &\quad + 12(90\beta + 96 + 21\beta^2)(\lambda_k - \lambda_i)^2(\lambda_l - \lambda_i)^2 \\ &\quad + 12(90\beta + 96 + 21\beta^2)(\lambda_j - \lambda_i)^2(\lambda_k - \lambda_i)^2 \\ &\quad + 2(90\beta + 96 + 21\beta^2)(\lambda_j - \lambda_i)^4 \\ &\quad + 12(106\beta + 16 + 63\beta^2)(\lambda_j - \lambda_i)(\lambda_k - \lambda_i)^2(\lambda_l - \lambda_i), \end{aligned} \quad (\text{B.35})$$

$$\begin{aligned} \xi^{(3)}(\lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta) &= 24(90\beta + 96 + 21\beta^2)(\lambda_k - \lambda_i)(\lambda_l - \lambda_i)^2 \\ &\quad + 24(90\beta + 96 + 21\beta^2)(\lambda_j - \lambda_i)^2(\lambda_k - \lambda_i) \\ &\quad + 24(106\beta + 16 + 63\beta^2)(\lambda_j - \lambda_i)(\lambda_k - \lambda_i)(\lambda_l - \lambda_i), \end{aligned} \quad (\text{B.36})$$

$$\begin{aligned} \xi^{(4)}(\lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta) &= 24(90\beta + 96 + 21\beta^2)(\lambda_l - \lambda_i)^2 \\ &\quad + 24(90\beta + 96 + 21\beta^2)(\lambda_j - \lambda_i)^2 \\ &\quad + 24(106\beta + 16 + 63\beta^2)(\lambda_j - \lambda_i)(\lambda_l - \lambda_i), \end{aligned} \quad (\text{B.37})$$

Define

$$\begin{aligned}
\xi_0(\lambda_l; \lambda_j; \lambda_i; \beta) &\stackrel{\text{def}}{=} \xi(\lambda_l; \lambda_l; \lambda_j; \lambda_i; \beta), \\
\xi_1(\lambda_l; \lambda_j; \lambda_i; \beta) &\stackrel{\text{def}}{=} \xi^{(1)}(\lambda_l; \lambda_l; \lambda_j; \lambda_i; \beta), \\
\xi_2(\lambda_l; \lambda_j; \lambda_i; \beta) &\stackrel{\text{def}}{=} \xi^{(2)}(\lambda_l; \lambda_l; \lambda_j; \lambda_i; \beta), \\
\xi_3(\lambda_l; \lambda_j; \lambda_i; \beta) &\stackrel{\text{def}}{=} \xi^{(3)}(\lambda_l; \lambda_l; \lambda_j; \lambda_i; \beta), \\
\xi_4(\lambda_l; \lambda_j; \lambda_i; \beta) &\stackrel{\text{def}}{=} \xi^{(4)}(\lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta).
\end{aligned}$$

We will prove that the above equations are non-negative for all $\lambda_k \geq \lambda_l \geq \lambda_j \geq \lambda_i$ and $\beta \geq -\frac{2}{5}$ and, hence, deduce that $\xi(\lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta)$ is positive for all $\lambda_k \geq \lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Differentiating $\xi_0(\lambda_l; \lambda_j; \lambda_i; \beta)$ with respect to λ_l we obtain

$$\begin{aligned}
\xi_0(\lambda_l; \lambda_j; \lambda_i; \beta) &= (42\beta^2 + 192 + 180\beta)(\lambda_l - \lambda_i)^6 \\
&\quad + (42\beta^2 + 192 + 180\beta)(\lambda_j - \lambda_i)^2(\lambda_l - \lambda_i)^4 \\
&\quad + (212\beta + 32 + 126\beta^2)(\lambda_j - \lambda_i)(\lambda_l - \lambda_i)^5 \\
&\quad + (105\beta^2 + 208 + 286\beta)(\lambda_j - \lambda_i)^4(\lambda_l - \lambda_i)^2, \\
\xi_0^{(1)}(\lambda_l; \lambda_j; \lambda_i; \beta) &= 6(42\beta^2 + 192 + 180\beta)(\lambda_l - \lambda_i)^5 \\
&\quad + 4(42\beta^2 + 192 + 180\beta)(\lambda_j - \lambda_i)^2(\lambda_l - \lambda_i)^3 \\
&\quad + 5(212\beta + 32 + 126\beta^2)(\lambda_j - \lambda_i)(\lambda_l - \lambda_i)^4 \\
&\quad + 2(105\beta^2 + 208 + 286\beta)(\lambda_j - \lambda_i)^4(\lambda_l - \lambda_i), \\
\xi_0^{(2)}(\lambda_l; \lambda_j; \lambda_i; \beta) &= 30(42\beta^2 + 192 + 180\beta)(\lambda_l - \lambda_i)^4 \\
&\quad + 12(42\beta^2 + 192 + 180\beta)(\lambda_j - \lambda_i)^2(\lambda_l - \lambda_i)^2 \\
&\quad + 20(212\beta + 32 + 126\beta^2)(\lambda_j - \lambda_i)(\lambda_l - \lambda_i)^3 \\
&\quad + 2(105\beta^2 + 208 + 286\beta)(\lambda_j - \lambda_i)^4, \\
\xi_0^{(3)}(\lambda_l; \lambda_j; \lambda_i; \beta) &= 120(42\beta^2 + 192 + 180\beta)(\lambda_l - \lambda_i)^3 \\
&\quad + 24(42\beta^2 + 192 + 180\beta)(\lambda_j - \lambda_i)^2(\lambda_l - \lambda_i) \\
&\quad + 60(212\beta + 32 + 126\beta^2)(\lambda_j - \lambda_i)(\lambda_l - \lambda_i)^2, \\
\xi_0^{(4)}(\lambda_l; \lambda_j; \lambda_i; \beta) &= 360(42\beta^2 + 192 + 180\beta)(\lambda_l - \lambda_i)^2 \\
&\quad + 24(42\beta^2 + 192 + 180\beta)(\lambda_j - \lambda_i)^2 \\
&\quad + 120(212\beta + 32 + 126\beta^2)(\lambda_j - \lambda_i)(\lambda_l - \lambda_i), \\
\xi_0^{(5)}(\lambda_l; \lambda_j; \lambda_i; \beta) &= 720(42\beta^2 + 192 + 180\beta)(\lambda_l - \lambda_i) \\
&\quad + 120(212\beta + 32 + 126\beta^2)(\lambda_j - \lambda_i), \\
\xi_0^{(6)}(\lambda_l; \lambda_j; \lambda_i; \beta) &= 129600\beta + 138240 + 30240\beta^2 \\
&> 0 \text{ for all } \beta > -2.
\end{aligned}$$

Now

$$\begin{aligned}
 \xi_0(\lambda_j; \lambda_j; \lambda_i; \beta) &= 3(105\beta^2 + 208 + 286\beta)(\lambda_j - \lambda_i)^6 \geq 0 \text{ for all } \beta, \\
 \xi_0^{(1)}(\lambda_j; \lambda_j; \lambda_i; \beta) &= 12(105\beta^2 + 208 + 286\beta)(\lambda_j - \lambda_i)^5 \geq 0 \text{ for all } \beta, \\
 \xi_0^{(2)}(\lambda_j; \lambda_j; \lambda_i; \beta) &= 6(\lambda_j - \lambda_i)^4(749\beta^2 + 2062\beta + 1520) \geq 0 \text{ for all } \beta, \\
 \xi_0^{(3)}(\lambda_j; \lambda_j; \lambda_i; \beta) &= 168(\lambda_j - \lambda_i)^3(81\beta^2 + 176 + 230\beta) \geq 0 \text{ for all } \beta, \\
 \xi_0^{(4)}(\lambda_j; \lambda_j; \lambda_i; \beta) &= 48(\lambda_j - \lambda_i)^2(1970\beta + 1616 + 651\beta^2) \geq 0 \text{ for all } \beta, \\
 \xi_0^{(5)}(\lambda_j; \lambda_j; \lambda_i; \beta) &= 240(189\beta^2 + 592 + 646\beta)(\lambda_j - \lambda_i) \geq 0 \text{ for all } \beta,
 \end{aligned}$$

and, hence, $\xi_0(\lambda_l; \lambda_j; \lambda_i; \beta)$ is non-negative.

Differentiating $\xi_1(\lambda_l; \lambda_j; \lambda_i; \beta)$ with respect to λ_l we obtain

$$\begin{aligned}
 \xi_1(\lambda_l; \lambda_j; \lambda_i; \beta) &= (540\beta + 576 + 126\beta^2)(\lambda_l - \lambda_i)^5 \\
 &\quad + (360\beta + 384 + 84\beta^2)(\lambda_j - \lambda_i)^2(\lambda_l - \lambda_i)^3 \\
 &\quad + (530\beta + 80 + 315\beta^2)(\lambda_j - \lambda_i)(\lambda_l - \lambda_i)^4 \\
 &\quad + (105\beta^2 + 208 + 286\beta)(\lambda_j - \lambda_i)^4(\lambda_l - \lambda_i), \\
 \xi_1^{(1)}(\lambda_l; \lambda_j; \lambda_i; \beta) &= 5(540\beta + 576 + 126\beta^2)(\lambda_l - \lambda_i)^4 \\
 &\quad + 3(360\beta + 384 + 84\beta^2)(\lambda_j - \lambda_i)^2(\lambda_l - \lambda_i)^2 \\
 &\quad + 4(530\beta + 80 + 315\beta^2)(\lambda_j - \lambda_i)(\lambda_l - \lambda_i)^3 \\
 &\quad + (105\beta^2 + 208 + 286\beta)(\lambda_j - \lambda_i)^4, \\
 \xi_1^{(2)}(\lambda_l; \lambda_j; \lambda_i; \beta) &= 20(540\beta + 576 + 126\beta^2)(\lambda_l - \lambda_i)^3 \\
 &\quad + 6(360\beta + 384 + 84\beta^2)(\lambda_j - \lambda_i)^2(\lambda_l - \lambda_i) \\
 &\quad + 12(530\beta + 80 + 315\beta^2)(\lambda_j - \lambda_i)(\lambda_l - \lambda_i)^2, \\
 \xi_1^{(3)}(\lambda_l; \lambda_j; \lambda_i; \beta) &= 60(540\beta + 576 + 126\beta^2)(\lambda_l - \lambda_i)^2 \\
 &\quad + 6(360\beta + 384 + 84\beta^2)(\lambda_j - \lambda_i)^2 \\
 &\quad + 24(530\beta + 80 + 315\beta^2)(\lambda_j - \lambda_i)(\lambda_l - \lambda_i), \\
 \xi_1^{(4)}(\lambda_l; \lambda_j; \lambda_i; \beta) &= 120(540\beta + 576 + 126\beta^2)(\lambda_l - \lambda_i) \\
 &\quad + 24(530\beta + 80 + 315\beta^2)(\lambda_j - \lambda_i), \\
 \xi_1^{(5)}(\lambda_l; \lambda_j; \lambda_i; \beta) &= 64800\beta + 69120 + 15120\beta^2 \\
 &> 0 \text{ for all } \beta > -2.
 \end{aligned}$$

Now

$$\begin{aligned}
 \xi_1(\lambda_j; \lambda_j; \lambda_i; \beta) &= 6(105\beta^2 + 208 + 286\beta)(\lambda_j - \lambda_i)^5 \geq 0 \text{ for all } \beta, \\
 \xi_1^{(1)}(\lambda_j; \lambda_j; \lambda_i; \beta) &= 3(749\beta^2 + 2062\beta + 1520)(\lambda_j - \lambda_i)^4 \geq 0 \text{ for all } \beta, \\
 \xi_1^{(2)}(\lambda_j; \lambda_j; \lambda_i; \beta) &= 84(81\beta^2 + 176 + 230\beta)(\lambda_j - \lambda_i)^3 \geq 0 \text{ for all } \beta, \\
 \xi_1^{(3)}(\lambda_j; \lambda_j; \lambda_i; \beta) &= 24(1970\beta + 1616 + 651\beta^2)(\lambda_j - \lambda_i)^2 \geq 0 \text{ for all } \beta, \\
 \xi_1^{(4)}(\lambda_j; \lambda_j; \lambda_i; \beta) &= 120(189\beta^2 + 592 + 646\beta)(\lambda_j - \lambda_i) \geq 0 \text{ for all } \beta,
 \end{aligned}$$

and, hence, $\xi_1(\lambda_l; \lambda_j; \lambda_i; \beta)$ is non-negative.

Differentiating $\xi_2(\lambda_l; \lambda_j; \lambda_i; \beta)$ with respect to λ_l we obtain

$$\begin{aligned}\xi_2(\lambda_l; \lambda_j; \lambda_i; \beta) &= 14(90\beta + 96 + 21\beta^2)(\lambda_l - \lambda_i)^4 \\ &\quad + 12(90\beta + 96 + 21\beta^2)(\lambda_j - \lambda_i)^2(\lambda_l - \lambda_i)^2 \\ &\quad + 2(90\beta + 96 + 21\beta^2)(\lambda_j - \lambda_i)^4 \\ &\quad + 12(106\beta + 16 + 63\beta^2)(\lambda_j - \lambda_i)(\lambda_l - \lambda_i)^3, \\ \xi_2^{(1)}(\lambda_l; \lambda_j; \lambda_i; \beta) &= 56(90\beta + 96 + 21\beta^2)(\lambda_l - \lambda_i)^3 \\ &\quad + 24(90\beta + 96 + 21\beta^2)(\lambda_j - \lambda_i)^2(\lambda_l - \lambda_i) \\ &\quad + 36(106\beta + 16 + 63\beta^2)(\lambda_j - \lambda_i)(\lambda_l - \lambda_i)^2, \\ \xi_2^{(2)}(\lambda_l; \lambda_j; \lambda_i; \beta) &= 168(90\beta + 96 + 21\beta^2)(\lambda_l - \lambda_i)^2 \\ &\quad + 24(90\beta + 96 + 21\beta^2)(\lambda_j - \lambda_i)^2 \\ &\quad + 72(106\beta + 16 + 63\beta^2)(\lambda_j - \lambda_i)(\lambda_l - \lambda_i), \\ \xi_2^{(3)}(\lambda_l; \lambda_j; \lambda_i; \beta) &= 336(90\beta + 96 + 21\beta^2)(\lambda_l - \lambda_i) \\ &\quad + 72(106\beta + 16 + 63\beta^2)(\lambda_j - \lambda_i), \\ \xi_2^{(4)}(\lambda_l; \lambda_j; \lambda_i; \beta) &= 30240\beta + 32256 + 7056\beta^2 \\ &> 0 \text{ for all } \beta > -2.\end{aligned}$$

Now

$$\begin{aligned}\xi_2(\lambda_j; \lambda_j; \lambda_i; \beta) &= 48(79\beta + 28\beta^2 + 60)(\lambda_j - \lambda_i)^4 \geq 0 \text{ for all } \beta, \\ \xi_2^{(1)}(\lambda_j; \lambda_j; \lambda_i; \beta) &= 12(918\beta + 688 + 329\beta^2)(\lambda_j - \lambda_i)^3 \geq 0 \text{ for all } \beta, \\ \xi_2^{(2)}(\lambda_j; \lambda_j; \lambda_i; \beta) &= 72(119\beta^2 + 272 + 346\beta)(\lambda_j - \lambda_i)^2 \geq 0 \text{ for all } \beta, \\ \xi_2^{(3)}(\lambda_j; \lambda_j; \lambda_i; \beta) &= 72(526\beta + 464 + 161\beta^2)(\lambda_j - \lambda_i) \geq 0 \text{ for all } \beta,\end{aligned}$$

and, hence, $\xi_2(\lambda_l; \lambda_j; \lambda_i; \beta)$ is non-negative.

Differentiating $\xi_3(\lambda_l; \lambda_j; \lambda_i; \beta)$ with respect to λ_l we obtain

$$\begin{aligned}\xi_3(\lambda_l; \lambda_j; \lambda_i; \beta) &= 24(90\beta + 96 + 21\beta^2)(\lambda_l - \lambda_i)^3 \\ &\quad + 24(90\beta + 96 + 21\beta^2)(\lambda_j - \lambda_i)^2(\lambda_l - \lambda_i) \\ &\quad + 24(106\beta + 16 + 63\beta^2)(\lambda_j - \lambda_i)(\lambda_l - \lambda_i)^2, \\ \xi_3^{(1)}(\lambda_l; \lambda_j; \lambda_i; \beta) &= 72(90\beta + 96 + 21\beta^2)(\lambda_l - \lambda_i)^2 \\ &\quad + 24(90\beta + 96 + 21\beta^2)(\lambda_j - \lambda_i)^2 \\ &\quad + 48(106\beta + 16 + 63\beta^2)(\lambda_j - \lambda_i)(\lambda_l - \lambda_i), \\ \xi_3^{(2)}(\lambda_l; \lambda_j; \lambda_i; \beta) &= 144(90\beta + 96 + 21\beta^2)(\lambda_l - \lambda_i) \\ &\quad + 48(106\beta + 16 + 63\beta^2)(\lambda_j - \lambda_i), \\ \xi_3^{(3)}(\lambda_l; \lambda_j; \lambda_i; \beta) &= 12960\beta + 13824 + 3024\beta^2 \\ &> 0 \text{ for all } \beta > -2.\end{aligned}$$

Now

$$\begin{aligned}\xi_3(\lambda_j; \lambda_j; \lambda_i; \beta) &= 24(105\beta^2 + 208 + 286\beta)(\lambda_j - \lambda_i)^3 \geq 0 \text{ for all } \beta, \\ \xi_3^{(1)}(\lambda_j; \lambda_j; \lambda_i; \beta) &= 48(105\beta^2 + 208 + 286\beta)(\lambda_j - \lambda_i)^2 \geq 0 \text{ for all } \beta, \\ \xi_3^{(2)}(\lambda_j; \lambda_j; \lambda_i; \beta) &= 96(188\beta + 152 + 63\beta^2)(\lambda_j - \lambda_i) \geq 0 \text{ for all } \beta\end{aligned}$$

and, hence, $\xi_3(\lambda_l; \lambda_j; \lambda_i; \beta)$ is non-negative.

Differentiating $\xi_4(\lambda_l; \lambda_j; \lambda_i; \beta)$ with respect to λ_l we obtain

$$\begin{aligned}\xi_4(\lambda_l; \lambda_j; \lambda_i; \beta) &= 24(90\beta + 96 + 21\beta^2)(\lambda_l - \lambda_i)^2 \\ &\quad + 24(90\beta + 96 + 21\beta^2)(\lambda_j - \lambda_i)^2 \\ &\quad + 24(106\beta + 16 + 63\beta^2)(\lambda_j - \lambda_i)(\lambda_l - \lambda_i), \\ \xi_4^{(1)}(\lambda_l; \lambda_j; \lambda_i; \beta) &= 48(90\beta + 96 + 21\beta^2)(\lambda_l - \lambda_i) \\ &\quad + 24(106\beta + 16 + 63\beta^2)(\lambda_j - \lambda_i), \\ \xi_4^{(1)}(\lambda_l; \lambda_j; \lambda_i; \beta) &= 4320\beta + 4608 + 1008\beta^2 \\ &> 0 \text{ for all } \beta > -2.\end{aligned}$$

Now

$$\begin{aligned}\xi_4(\lambda_j; \lambda_j; \lambda_i; \beta) &= 24(105\beta^2 + 208 + 286\beta)(\lambda_j - \lambda_i)^2 \geq 0 \text{ for all } \beta, \\ \xi_4^{(1)}(\lambda_j; \lambda_j; \lambda_i; \beta) &= 24(105\beta^2 + 208 + 286\beta)(\lambda_j - \lambda_i) \geq 0 \text{ for all } \beta\end{aligned}$$

and, hence, $\xi_4(\lambda_l; \lambda_j; \lambda_i; \beta)$ is non-negative.

This completes the proof. □

Lemma B.9 Let

$$\begin{aligned}\xi(\lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta) &= 3(\beta^3 + 64 + 60\beta + 12\beta^2)\hat{s}(0, 2, 3) \\ &\quad + (96\beta^2 + 9\beta^3 + 32 + 236\beta)\hat{s}(1, 1, 3),\end{aligned}$$

where

$$\hat{s}(a, b, c) = \sum_{\substack{(n_1, n_2, n_3) \\ \in \text{perm}(a, b, c)}} (\lambda_j - \lambda_i)^{n_1} (\lambda_k - \lambda_i)^{n_2} (\lambda_l - \lambda_i)^{n_3}.$$

The expression $\xi(\lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta)$ is non-negative for all $\lambda_k \geq \lambda_l \geq \lambda_j \geq \lambda_i$, and $\beta \geq -\frac{2}{5}$.

Proof. Differentiating $\xi(\lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta)$ with respect to λ_k we obtain

$$\begin{aligned}
\xi^{(1)}(\lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta) &= (96\beta^2 + 9\beta^3 + 32 + 236\beta)(\lambda_l - \lambda_i)(\lambda_j - \lambda_i)^3 \\
&\quad + (96\beta^2 + 9\beta^3 + 32 + 236\beta)(\lambda_l - \lambda_i)^3(\lambda_j - \lambda_i) \\
&\quad + 3(96\beta^2 + 9\beta^3 + 32 + 236\beta)(\lambda_k - \lambda_i)^2(\lambda_l - \lambda_i)(\lambda_j - \lambda_i) \\
&\quad + 2(36\beta^2 + 3\beta^3 + 192 + 180\beta)(\lambda_k - \lambda_i)(\lambda_j - \lambda_i)^3 \\
&\quad + 2(36\beta^2 + 3\beta^3 + 192 + 180\beta)(\lambda_k - \lambda_i)(\lambda_l - \lambda_i)^3 \\
&\quad + 3(36\beta^2 + 3\beta^3 + 192 + 180\beta)(\lambda_k - \lambda_i)^2(\lambda_j - \lambda_i)^2 \\
&\quad + 3(36\beta^2 + 3\beta^3 + 192 + 180\beta)(\lambda_k - \lambda_i)^2(\lambda_l - \lambda_i)^2, \\
\xi^{(2)}(\lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta) &= 6(96\beta^2 + 9\beta^3 + 32 + 236\beta)(\lambda_k - \lambda_i)(\lambda_l - \lambda_i)(\lambda_j - \lambda_i) \\
&\quad + 2(36\beta^2 + 3\beta^3 + 192 + 180\beta)(\lambda_j - \lambda_i)^3 \\
&\quad + 2(36\beta^2 + 3\beta^3 + 192 + 180\beta)(\lambda_l - \lambda_i)^3 \\
&\quad + 6(36\beta^2 + 3\beta^3 + 192 + 180\beta)(\lambda_k - \lambda_i)(\lambda_j - \lambda_i)^2 \\
&\quad + 6(36\beta^2 + 3\beta^3 + 192 + 180\beta)(\lambda_k - \lambda_i)(\lambda_l - \lambda_i)^2, \\
\xi^{(3)}(\lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta) &= 6(96\beta^2 + 9\beta^3 + 32 + 236\beta)(\lambda_l - \lambda_i)(\lambda_j - \lambda_i) \\
&\quad + 6(36\beta^2 + 3\beta^3 + 192 + 180\beta)(\lambda_j - \lambda_i)^2 \\
&\quad + 6(36\beta^2 + 3\beta^3 + 192 + 180\beta)(\lambda_l - \lambda_i)^2,
\end{aligned}$$

Define

$$\begin{aligned}
\xi_0(\lambda_l; \lambda_j; \lambda_i; \beta) &\stackrel{\text{def}}{=} \xi(\lambda_l; \lambda_l; \lambda_j; \lambda_i; \beta), \\
\xi_1(\lambda_l; \lambda_j; \lambda_i; \beta) &\stackrel{\text{def}}{=} \xi^{(1)}(\lambda_l; \lambda_l; \lambda_j; \lambda_i; \beta), \\
\xi_2(\lambda_l; \lambda_j; \lambda_i; \beta) &\stackrel{\text{def}}{=} \xi^{(2)}(\lambda_l; \lambda_l; \lambda_j; \lambda_i; \beta), \\
\xi_3(\lambda_l; \lambda_j; \lambda_i; \beta) &\stackrel{\text{def}}{=} \xi^{(3)}(\lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta).
\end{aligned}$$

We will prove that the above equations are non-negative for all $\lambda_k \geq \lambda_l \geq \lambda_j \geq \lambda_i$ and $\beta \geq -\frac{2}{5}$ and, hence, deduce that $\xi(\lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta)$ is positive for all $\lambda_k \geq \lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Differentiating $\xi_0(\lambda_l; \lambda_j; \lambda_i; \beta)$ with respect to λ_l we obtain

$$\begin{aligned}
\xi_0(\lambda_l; \lambda_j; \lambda_i; \beta) &= (96\beta^2 + 9\beta^3 + 32 + 236\beta)(\lambda_l - \lambda_i)^2(\lambda_j - \lambda_i)^3 \\
&\quad + 2(96\beta^2 + 9\beta^3 + 32 + 236\beta)(\lambda_l - \lambda_i)^4(\lambda_j - \lambda_i) \\
&\quad + 2(36\beta^2 + 3\beta^3 + 192 + 180\beta)(\lambda_l - \lambda_i)^2(\lambda_j - \lambda_i)^3 \\
&\quad + 2(36\beta^2 + 3\beta^3 + 192 + 180\beta)(\lambda_l - \lambda_i)^3(\lambda_j - \lambda_i)^2 \\
&\quad + 2(36\beta^2 + 3\beta^3 + 192 + 180\beta)(\lambda_l - \lambda_i)^5, \\
\xi_0^{(1)}(\lambda_l; \lambda_j; \lambda_i; \beta) &= 2(96\beta^2 + 9\beta^3 + 32 + 236\beta)(\lambda_l - \lambda_i)(\lambda_j - \lambda_i)^3 \\
&\quad + 8(96\beta^2 + 9\beta^3 + 32 + 236\beta)(\lambda_l - \lambda_i)^3(\lambda_j - \lambda_i) \\
&\quad + 4(36\beta^2 + 3\beta^3 + 192 + 180\beta)(\lambda_l - \lambda_i)(\lambda_j - \lambda_i)^3 \\
&\quad + 6(36\beta^2 + 3\beta^3 + 192 + 180\beta)(\lambda_l - \lambda_i)^2(\lambda_j - \lambda_i)^2 \\
&\quad + 10(36\beta^2 + 3\beta^3 + 192 + 180\beta)(\lambda_l - \lambda_i)^4, \\
\xi_0^{(2)}(\lambda_l; \lambda_j; \lambda_i; \beta) &= 2(96\beta^2 + 9\beta^3 + 32 + 236\beta)(\lambda_j - \lambda_i)^3 \\
&\quad + 24(96\beta^2 + 9\beta^3 + 32 + 236\beta)(\lambda_l - \lambda_i)^2(\lambda_j - \lambda_i) \\
&\quad + 4(36\beta^2 + 3\beta^3 + 192 + 180\beta)(\lambda_j - \lambda_i)^3 \\
&\quad + 12(36\beta^2 + 3\beta^3 + 192 + 180\beta)(\lambda_l - \lambda_i)(\lambda_j - \lambda_i)^2 \\
&\quad + 40(36\beta^2 + 3\beta^3 + 192 + 180\beta)(\lambda_l - \lambda_i)^3, \\
\xi_0^{(3)}(\lambda_l; \lambda_j; \lambda_i; \beta) &= 48(96\beta^2 + 9\beta^3 + 32 + 236\beta)(\lambda_l - \lambda_i)(\lambda_j - \lambda_i) \\
&\quad + 12(36\beta^2 + 3\beta^3 + 192 + 180\beta)(\lambda_j - \lambda_i)^2 \\
&\quad + 120(36\beta^2 + 3\beta^3 + 192 + 180\beta)(\lambda_l - \lambda_i)^2, \\
\xi_0^{(4)}(\lambda_l; \lambda_j; \lambda_i; \beta) &= 48(96\beta^2 + 9\beta^3 + 32 + 236\beta)(\lambda_j - \lambda_i) \\
&\quad + 240(36\beta^2 + 3\beta^3 + 192 + 180\beta)(\lambda_l - \lambda_i), \\
\xi_0^{(5)}(\lambda_l; \lambda_j; \lambda_i; \beta) &= 8640\beta^2 + 720\beta^3 + 46080 + 43200\beta \\
&> 0 \text{ for all } \beta > -1.4242.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_0(\lambda_j; \lambda_j; \lambda_i; \beta) &= 3(\lambda_j - \lambda_i)^5(168\beta^2 + 15\beta^3 + 416 + 596\beta) \geq 0 \text{ for all } \beta \geq -0.9145, \\
\xi_0^{(1)}(\lambda_j; \lambda_j; \lambda_i; \beta) &= 10(\lambda_j - \lambda_i)^4(168\beta^2 + 15\beta^3 + 416 + 596\beta) \geq 0 \text{ for all } \beta \geq -0.9145, \\
\xi_0^{(2)}(\lambda_j; \lambda_j; \lambda_i; \beta) &= 2(\lambda_j - \lambda_i)^3(201\beta^3 + 8108\beta + 5792 + 2256\beta^2) \geq 0 \text{ for all } \beta \geq -0.9393, \\
\xi_0^{(3)}(\lambda_j; \lambda_j; \lambda_i; \beta) &= 12(\lambda_j - \lambda_i)^2(780\beta^2 + 2924\beta + 2240 + 69\beta^3) \geq 0 \text{ for all } \beta \geq -1.0173, \\
\xi_0^{(4)}(\lambda_j; \lambda_j; \lambda_i; \beta) &= 192(69\beta^2 + 6\beta^3 + 248 + 284\beta)(\lambda_j - \lambda_i) \geq 0 \text{ for all } \beta \geq -1.1738
\end{aligned}$$

and, hence, $\xi_0(\lambda_l; \lambda_j; \lambda_i; \beta)$ is non-negative.

Differentiating $\xi_1(\lambda_l; \lambda_j; \lambda_i; \beta)$ with respect to λ_l we obtain

$$\begin{aligned}
\xi_1(\lambda_l; \lambda_j; \lambda_i; \beta) &= (96\beta^2 + 9\beta^3 + 32 + 236\beta)(\lambda_l - \lambda_i)(\lambda_j - \lambda_i)^3 \\
&\quad + 4(96\beta^2 + 9\beta^3 + 32 + 236\beta)(\lambda_l - \lambda_i)^3(\lambda_j - \lambda_i) \\
&\quad + 2(36\beta^2 + 3\beta^3 + 192 + 180\beta)(\lambda_l - \lambda_i)(\lambda_j - \lambda_i)^3 \\
&\quad + 5(36\beta^2 + 3\beta^3 + 192 + 180\beta)(\lambda_l - \lambda_i)^4 \\
&\quad + 3(36\beta^2 + 3\beta^3 + 192 + 180\beta)(\lambda_l - \lambda_i)^2(\lambda_j - \lambda_i)^2, \\
\xi_1^{(1)}(\lambda_l; \lambda_j; \lambda_i; \beta) &= (96\beta^2 + 9\beta^3 + 32 + 236\beta)(\lambda_j - \lambda_i)^3 + 12(96\beta^2 \\
&\quad + 9\beta^3 + 32 + 236\beta)(\lambda_l - \lambda_i)^2(\lambda_j - \lambda_i) \\
&\quad + 2(36\beta^2 + 3\beta^3 + 192 + 180\beta)(\lambda_j - \lambda_i)^3 \\
&\quad + 20(36\beta^2 + 3\beta^3 + 192 + 180\beta)(\lambda_l - \lambda_i)^3 \\
&\quad + 6(36\beta^2 + 3\beta^3 + 192 + 180\beta)(\lambda_l - \lambda_i)(\lambda_j - \lambda_i)^2, \\
\xi_1^{(2)}(\lambda_l; \lambda_j; \lambda_i; \beta) &= 24(96\beta^2 + 9\beta^3 + 32 + 236\beta)(\lambda_l - \lambda_i)(\lambda_j - \lambda_i) \\
&\quad + 60(36\beta^2 + 3\beta^3 + 192 + 180\beta)(\lambda_l - \lambda_i)^2 \\
&\quad + 6(36\beta^2 + 3\beta^3 + 192 + 180\beta)(\lambda_j - \lambda_i)^2, \\
\xi_1^{(3)}(\lambda_l; \lambda_j; \lambda_i; \beta) &= 24(96\beta^2 + 9\beta^3 + 32 + 236\beta)(\lambda_j - \lambda_i) \\
&\quad + 120(36\beta^2 + 3\beta^3 + 192 + 180\beta)(\lambda_l - \lambda_i), \\
\xi_1^{(4)}(\lambda_l; \lambda_j; \lambda_i; \beta) &= 4320\beta^2 + 360\beta^3 + 23040 + 21600\beta \\
&> 0 \text{ for all } \beta > -1.4242.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_1(\lambda_j; \lambda_j; \lambda_i; \beta) &= 5(\lambda_j - \lambda_i)^4(168\beta^2 + 15\beta^3 + 416 + 596\beta) \geq 0 \text{ for all } \beta \geq -0.9145, \\
\xi_1^{(1)}(\lambda_j; \lambda_j; \lambda_i; \beta) &= (\lambda_j - \lambda_i)^3(201\beta^3 + 8108\beta + 5792 + 2256\beta^2) \geq 0 \text{ for all } \beta \geq -0.9393, \\
\xi_1^{(2)}(\lambda_j; \lambda_j; \lambda_i; \beta) &= 6(\lambda_j - \lambda_i)^2(780\beta^2 + 2924\beta + 2240 + 69\beta^3) \geq 0 \text{ for all } \beta \geq -1.0173, \\
\xi_1^{(3)}(\lambda_j; \lambda_j; \lambda_i; \beta) &= 96(69\beta^2 + 6\beta^3 + 248 + 284\beta)(\lambda_j - \lambda_i) \geq 0 \text{ for all } \beta \geq -1.1738,
\end{aligned}$$

and, hence, $\xi_1(\lambda_l; \lambda_j; \lambda_i; \beta)$ is non-negative.

Differentiating $\xi_2(\lambda_l; \lambda_j; \lambda_i; \beta)$ with respect to λ_l we obtain

$$\begin{aligned}
\xi_2(\lambda_l; \lambda_j; \lambda_i; \beta) &= 6(96\beta^2 + 9\beta^3 + 32 + 236\beta)(\lambda_l - \lambda_i)^2(\lambda_j - \lambda_i) \\
&\quad + 2(36\beta^2 + 3\beta^3 + 192 + 180\beta)(\lambda_j - \lambda_i)^3 \\
&\quad + 8(36\beta^2 + 3\beta^3 + 192 + 180\beta)(\lambda_l - \lambda_i)^3 \\
&\quad + 6(36\beta^2 + 3\beta^3 + 192 + 180\beta)(\lambda_l - \lambda_i)(\lambda_j - \lambda_i)^2, \\
\xi_2^{(1)}(\lambda_l; \lambda_j; \lambda_i; \beta) &= 12(96\beta^2 + 9\beta^3 + 32 + 236\beta)(\lambda_l - \lambda_i)(\lambda_j - \lambda_i) \\
&\quad + 24(36\beta^2 + 3\beta^3 + 192 + 180\beta)(\lambda_l - \lambda_i)^2 \\
&\quad + 6(36\beta^2 + 3\beta^3 + 192 + 180\beta)(\lambda_j - \lambda_i)^2, \\
\xi_2^{(2)}(\lambda_l; \lambda_j; \lambda_i; \beta) &= 12(96\beta^2 + 9\beta^3 + 32 + 236\beta)(\lambda_j - \lambda_i) \\
&\quad + 48(36\beta^2 + 3\beta^3 + 192 + 180\beta)(\lambda_l - \lambda_i), \\
\xi_2^{(3)}(\lambda_l; \lambda_j; \lambda_i; \beta) &= 1728\beta^2 + 144\beta^3 + 9216 + 8640\beta \\
&> 0 \text{ for all } \beta > -1.4242.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_2(\lambda_j; \lambda_j; \lambda_i; \beta) &= 6(\lambda_j - \lambda_i)^3(192\beta^2 + 17\beta^3 + 544 + 716\beta) \geq 0 \text{ for all } \beta \geq -1.0079, \\
\xi_2^{(1)}(\lambda_j; \lambda_j; \lambda_i; \beta) &= 6(\lambda_j - \lambda_i)^2(372\beta^2 + 1372\beta + 1024 + 33\beta^3) \geq 0 \text{ for all } \beta \geq -0.9877, \\
\xi_2^{(2)}(\lambda_j; \lambda_j; \lambda_i; \beta) &= 12(240\beta^2 + 21\beta^3 + 800 + 956\beta)(\lambda_j - \lambda_i) \geq 0 \text{ for all } \beta \geq -1.1217
\end{aligned}$$

and, hence, $\xi_2(\lambda_l; \lambda_j; \lambda_i; \beta)$ is non-negative.

Differentiating $\xi_3(\lambda_l; \lambda_j; \lambda_i; \beta)$ with respect to λ_l we obtain

$$\begin{aligned}
\xi_3(\lambda_l; \lambda_j; \lambda_i; \beta) &= 6(96\beta^2 + 9\beta^3 + 32 + 236\beta)(\lambda_l - \lambda_i)(\lambda_j - \lambda_i) \\
&\quad + 6(36\beta^2 + 3\beta^3 + 192 + 180\beta)(\lambda_j - \lambda_i)^2 \\
&\quad + 6(36\beta^2 + 3\beta^3 + 192 + 180\beta)(\lambda_l - \lambda_i)^2, \\
\xi_3^{(1)}(\lambda_l; \lambda_j; \lambda_i; \beta) &= 6(96\beta^2 + 9\beta^3 + 32 + 236\beta)(\lambda_j - \lambda_i) \\
&\quad + 12(36\beta^2 + 3\beta^3 + 192 + 180\beta)(\lambda_l - \lambda_i), \\
\xi_3^{(2)}(\lambda_l; \lambda_j; \lambda_i; \beta) &= 432\beta^2 + 36\beta^3 + 2304 + 2160\beta \\
&> 0 \text{ for all } \beta > -1.4242.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_3(\lambda_j; \lambda_j; \lambda_i; \beta) &= 6(\lambda_j - \lambda_i)^2(168\beta^2 + 15\beta^3 + 416 + 596\beta) \geq 0 \text{ for all } \beta \geq -0.9145, \\
\xi_3^{(1)}(\lambda_j; \lambda_j; \lambda_i; \beta) &= 6(168\beta^2 + 15\beta^3 + 416 + 596\beta)(\lambda_j - \lambda_i) \geq 0 \text{ for all } \beta \geq -0.9145
\end{aligned}$$

and, hence, $\xi_3(\lambda_l; \lambda_j; \lambda_i; \beta)$ is non-negative.

This completes the proof.

□

Lemma B.10 Let

$$\begin{aligned}\xi(\lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta) &= (15\beta^2 + 44\beta + \beta^3 + 24)\hat{s}(0, 0, 3) \\ &\quad + 3(3\beta^3 + 42 + 47\beta + 20\beta^2)\hat{s}(0, 1, 2) \\ &\quad + 3(9\beta^3 + 45\beta^2 - 24 + 46\beta)\hat{s}(1, 1, 1),\end{aligned}$$

where

$$\hat{s}(a, b, c) = \sum_{\substack{(n_1, n_2, n_3) \\ \in \text{perm}(a, b, c)}} (\lambda_j - \lambda_i)^{n_1} (\lambda_k - \lambda_i)^{n_2} (\lambda_l - \lambda_i)^{n_3}.$$

The expression $\xi(\lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta)$ is non-negative for all $\lambda_k \geq \lambda_l \geq \lambda_j \geq \lambda_i$, and $\beta \geq -\frac{2}{5}$.

Proof. Differentiating $\xi(\lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta)$ with respect to λ_k we obtain

$$\begin{aligned}\xi^{(1)}(\lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta) &= 3(15\beta^2 + \beta^3 + 24 + 44\beta)(\lambda_k - \lambda_i)^2 \\ &\quad + 2(60\beta^2 + 9\beta^3 + 126 + 141\beta)(\lambda_j - \lambda_i)^2 \\ &\quad + (60\beta^2 + 9\beta^3 + 126 + 141\beta)(\lambda_l - \lambda_i)^2 \\ &\quad + 2(60\beta^2 + 9\beta^3 + 126 + 141\beta)(\lambda_k - \lambda_i)(\lambda_j - \lambda_i) \\ &\quad + 2(60\beta^2 + 9\beta^3 + 126 + 141\beta)(\lambda_k - \lambda_i)(\lambda_l - \lambda_i) \\ &\quad + (135\beta^2 + 27\beta^3 - 72 + 138\beta)(\lambda_l - \lambda_i)(\lambda_j - \lambda_i), \\ \xi^{(2)}(\lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta) &= 6(15\beta^2 + \beta^3 + 24 + 44\beta)(\lambda_k - \lambda_i) \\ &\quad + 2(60\beta^2 + 9\beta^3 + 126 + 141\beta)(\lambda_j - \lambda_i) \\ &\quad + 2(60\beta^2 + 9\beta^3 + 126 + 141\beta)(\lambda_l - \lambda_i), \\ \xi^{(3)}(\lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta) &= 90\beta^2 + 6\beta^3 + 144 + 264\beta \\ &> 0 \text{ for all } \beta > -0.7085,\end{aligned}$$

Define

$$\begin{aligned}\xi_0(\lambda_l; \lambda_j; \lambda_i; \beta) &\stackrel{\text{def}}{=} \xi(\lambda_l; \lambda_l; \lambda_j; \lambda_i; \beta), \\ \xi_1(\lambda_l; \lambda_j; \lambda_i; \beta) &\stackrel{\text{def}}{=} \xi^{(1)}(\lambda_l; \lambda_l; \lambda_j; \lambda_i; \beta), \\ \xi_2(\lambda_l; \lambda_j; \lambda_i; \beta) &\stackrel{\text{def}}{=} \xi^{(2)}(\lambda_l; \lambda_l; \lambda_j; \lambda_i; \beta).\end{aligned}$$

We will prove that the above equations are non-negative for all $\lambda_k \geq \lambda_l \geq \lambda_j \geq \lambda_i$ and $\beta \geq -\frac{2}{5}$ and, hence, deduce that $\xi(\lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta)$ is positive for all $\lambda_k \geq \lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Differentiating $\xi_0(\lambda_l; \lambda_j; \lambda_i; \beta)$ with respect to λ_l we obtain

$$\begin{aligned}
\xi_0(\lambda_l; \lambda_j; \lambda_i; \beta) &= (15\beta^2 + \beta^3 + 24 + 44\beta)(\lambda_j - \lambda_i)^3 \\
&\quad + 2(15\beta^2 + \beta^3 + 24 + 44\beta)(\lambda_l - \lambda_i)^3 \\
&\quad + 3(60\beta^2 + 9\beta^3 + 126 + 141\beta)(\lambda_l - \lambda_i)(\lambda_j - \lambda_i)^2 \\
&\quad + 2(60\beta^2 + 9\beta^3 + 126 + 141\beta)(\lambda_l - \lambda_i)^3 \\
&\quad + (60\beta^2 + 9\beta^3 + 126 + 141\beta)(\lambda_l - \lambda_i)^2(\lambda_j - \lambda_i) \\
&\quad + (135\beta^2 + 27\beta^3 - 72 + 138\beta)(\lambda_l - \lambda_i)^2(\lambda_j - \lambda_i), \\
\xi_0^{(1)}(\lambda_l; \lambda_j; \lambda_i; \beta) &= 6(15\beta^2 + \beta^3 + 24 + 44\beta)(\lambda_l - \lambda_i)^2 \\
&\quad + 3(60\beta^2 + 9\beta^3 + 126 + 141\beta)(\lambda_j - \lambda_i)^2 \\
&\quad + 6(60\beta^2 + 9\beta^3 + 126 + 141\beta)(\lambda_l - \lambda_i)^2 \\
&\quad + 2(60\beta^2 + 9\beta^3 + 126 + 141\beta)(\lambda_l - \lambda_i)(\lambda_j - \lambda_i) \\
&\quad + 2(135\beta^2 + 27\beta^3 - 72 + 138\beta)(\lambda_l - \lambda_i)(\lambda_j - \lambda_i), \\
\xi_0^{(2)}(\lambda_l; \lambda_j; \lambda_i; \beta) &= 12(15\beta^2 + \beta^3 + 24 + 44\beta)(\lambda_l - \lambda_i) \\
&\quad + 12(60\beta^2 + 9\beta^3 + 126 + 141\beta)(\lambda_l - \lambda_i) \\
&\quad + 2(60\beta^2 + 9\beta^3 + 126 + 141\beta)(\lambda_j - \lambda_i) \\
&\quad + 2(135\beta^2 + 27\beta^3 - 72 + 138\beta)(\lambda_j - \lambda_i), \\
\xi_0^{(3)}(\lambda_l; \lambda_j; \lambda_i; \beta) &= 900\beta^2 + 120\beta^3 + 1800 + 2220\beta \\
&> 0 \text{ for all } \beta > -2.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_0(\lambda_j; \lambda_j; \lambda_i; \beta) &= 12(\lambda_j - \lambda_i)^3(\beta + 3)(7\beta^2 + 24\beta + 21) \geq 0 \text{ for all } \beta \geq -3, \\
\xi_0^{(1)}(\lambda_j; \lambda_j; \lambda_i; \beta) &= 3(\lambda_j - \lambda_i)^2(\beta + 3)(53\beta^2 + 181\beta + 154) \geq 0 \text{ for all } \beta \geq -1.6073, \\
\xi_0^{(2)}(\lambda_j; \lambda_j; \lambda_i; \beta) &= 6(\beta + 3)(32\beta^2 + 119\beta + 106)(\lambda_j - \lambda_i) \geq 0 \text{ for all } \beta \geq -1.4789
\end{aligned}$$

and, hence, $\xi_0(\lambda_l; \lambda_j; \lambda_i; \beta)$ is non-negative.

Differentiating $\xi_1(\lambda_l; \lambda_j; \lambda_i; \beta)$ with respect to λ_l we obtain

$$\begin{aligned}
\xi_1(\lambda_l; \lambda_j; \lambda_i; \beta) &= 3(15\beta^2 + \beta^3 + 24 + 44\beta)(\lambda_l - \lambda_i)^2 \\
&\quad + 2(60\beta^2 + 9\beta^3 + 126 + 141\beta)(\lambda_j - \lambda_i)^2 \\
&\quad + 3(60\beta^2 + 9\beta^3 + 126 + 141\beta)(\lambda_l - \lambda_i)^2 \\
&\quad + 2(60\beta^2 + 9\beta^3 + 126 + 141\beta)(\lambda_l - \lambda_i)(\lambda_j - \lambda_i) \\
&\quad + (135\beta^2 + 27\beta^3 - 72 + 138\beta)(\lambda_l - \lambda_i)(\lambda_j - \lambda_i), \\
\xi_1^{(1)}(\lambda_l; \lambda_j; \lambda_i; \beta) &= 6(15\beta^2 + \beta^3 + 24 + 44\beta)(\lambda_l - \lambda_i) \\
&\quad + 6(60\beta^2 + 9\beta^3 + 126 + 141\beta)(\lambda_l - \lambda_i) \\
&\quad + 2(60\beta^2 + 9\beta^3 + 126 + 141\beta)(\lambda_j - \lambda_i) \\
&\quad + (135\beta^2 + 27\beta^3 - 72 + 138\beta)(\lambda_j - \lambda_i),
\end{aligned}$$

$$\begin{aligned}\xi_1^{(2)}(\lambda_l; \lambda_j; \lambda_i; \beta) &= 450\beta^2 + 60\beta^3 + 900 + 1110\beta \\ &> 0 \text{ for all } \beta > -2.\end{aligned}$$

Now

$$\begin{aligned}\xi_1(\lambda_j; \lambda_j; \lambda_i; \beta) &= 3(\lambda_j - \lambda_i)^2(\beta + 3)(31\beta^2 + 107\beta + 98) \geq 0 \text{ for all } \beta \geq -3, \\ \xi_1^{(1)}(\lambda_j; \lambda_j; \lambda_i; \beta) &= 15(\beta + 3)(\beta + 2)(7\beta + 12)(\lambda_j - \lambda_i) \geq 0 \text{ for all } \beta \geq -1.7143\end{aligned}$$

and, hence, $\xi_1(\lambda_l; \lambda_j; \lambda_i; \beta)$ is non-negative.

Differentiating $\xi_2(\lambda_l; \lambda_j; \lambda_i; \beta)$ with respect to λ_l we obtain

$$\begin{aligned}\xi_2(\lambda_l; \lambda_j; \lambda_i; \beta) &= 6(15\beta^2 + \beta^3 + 24 + 44\beta)(\lambda_l - \lambda_i) \\ &\quad + 2(60\beta^2 + 9\beta^3 + 126 + 141\beta)(\lambda_j - \lambda_i) \\ &\quad + 2(60\beta^2 + 9\beta^3 + 126 + 141\beta)(\lambda_l - \lambda_i), \\ \xi_2^{(1)}(\lambda_l; \lambda_j; \lambda_i; \beta) &= 210\beta^2 + 24\beta^3 + 396 + 546\beta \\ &> 0 \text{ for all } \beta > -1.2120.\end{aligned}$$

Now

$$\xi_2(\lambda_j; \lambda_j; \lambda_i; \beta) = 6(\beta + 3)(7\beta^2 + 34\beta + 36)(\lambda_j - \lambda_i) \geq 0 \text{ for all } \beta \geq -1.5596$$

and, hence, $\xi_2(\lambda_l; \lambda_j; \lambda_i; \beta)$ is non-negative.

This completes the proof. □

Lemma B.11 Let

$$\begin{aligned}\xi(\lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta) &= 30t(0, 4, 4, 4) + (8\beta - 16)t(1, 3, 4, 4) \\ &\quad + (9\beta - 18)t(2, 2, 4, 4) + (3\beta^2 - 18\beta + 24)t(2, 3, 3, 4) \\ &\quad + (2\beta^3 - 24\beta^2 + 88\beta - 96)t(3, 3, 3, 3),\end{aligned}$$

where

$$t(a, b, c, d) = \sum_{\substack{(n_1, n_2, n_3, n_4) \\ \in \text{perm}(a, b, c, d)}} \lambda_i^{n_1} \lambda_j^{n_2} \lambda_l^{n_3} \lambda_k^{n_4}.$$

The expression $\xi(\lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta)$ is positive for all $\lambda_k \geq \lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Proof. Let

$$\hat{s}(a, b, c) = \sum_{\substack{(n_1, n_2, n_3) \\ \in \text{perm}(a, b, c)}} \lambda_i^{n_1} \lambda_j^{n_2} \lambda_l^{n_3}.$$

Differentiating $\xi(\lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta)$ with respect to λ_k we obtain

$$\begin{aligned} \xi^{(1)}(\lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta) &= 120\lambda_k^3 \hat{s}(0, 4, 4) \\ &\quad + 2(18\beta - 36)\lambda_k^3 \hat{s}(2, 2, 4) \\ &\quad + 2(6\beta^2 - 36\beta + 48)\lambda_k^3 \hat{s}(2, 3, 3) \\ &\quad + 3(2\beta^3 - 24\beta^2 + 88\beta - 96)\lambda_k^2 \hat{s}(3, 3, 3) \\ &\quad + 3(3\beta^2 - 18\beta + 24)\lambda_k^2 \hat{s}(2, 3, 4) \\ &\quad + 3(8\beta - 16)\lambda_k^2 \hat{s}(1, 4, 4) \\ &\quad + 2(16\beta - 32)\lambda_k^3 \hat{s}(1, 3, 4) \\ &\quad + (18\beta - 36)\lambda_k \hat{s}(2, 3, 4) \\ &\quad + (6\beta^2 - 36\beta + 48)\lambda_k \hat{s}(3, 3, 4) \\ &\quad + (8\beta - 16)\hat{s}(3, 4, 4), \\ \xi^{(2)}(\lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta) &= 360\lambda_k^2 \hat{s}(0, 4, 4) \\ &\quad + 6(18\beta - 36)\lambda_k^2 \hat{s}(2, 2, 4) \\ &\quad + 6(6\beta^2 - 36\beta + 48)\lambda_k^2 \hat{s}(2, 3, 3) \\ &\quad + 6(16\beta - 32)\lambda_k^2 \hat{s}(1, 3, 4) \\ &\quad + 3(16\beta - 32)\lambda_k \hat{s}(1, 4, 4) \\ &\quad + 3(4\beta^3 - 48\beta^2 + 176\beta - 192)\lambda_k \hat{s}(3, 3, 3) \\ &\quad + 3(6\beta^2 - 36\beta + 48)\lambda_k \hat{s}(2, 3, 4) \\ &\quad + (6\beta^2 - 36\beta + 48)\hat{s}(3, 3, 4) \\ &\quad + (18\beta - 36)\hat{s}(2, 4, 4), \\ \xi^{(3)}(\lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta) &= 720\lambda_k \hat{s}(0, 4, 4) \\ &\quad + 12(18\beta - 36)\lambda_k \hat{s}(2, 2, 4) \\ &\quad + 12(16\beta - 32)\lambda_k \hat{s}(1, 3, 4) \\ &\quad + 12(6\beta^2 - 36\beta + 48)\lambda_k \hat{s}(2, 3, 3) \\ &\quad + (48\beta - 96)\hat{s}(1, 4, 4) \\ &\quad + (12\beta^3 - 144\beta^2 + 528\beta - 576)\hat{s}(3, 3, 3) \\ &\quad + (18\beta^2 - 108\beta + 144)\hat{s}(2, 3, 4), \\ \xi^{(4)}(\lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta) &= 720\hat{s}(0, 4, 4) \\ &\quad + 12(18\beta - 36)\hat{s}(2, 2, 4) \\ &\quad + 12(16\beta - 32)\hat{s}(1, 3, 4) \\ &\quad + 12(6\beta^2 - 36\beta + 48)\hat{s}(2, 3, 3). \end{aligned}$$

Define

$$\begin{aligned}
\xi_0(\lambda_l; \lambda_j; \lambda_i; \beta) &\stackrel{\text{def}}{=} \xi(\lambda_l; \lambda_l; \lambda_j; \lambda_i; \beta)/\lambda_l^4, \\
\xi_1(\lambda_l; \lambda_j; \lambda_i; \beta) &\stackrel{\text{def}}{=} \xi^{(1)}(\lambda_l; \lambda_l; \lambda_j; \lambda_i; \beta)/\lambda_l^3, \\
\xi_2(\lambda_l; \lambda_j; \lambda_i; \beta) &\stackrel{\text{def}}{=} \xi^{(2)}(\lambda_l; \lambda_l; \lambda_j; \lambda_i; \beta)/\lambda_l^2, \\
\xi_3(\lambda_l; \lambda_j; \lambda_i; \beta) &\stackrel{\text{def}}{=} \xi^{(3)}(\lambda_l; \lambda_l; \lambda_j; \lambda_i; \beta)/\lambda_l, \\
\xi_4(\lambda_l; \lambda_j; \lambda_i; \beta) &\stackrel{\text{def}}{=} \xi^{(4)}(\lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta).
\end{aligned}$$

We will prove that the above equations are non-negative for all $\lambda_k \geq \lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$ and, hence, deduce that $\xi(\lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta)$ is positive for all $\lambda_k \geq \lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Differentiating $\xi_0(\lambda_l; \lambda_j; \lambda_i; \beta)$ with respect to λ_l we obtain

$$\begin{aligned}
\xi_0(\lambda_l; \lambda_j; \lambda_i; \beta) &= 30\lambda_j^4\lambda_l^4 + 30\lambda_i^4\lambda_l^4 \\
&\quad + (9\beta - 18)\lambda_i^2\lambda_j^2\lambda_l^4 \\
&\quad + (8\beta - 16)\lambda_l^4(\lambda_i\lambda_j^3 + \lambda_i^3\lambda_j) \\
&\quad + (16\beta - 32)\lambda_l^3(\lambda_i\lambda_j^4 + \lambda_i^4\lambda_j) \\
&\quad + (6\beta^2 - 36\beta + 48)\lambda_l^3(\lambda_i^2\lambda_j^3 + \lambda_i^3\lambda_j^2) \\
&\quad + (3\beta^2 - 12)\lambda_l^2(\lambda_i^2\lambda_j^4 + \lambda_i^4\lambda_j^2) \\
&\quad + (-48 - 18\beta^2 + 52\beta + 2\beta^3)\lambda_l^2\lambda_i^3\lambda_j^3 \\
&\quad + (-20\beta + 6\beta^2 + 16)\lambda_l(\lambda_i^3\lambda_j^4 + \lambda_i^4\lambda_j^3) \\
&\quad + (10 + 25\beta)\lambda_i^4\lambda_j^4, \\
\xi_0^{(1)}(\lambda_l; \lambda_j; \lambda_i; \beta) &= 120\lambda_l^3(\lambda_j^4 + \lambda_i^4) \\
&\quad + (-72 + 36\beta)\lambda_i^2\lambda_j^2\lambda_l^3 \\
&\quad + (-64 + 32\beta)\lambda_l^3(\lambda_i\lambda_j^3 + \lambda_i^3\lambda_j) \\
&\quad + (48\beta - 96)\lambda_l^2(\lambda_i\lambda_j^4 + \lambda_i^4\lambda_j) \\
&\quad + (18\beta^2 - 108\beta + 144)\lambda_l^2(\lambda_i^3\lambda_j^2 + \lambda_i^2\lambda_j^3) \\
&\quad + (6\beta^2 - 24)\lambda_l(\lambda_i^2\lambda_j^4 + \lambda_i^4\lambda_j^2) \\
&\quad + (-36\beta^2 + 4\beta^3 - 96 + 104\beta)\lambda_i^3\lambda_j^3\lambda_l \\
&\quad + (-20\beta + 6\beta^2 + 16)(\lambda_i^3\lambda_j^4 + \lambda_i^4\lambda_j^3), \\
\xi_0^{(2)}(\lambda_l; \lambda_j; \lambda_i; \beta) &= 360\lambda_l^2(\lambda_i^4 + \lambda_j^4) \\
&\quad + (-216 + 108\beta)\lambda_l^2\lambda_i^2\lambda_j^2 \\
&\quad + (-192 + 96\beta)\lambda_l^2(\lambda_i^3\lambda_j + \lambda_i\lambda_j^3) \\
&\quad + (36\beta^2 + 288 - 216\beta)\lambda_l(\lambda_i^2\lambda_j^3 + \lambda_i^3\lambda_j^2) \\
&\quad + (-192 + 96\beta)\lambda_l(\lambda_i^4\lambda_j + \lambda_i\lambda_j^4) \\
&\quad + (6\beta^2 - 24)(\lambda_i^4\lambda_j^2 + \lambda_i^2\lambda_j^4) \\
&\quad + (-36\beta^2 + 4\beta^3 - 96 + 104\beta)\lambda_i^3\lambda_j^3,
\end{aligned}$$

$$\begin{aligned}
\xi_0^{(3)}(\lambda_l; \lambda_j; \lambda_i; \beta) &= 720\lambda_l(\lambda_i^4 + \lambda_j^4) \\
&\quad + (192\beta - 384)\lambda_l(\lambda_i\lambda_j^3 + \lambda_i^3\lambda_j) \\
&\quad + (-432 + 216\beta)\lambda_l\lambda_i^2\lambda_j^2 \\
&\quad + (-192 + 96\beta)(\lambda_i\lambda_j^4 + \lambda_i^4\lambda_j) \\
&\quad + (36\beta^2 + 288 - 216\beta)(\lambda_i^2\lambda_j^3 + \lambda_i^3\lambda_j^2), \\
\xi_0^{(4)}(\lambda_l; \lambda_j; \lambda_i; \beta) &= 720(\lambda_j^4 + \lambda_i^4) + (-432 + 216\beta)\lambda_i^2\lambda_j^2 \\
&\quad + (192\beta - 384)(\lambda_i\lambda_j^3 + \lambda_i^3\lambda_j).
\end{aligned}$$

Define $\xi_{04}(\lambda_j; \lambda_i; \beta) = \xi_0^{(4)}(\lambda_l; \lambda_j; \lambda_i; \beta)$. We wish to show that ξ_{04} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{04} with respect to λ_j gives

$$\begin{aligned}
\xi_{04}(\lambda_j; \lambda_i; \beta) &= 720(\lambda_j^4 + \lambda_i^4) + (-432 + 216\beta)\lambda_i^2\lambda_j^2 \\
&\quad + (192\beta - 384)(\lambda_i\lambda_j^3 + \lambda_i^3\lambda_j), \\
\xi_{04}^{(1)}(\lambda_j; \lambda_i; \beta) &= 2880\lambda_j^3 + 3(-384 + 192\beta)\lambda_i\lambda_j^2 \\
&\quad + 2(-432 + 216\beta)\lambda_i^2\lambda_j + (-384 + 192\beta)\lambda_i^3, \\
\xi_{04}^{(2)}(\lambda_j; \lambda_i; \beta) &= 8640\lambda_j^2 + 6(-384 + 192\beta)\lambda_i\lambda_j \\
&\quad + 2(-432 + 216\beta)\lambda_i^2, \\
\xi_{04}^{(3)}(\lambda_j; \lambda_i; \beta) &= 17280\lambda_j + 6(-384 + 192\beta)\lambda_i, \\
\xi_{04}^{(4)}(\lambda_j; \lambda_i; \beta) &= 17280 > 0.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{04}(\lambda_i; \lambda_i; \beta) &= 120\lambda_i^4(2 + 5\beta) \geq 0 \text{ for all } \beta \geq -\frac{2}{5}, \\
\xi_{04}^{(1)}(\lambda_i; \lambda_i; \beta) &= 240\lambda_i^3(2 + 5\beta) \geq 0 \text{ for all } \beta \geq -\frac{2}{5}, \\
\xi_{04}^{(2)}(\lambda_i; \lambda_i; \beta) &= 144\lambda_i^2(38 + 11\beta) \geq 0 \text{ for all } \beta \geq -3.4545, \\
\xi_{04}^{(3)}(\lambda_i; \lambda_i; \beta) &= 1152\lambda_i(13 + \beta) \geq 0 \text{ for all } \beta \geq -13.
\end{aligned}$$

Hence, $\xi_0^{(4)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{03}(\lambda_j; \lambda_i; \beta) = \xi_0^{(3)}(\lambda_j; \lambda_j; \lambda_i; \beta)/\lambda_j$. We wish to show that ξ_{03} is non-negative

for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{03} with respect to λ_j gives

$$\begin{aligned}\xi_{03}(\lambda_j; \lambda_i; \beta) &= 720\lambda_j^4 + (-576 + 288\beta)\lambda_i\lambda_j^3 \\ &\quad + (36\beta^2 - 144)\lambda_i^2\lambda_j^2 + (36\beta^2 - 96 - 24\beta)\lambda_i^3\lambda_j \\ &\quad + (528 + 96\beta)\lambda_i^4, \\ \xi_{03}^{(1)}(\lambda_j; \lambda_i; \beta) &= 2880\lambda_j^3 + 3(-576 + 288\beta)\lambda_i\lambda_j^2 \\ &\quad + 2(36\beta^2 - 144)\lambda_i^2\lambda_j + (36\beta^2 - 96 - 24\beta)\lambda_i^3, \\ \xi_{03}^{(2)}(\lambda_j; \lambda_i; \beta) &= 8640\lambda_j^2 + 6(-576 + 288\beta)\lambda_i\lambda_j \\ &\quad + 2(36\beta^2 - 144)\lambda_i^2, \\ \xi_{03}^{(3)}(\lambda_j; \lambda_i; \beta) &= 17280\lambda_j + 6(-576 + 288\beta)\lambda_i, \\ \xi_{03}^{(4)}(\lambda_j; \lambda_i; \beta) &= 17280 > 0.\end{aligned}$$

Now

$$\begin{aligned}\xi_{03}(\lambda_i; \lambda_i; \beta) &= 72\lambda_i^4(\beta + 3)(\beta + 2) \geq 0 \text{ for all } \beta \geq -2, \\ \xi_{03}^{(1)}(\lambda_i; \lambda_i; \beta) &= 12\lambda_i^3(64 + 70\beta + 9\beta^2) \geq 0 \text{ for all } \beta \geq -1.0583, \\ \xi_{03}^{(2)}(\lambda_i; \lambda_i; \beta) &= 72\lambda_i^2(68 + 24\beta + \beta^2) \geq 0 \text{ for all } \beta \geq -3.2822, \\ \xi_{03}^{(3)}(\lambda_i; \lambda_i; \beta) &= 1728\lambda_i(8 + \beta) \geq 0 \text{ for all } \beta \geq -8.\end{aligned}$$

Hence, $\xi_0^{(3)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{02}(\lambda_j; \lambda_i; \beta) = \xi_0^{(2)}(\lambda_j; \lambda_j; \lambda_i; \beta)/\lambda_j^2$. We wish to show that ξ_{02} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{02} with respect to λ_j gives

$$\begin{aligned}\xi_{02}(\lambda_j; \lambda_i; \beta) &= 360\lambda_j^4 + (-384 + 192\beta)\lambda_i\lambda_j^3 \\ &\quad + (42\beta^2 - 108\beta + 48)\lambda_i^2\lambda_j^2 + (-16\beta + 4\beta^3)\lambda_i^3\lambda_j \\ &\quad + (144 + 6\beta^2 + 96\beta)\lambda_i^4, \\ \xi_{02}^{(1)}(\lambda_j; \lambda_i; \beta) &= 1440\lambda_j^3 + 3(-384 + 192\beta)\lambda_i\lambda_j^2 \\ &\quad + 2(42\beta^2 - 108\beta + 48)\lambda_i^2\lambda_j + (-16\beta + 4\beta^3)\lambda_i^3, \\ \xi_{02}^{(2)}(\lambda_j; \lambda_i; \beta) &= 4320\lambda_j^2 + 6(-384 + 192\beta)\lambda_i\lambda_j \\ &\quad + 2(42\beta^2 - 108\beta + 48)\lambda_i^2, \\ \xi_{02}^{(3)}(\lambda_j; \lambda_i; \beta) &= 8640\lambda_j + 6(-384 + 192\beta)\lambda_i, \\ \xi_{02}^{(4)}(\lambda_j; \lambda_i; \beta) &= 8640 > 0.\end{aligned}$$

Now

$$\begin{aligned}\xi_{02}(\lambda_i; \lambda_i; \beta) &= 4\lambda_i^4(\beta + 7)(\beta + 3)(\beta + 2) \geq 0 \text{ for all } \beta \geq -2, \\ \xi_{02}^{(1)}(\lambda_i; \lambda_i; \beta) &= 4\lambda_i^3(\beta + 3)(\beta + 2)(\beta + 16) \geq 0 \text{ for all } \beta \geq -2, \\ \xi_{02}^{(2)}(\lambda_i; \lambda_i; \beta) &= 12\lambda_i^2(8 + \beta)(7\beta + 22) \geq 0 \text{ for all } \beta \geq -\frac{22}{7}, \\ \xi_{02}^{(3)}(\lambda_i; \lambda_i; \beta) &= 576\lambda_i(11 + 2\beta) \geq 0 \text{ for all } \beta \geq -\frac{11}{2}.\end{aligned}$$

Hence, $\xi_0^{(2)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{01}(\lambda_j; \lambda_i; \beta) = \xi_0^{(1)}(\lambda_j; \lambda_j; \lambda_i; \beta)/\lambda_j^3$. We wish to show that ξ_{01} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{01} with respect to λ_j gives

$$\begin{aligned}\xi_{01}(\lambda_j; \lambda_i; \beta) &= 120\lambda_j^4 + (-160 + 80\beta)\lambda_i\lambda_j^3 \\ &\quad + (-72\beta + 48 + 24\beta^2)\lambda_i^2\lambda_j^2 + (8\beta + 4\beta^3 - 12\beta^2)\lambda_i^3\lambda_j \\ &\quad + (28\beta + 16 + 12\beta^2)\lambda_i^4, \\ \xi_{01}^{(1)}(\lambda_j; \lambda_i; \beta) &= 480\lambda_j^3 + 3(-160 + 80\beta)\lambda_i\lambda_j^2 \\ &\quad + 2(-72\beta + 48 + 24\beta^2)\lambda_i^2\lambda_j + (8\beta + 4\beta^3 - 12\beta^2)\lambda_i^3, \\ \xi_{01}^{(2)}(\lambda_j; \lambda_i; \beta) &= 1440\lambda_j^2 + 6(-160 + 80\beta)\lambda_i\lambda_j \\ &\quad + 2(-72\beta + 48 + 24\beta^2)\lambda_i^2, \\ \xi_{01}^{(3)}(\lambda_j; \lambda_i; \beta) &= 2880\lambda_j + 6(-160 + 80\beta)\lambda_i, \\ \xi_{01}^{(4)}(\lambda_j; \lambda_i; \beta) &= 2880 > 0.\end{aligned}$$

Now

$$\begin{aligned}\xi_{01}(\lambda_i; \lambda_i; \beta) &= 4\lambda_i^4(\beta + 3)(\beta + 2)(\beta + 1) \geq 0 \text{ for all } \beta \geq -1, \\ \xi_{01}^{(1)}(\lambda_i; \lambda_i; \beta) &= 4\lambda_i^3(\beta + 4)(\beta + 3)(\beta + 2) \geq 0 \text{ for all } \beta \geq -2, \\ \xi_{01}^{(2)}(\lambda_i; \lambda_i; \beta) &= 48\lambda_i^2(\beta + 4)(\beta + 3) \geq 0 \text{ for all } \beta \geq -3, \\ \xi_{01}^{(3)}(\lambda_i; \lambda_i; \beta) &= 480\lambda_i(\beta + 4) \geq 0 \text{ for all } \beta \geq -4.\end{aligned}$$

Hence, $\xi_0^{(1)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{00}(\lambda_j; \lambda_i; \beta) = \xi_0(\lambda_j; \lambda_j; \lambda_i; \beta)/\lambda_j^4$. We wish to show that ξ_{00} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{00} with respect to λ_j gives

$$\begin{aligned}\xi_{00}(\lambda_j; \lambda_i; \beta) &= 30\lambda_j^4 + (24\beta - 48)\lambda_i\lambda_j^3 \\ &\quad + (-27\beta + 9\beta^2 + 18)\lambda_i^2\lambda_j^2 + (-6\beta^2 + 2\beta^3 + 4\beta)\lambda_i^3\lambda_j \\ &\quad + (9\beta^2 + 12 + 21\beta)\lambda_i^4, \\ \xi_{00}^{(1)}(\lambda_j; \lambda_i; \beta) &= 120\lambda_j^3 + 3(24\beta - 48)\lambda_i\lambda_j^2 \\ &\quad + 2(-27\beta + 9\beta^2 + 18)\lambda_i^2\lambda_j + (-6\beta^2 + 2\beta^3 + 4\beta)\lambda_i^3, \\ \xi_{00}^{(2)}(\lambda_j; \lambda_i; \beta) &= 360\lambda_j^2 + 6(24\beta - 48)\lambda_i\lambda_j \\ &\quad + 2(-27\beta + 9\beta^2 + 18)\lambda_i^2, \\ \xi_{00}^{(3)}(\lambda_j; \lambda_i; \beta) &= 720\lambda_j + 6(24\beta - 48)\lambda_i, \\ \xi_{00}^{(4)}(\lambda_j; \lambda_i; \beta) &= 720 > 0.\end{aligned}$$

Now

$$\begin{aligned}\xi_{00}(\lambda_i; \lambda_i; \beta) &= 2\lambda_i^4(\beta + 3)(\beta + 2)(\beta + 1) > 0 \text{ for all } \beta > -1, \\ \xi_{00}^{(1)}(\lambda_i; \lambda_i; \beta) &= 2\lambda_i^3(\beta + 3)(\beta + 2)(\beta + 1) > 0 \text{ for all } \beta > -1, \\ \xi_{00}^{(2)}(\lambda_i; \lambda_i; \beta) &= 18\lambda_i^2(\beta + 3)(\beta + 2) > 0 \text{ for all } \beta > -2, \\ \xi_{00}^{(3)}(\lambda_i; \lambda_i; \beta) &= 144\lambda_i(\beta + 3) > 0 \text{ for all } \beta > -3.\end{aligned}$$

Hence, ξ_0 is positive for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$. This implies that $\xi(\lambda_l; \lambda_l; \lambda_j; \lambda_i; \beta)$ is positive for all $\lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Differentiating $\xi_1(\lambda_l; \lambda_j; \lambda_i; \beta)$ with respect to λ_l we obtain

$$\begin{aligned}
\xi_1(\lambda_l; \lambda_j; \lambda_i; \beta) &= 120\lambda_l^4(\lambda_j^4 + \lambda_i^4) + (-72 + 36\beta)\lambda_l^4\lambda_i^2\lambda_j^2 \\
&\quad + (32\beta - 64)\lambda_l^4(\lambda_i\lambda_j^3 + \lambda_i^3\lambda_j) + (-112 + 56\beta)\lambda_l^3(\lambda_i\lambda_j^4 + \lambda_i^4\lambda_j) \\
&\quad + (21\beta^2 - 126\beta + 168)\lambda_l^3(\lambda_i^2\lambda_j^3 + \lambda_i^3\lambda_j^2) \\
&\quad + (-144 + 156\beta + 6\beta^3 - 54\beta^2)\lambda_i^3\lambda_j^3\lambda_l^2 \\
&\quad + (-36 + 9\beta^2)\lambda_l^2(\lambda_i^2\lambda_j^4 + \lambda_i^4\lambda_j^2) \\
&\quad + (15\beta^2 + 40 - 50\beta)\lambda_l(\lambda_i^3\lambda_j^4 + \lambda_i^4\lambda_j^3) \\
&\quad + (50\beta + 20)\lambda_l^4\lambda_j^4, \\
\xi_1^{(1)}(\lambda_l; \lambda_j; \lambda_i; \beta) &= 2(-36 + 9\beta^2)\lambda_l\lambda_i^2\lambda_j^4 + (15\beta^2 + 40 - 50\beta)\lambda_i^3\lambda_j^4 \\
&\quad + 3(-112 + 56\beta)\lambda_l^2\lambda_i + 480\lambda_l^3\lambda_j^4 + (15\beta^2 + 40 - 50\beta)\lambda_l^4\lambda_j^3 \\
&\quad + 4(32\beta - 64)\lambda_l^3\lambda_i\lambda_j^3 + 2(-144 + 156\beta + 6\beta^3 - 54\beta^2)\lambda_l\lambda_i^3\lambda_j^3 \\
&\quad + 3(21\beta^2 - 126\beta + 168)\lambda_l^2\lambda_i^2\lambda_j^3 + 2(-36 + 9\beta^2)\lambda_l\lambda_i^4\lambda_j^2 \\
&\quad + 3(21\beta^2 - 126\beta + 168)\lambda_l^2\lambda_i^3\lambda_j^2 + 4(-72 + 36\beta)\lambda_l^3\lambda_i^2\lambda_j^2 \\
&\quad + 3(-112 + 56\beta)\lambda_l^2\lambda_i^4\lambda_j + 4(32\beta - 64)\lambda_l^3\lambda_i^3\lambda_j \\
&\quad + 480\lambda_l^4\lambda_i^3, \\
\xi_1^{(2)}(\lambda_l; \lambda_j; \lambda_i; \beta) &= 2(-36 + 9\beta^2)\lambda_i^2\lambda_j^4 + 6(-112 + 56\beta)\lambda_l\lambda_i\lambda_j^4 \\
&\quad + 1440\lambda_l^2\lambda_j^4 + 12(32\beta - 64)\lambda_l^2\lambda_i\lambda_j^3 + 2(-144 + 156\beta + 6\beta^3 - 54\beta^2)\lambda_i^3\lambda_j^3 \\
&\quad + 6(21\beta^2 - 126\beta + 168)\lambda_l\lambda_i^2\lambda_j^3 + 2(-36 + 9\beta^2)\lambda_i^4\lambda_j^2 \\
&\quad + 6(21\beta^2 - 126\beta + 168)\lambda_l\lambda_i^3\lambda_j^2 + 12(-72 + 36\beta)\lambda_l^2\lambda_i^2\lambda_j^2 \\
&\quad + 6(-112 + 56\beta)\lambda_l\lambda_i^4\lambda_j + 12(32\beta - 64)\lambda_l^2\lambda_i^3\lambda_j \\
&\quad + 1440\lambda_i^4\lambda_l^2, \\
\xi_1^{(3)}(\lambda_l; \lambda_j; \lambda_i; \beta) &= 6(-112 + 56\beta)\lambda_i\lambda_j^4 + 2880\lambda_l\lambda_j^4 \\
&\quad + 24(32\beta - 64)\lambda_l\lambda_i\lambda_j^3 + 6(21\beta^2 - 126\beta + 168)\lambda_i^2\lambda_j^3 \\
&\quad + 6(21\beta^2 - 126\beta + 168)\lambda_i^3\lambda_j^2 + 24(-72 + 36\beta)\lambda_l\lambda_i^2\lambda_j^2 \\
&\quad + 6(-112 + 56\beta)\lambda_i^4\lambda_j + 24(32\beta - 64)\lambda_l\lambda_i^3\lambda_j \\
&\quad + 2880\lambda_i^4\lambda_l, \\
\xi_1^{(4)}(\lambda_l; \lambda_j; \lambda_i; \beta) &= 2880\lambda_j^4 + 24(32\beta - 64)\lambda_i\lambda_j^3 \\
&\quad + 24(-72 + 36\beta)\lambda_i^2\lambda_j^2 + 24(32\beta - 64)\lambda_i^3\lambda_j \\
&\quad + 2880\lambda_i^4.
\end{aligned}$$

Define $\xi_{14}(\lambda_j; \lambda_i; \beta) = \xi_1^{(4)}(\lambda_l; \lambda_j; \lambda_i; \beta)$. We wish to show that ξ_{14} is non-negative for all

$\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{14} with respect to λ_j gives

$$\begin{aligned}\xi_{14}(\lambda_j; \lambda_i; \beta) &= 2880\lambda_j^4 + 24(32\beta - 64)\lambda_i\lambda_j^3 \\ &\quad + 24(-72 + 36\beta)\lambda_i^2\lambda_j^2 + 24(32\beta - 64)\lambda_i^3\lambda_j \\ &\quad + 2880\lambda_i^4, \\ \xi_{14}^{(1)}(\lambda_j; \lambda_i; \beta) &= 11520\lambda_j^3 + 72(32\beta - 64)\lambda_i\lambda_j^2 \\ &\quad + 48(-72 + 36\beta)\lambda_i^2\lambda_j + 24(32\beta - 64)\lambda_i^3, \\ \xi_{14}^{(2)}(\lambda_j; \lambda_i; \beta) &= 34560\lambda_j^2 + 144(32\beta - 64)\lambda_i\lambda_j \\ &\quad + 48(-72 + 36\beta)\lambda_i^2, \\ \xi_{14}^{(3)}(\lambda_j; \lambda_i; \beta) &= 69120\lambda_j + 144(32\beta - 64)\lambda_i, \\ \xi_{14}^{(4)}(\lambda_j; \lambda_i; \beta) &= 69120 > 0.\end{aligned}$$

Now

$$\begin{aligned}\xi_{14}(\lambda_i; \lambda_i; \beta) &= 480\lambda_i^4(2 + 5\beta) \geq 0 \text{ for all } \beta \geq -\frac{2}{5}, \\ \xi_{14}^{(1)}(\lambda_i; \lambda_i; \beta) &= 960\lambda_i^3(2 + 5\beta) \geq 0 \text{ for all } \beta \geq -\frac{2}{5}, \\ \xi_{14}^{(2)}(\lambda_i; \lambda_i; \beta) &= 576\lambda_i^2(38 + 11\beta) \geq 0 \text{ for all } \beta \geq -3.4545, \\ \xi_{14}^{(3)}(\lambda_i; \lambda_i; \beta) &= 4608\lambda_i(13 + \beta) \geq 0 \text{ for all } \beta \geq -13.\end{aligned}$$

Hence, $\xi_1^{(4)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{13}(\lambda_j; \lambda_i; \beta) = \xi_1^{(3)}(\lambda_j; \lambda_j; \lambda_i; \beta)/\lambda_j$. We wish to show that ξ_{13} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{13} with respect to λ_j gives

$$\begin{aligned}\xi_{13}(\lambda_j; \lambda_i; \beta) &= (2208 + 336\beta)\lambda_i^4 + (-528 + 12\beta + 126\beta^2)\lambda_j\lambda_i^3 \\ &\quad + (108\beta - 720 + 126\beta^2)\lambda_j^2\lambda_i^2 + (-2208 + 1104\beta)\lambda_j^3\lambda_i \\ &\quad + 2880\lambda_j^4, \\ \xi_{13}^{(1)}(\lambda_j; \lambda_i; \beta) &= (-528 + 12\beta + 126\beta^2)\lambda_i^3 + 2(108\beta - 720 + 126\beta^2)\lambda_j\lambda_i^2 \\ &\quad + 3(-2208 + 1104\beta)\lambda_j^2\lambda_i + 11520\lambda_j^3, \\ \xi_{13}^{(2)}(\lambda_j; \lambda_i; \beta) &= 2(108\beta - 720 + 126\beta^2)\lambda_i^2 + 6(-2208 + 1104\beta)\lambda_j\lambda_i \\ &\quad + 34560\lambda_j^2, \\ \xi_{13}^{(3)}(\lambda_j; \lambda_i; \beta) &= 6(-2208 + 1104\beta)\lambda_i + 69120\lambda_j, \\ \xi_{13}^{(4)}(\lambda_j; \lambda_i; \beta) &= 69120 > 0.\end{aligned}$$

Now

$$\begin{aligned}\xi_{13}(\lambda_i; \lambda_i; \beta) &= 12\lambda_i^4(7\beta + 34)(3\beta + 4) \geq 0 \text{ for all } \beta \geq -\frac{4}{3}, \\ \xi_{13}^{(1)}(\lambda_i; \lambda_i; \beta) &= 6\lambda_i^3(488 + 590\beta + 63\beta^2) \geq 0 \text{ for all } \beta \geq -0.9169, \\ \xi_{13}^{(2)}(\lambda_i; \lambda_i; \beta) &= 36\lambda_i^2(190\beta + 552 + 7\beta^2) \geq 0 \text{ for all } \beta \geq -3.3086, \\ \xi_{13}^{(3)}(\lambda_i; \lambda_i; \beta) &= 288\lambda_i(194 + 23\beta) \geq 0 \text{ for all } \beta \geq -8.4348.\end{aligned}$$

Hence, $\xi_1^{(3)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{12}(\lambda_j; \lambda_i; \beta) = \xi_1^{(2)}(\lambda_j; \lambda_j; \lambda_i; \beta)/\lambda_j^2$. We wish to show that ξ_{12} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{12} with respect to λ_j gives

$$\begin{aligned}\xi_{12}(\lambda_j; \lambda_i; \beta) &= 1440\lambda_j^4 + (720\beta - 1440)\lambda_i\lambda_j^3 \\ &\quad + (72 - 324\beta + 144\beta^2)\lambda_i^2\lambda_j^2 + (12\beta^3 - 48 + 18\beta^2 - 60\beta)\lambda_i^3\lambda_j \\ &\quad + (696 + 336\beta + 18\beta^2)\lambda_i^4, \\ \xi_{12}^{(1)}(\lambda_j; \lambda_i; \beta) &= 5760\lambda_j^3 + 3(720\beta - 1440)\lambda_i\lambda_j^2 \\ &\quad + 2(72 - 324\beta + 144\beta^2)\lambda_i^2\lambda_j + (12\beta^3 - 48 + 18\beta^2 - 60\beta)\lambda_i^3, \\ \xi_{12}^{(2)}(\lambda_j; \lambda_i; \beta) &= 17280\lambda_j^2 + 6(720\beta - 1440)\lambda_i\lambda_j \\ &\quad + 2(72 - 324\beta + 144\beta^2)\lambda_i^2, \\ \xi_{12}^{(3)}(\lambda_j; \lambda_i; \beta) &= 34560\lambda_j + 6(720\beta - 1440)\lambda_i, \\ \xi_{12}^{(4)}(\lambda_j; \lambda_i; \beta) &= 34560 > 0.\end{aligned}$$

Now

$$\begin{aligned}\xi_{12}(\lambda_i; \lambda_i; \beta) &= 12\lambda_i^4(\beta + 10)(\beta + 3)(\beta + 2) \geq 0 \text{ for all } \beta \geq -2, \\ \xi_{12}^{(1)}(\lambda_i; \lambda_i; \beta) &= 6\lambda_i^3(256 + 242\beta + 51\beta^2 + 2\beta^3) \geq 0 \text{ for all } \beta \geq -1.5098, \\ \xi_{12}^{(2)}(\lambda_i; \lambda_i; \beta) &= 72\lambda_i^2(122 + 51\beta + 4\beta^2) \geq 0 \text{ for all } \beta \geq -3.1906, \\ \xi_{12}^{(3)}(\lambda_i; \lambda_i; \beta) &= 4320\lambda_i(6 + \beta) \geq 0 \text{ for all } \beta \geq -6.\end{aligned}$$

Hence, $\xi_1^{(2)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{11}(\lambda_j; \lambda_i; \beta) = \xi_1^{(1)}(\lambda_j; \lambda_j; \lambda_i; \beta)/\lambda_j^3$. We wish to show that ξ_{11} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{11} with respect to λ_j gives

$$\begin{aligned}\xi_{11}(\lambda_j; \lambda_i; \beta) &= 480\lambda_j^4 + (296\beta - 592)\lambda_i\lambda_j^3 \\ &\quad + (144 - 234\beta + 81\beta^2)\lambda_i^2\lambda_j^2 + (12\beta - 30\beta^2 + 12\beta^3)\lambda_i^3\lambda_j \\ &\quad + (112 + 118\beta + 33\beta^2)\lambda_i^4, \\ \xi_{11}^{(1)}(\lambda_j; \lambda_i; \beta) &= 1920\lambda_j^3 + 3(296\beta - 592)\lambda_i\lambda_j^2 + 2(144 - 234\beta + 81\beta^2)\lambda_i^2\lambda_j \\ &\quad + (12\beta - 30\beta^2 + 12\beta^3)\lambda_i^3, \\ \xi_{11}^{(2)}(\lambda_j; \lambda_i; \beta) &= 5760\lambda_j^2 + 6(296\beta - 592)\lambda_i\lambda_j + 2(144 - 234\beta + 81\beta^2)\lambda_i^2, \\ \xi_{11}^{(3)}(\lambda_j; \lambda_i; \beta) &= 11520\lambda_j + 6(296\beta - 592)\lambda_i, \\ \xi_{11}^{(4)}(\lambda_j; \lambda_i; \beta) &= 11520 > 0.\end{aligned}$$

Now

$$\begin{aligned}\xi_{11}(\lambda_i; \lambda_i; \beta) &= 12\lambda_i^4(\beta + 3)(\beta + 2)^2 \geq 0 \text{ for all } \beta \geq -2, \\ \xi_{11}^{(1)}(\lambda_i; \lambda_i; \beta) &= 12\lambda_i^3(\beta + 3)(\beta + 2)(6 + \beta) \geq 0 \text{ for all } \beta \geq -2, \\ \xi_{11}^{(2)}(\lambda_i; \lambda_i; \beta) &= 6\lambda_i^2(416 + 218\beta + 27\beta^2) \geq 0 \text{ for all } \beta \geq -3.0935, \\ \xi_{11}^{(3)}(\lambda_i; \lambda_i; \beta) &= 48\lambda_i(166 + 37\beta) \geq 0 \text{ for all } \beta \geq -\frac{166}{37}.\end{aligned}$$

Hence, $\xi_1^{(1)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{10}(\lambda_j; \lambda_i; \beta) = \xi_1(\lambda_j; \lambda_j; \lambda_i; \beta)/\lambda_j^4$. We wish to show that ξ_{10} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{10} with respect to λ_j gives

$$\begin{aligned}\xi_{10}(\lambda_j; \lambda_i; \beta) &= 120\lambda_j^4 + (88\beta - 176)\lambda_i\lambda_j^3 \\ &\quad + (60 - 90\beta + 30\beta^2)\lambda_i^2\lambda_j^2 + (12\beta - 18\beta^2 + 6\beta^3)\lambda_i^3\lambda_j \\ &\quad + (32 + 56\beta + 24\beta^2)\lambda_i^4, \\ \xi_{10}^{(1)}(\lambda_j; \lambda_i; \beta) &= 480\lambda_j^3 + 3(88\beta - 176)\lambda_i\lambda_j^2 + 2(60 - 90\beta + 30\beta^2)\lambda_i^2\lambda_j \\ &\quad + (12\beta - 18\beta^2 + 6\beta^3)\lambda_i^3, \\ \xi_{10}^{(2)}(\lambda_j; \lambda_i; \beta) &= 1440\lambda_j^2 + 6(88\beta - 176)\lambda_i\lambda_j \\ &\quad + 2(60 - 90\beta + 30\beta^2)\lambda_i^2, \\ \xi_{10}^{(3)}(\lambda_j; \lambda_i; \beta) &= 2880\lambda_j + 6(88\beta - 176)\lambda_i, \\ \xi_{10}^{(4)}(\lambda_j; \lambda_i; \beta) &= 2880 > 0.\end{aligned}$$

Now

$$\begin{aligned}\xi_{10}(\lambda_i; \lambda_i; \beta) &= 6\lambda_i^4(\beta + 3)(\beta + 2)(\beta + 1) > 0 \text{ for all } \beta > -1, \\ \xi_{10}^{(1)}(\lambda_i; \lambda_i; \beta) &= 6\lambda_i^3(\beta + 3)(\beta + 2)^2 > 0 \text{ for all } \beta > -2, \\ \xi_{10}^{(2)}(\lambda_i; \lambda_i; \beta) &= 12\lambda_i^2(\beta + 3)(5\beta + 14) > 0 \text{ for all } \beta > -\frac{14}{5}, \\ \xi_{10}^{(3)}(\lambda_i; \lambda_i; \beta) &= 48\lambda_i(38 + 11\beta) > 0 \text{ for all } \beta > -\frac{38}{11}.\end{aligned}$$

Hence, ξ_1 is positive for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$. This implies that $\xi(\lambda_l; \lambda_l; \lambda_j; \lambda_i; \beta)$ is positive for all $\lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Differentiating $\xi_2(\lambda_l; \lambda_j; \lambda_i; \beta)$ with respect to λ_l we obtain

$$\begin{aligned}\xi_2(\lambda_l; \lambda_j; \lambda_i; \beta) &= 360\lambda_j^4\lambda_l^4 + (-216 + 108\beta)\lambda_i^2\lambda_j^2\lambda_l^4 \\ &\quad + (96\beta - 192)\lambda_i\lambda_j^3\lambda_l^4 + (96\beta - 192)\lambda_i^3\lambda_j\lambda_l^4 \\ &\quad + 360\lambda_i^4\lambda_l^4 + (-288 + 144\beta)\lambda_i\lambda_j^4\lambda_l^3 \\ &\quad + (-288 + 144\beta)\lambda_i^4\lambda_j\lambda_l^3 + (-324\beta + 54\beta^2 + 432)\lambda_i^2\lambda_j^3\lambda_l^3 \\ &\quad + (-324\beta + 54\beta^2 + 432)\lambda_i^3\lambda_j^2\lambda_l^3 + (18\beta^2 + 18\beta - 108)\lambda_i^2\lambda_j^4\lambda_l^2 \\ &\quad + (-102\beta^2 - 240 + 276\beta + 12\beta^3)\lambda_i^3\lambda_j^3\lambda_l^2 + (18\beta^2 + 18\beta - 108)\lambda_i^4\lambda_j^2\lambda_l^2 \\ &\quad + (-48\beta + 24\beta^2)\lambda_i^3\lambda_j^4\lambda_l + (-48\beta + 24\beta^2)\lambda_i^4\lambda_j^3\lambda_l \\ &\quad + (228 + 66\beta)\lambda_i^4\lambda_j^4,\end{aligned}$$

$$\begin{aligned}
\xi_2^{(1)}(\lambda_i; \lambda_j; \lambda_i; \beta) &= 4(360\lambda_j^4 + (-216 + 108\beta)\lambda_i^2\lambda_j^2 \\
&\quad + (96\beta - 192)\lambda_i\lambda_j^3 + (96\beta - 192)\lambda_i^3\lambda_j \\
&\quad + 360\lambda_i^4\lambda_j^3 + 3((-288 + 144\beta)\lambda_i\lambda_j^4 \\
&\quad + (-288 + 144\beta)\lambda_i^4\lambda_j + (-324\beta + 54\beta^2 + 432)\lambda_i^2\lambda_j^3 \\
&\quad + (-324\beta + 54\beta^2 + 432)\lambda_i^3\lambda_j^2)\lambda_i^2 + 2((18\beta^2 + 18\beta - 108)\lambda_i^2\lambda_j^4 \\
&\quad + (-102\beta^2 - 240 + 276\beta + 12\beta^3)\lambda_i^3\lambda_j^3 + (18\beta^2 + 18\beta - 108)\lambda_i^4\lambda_j^2)\lambda_i \\
&\quad + (-48\beta + 24\beta^2)\lambda_i^3\lambda_j^4 + (-48\beta + 24\beta^2)\lambda_i^4\lambda_j^3, \\
\xi_2^{(2)}(\lambda_i; \lambda_j; \lambda_i; \beta) &= 4320\lambda_j^4\lambda_i^2 + 12(-216 + 108\beta)\lambda_i^2\lambda_j^2\lambda_i^2 \\
&\quad + 12(96\beta - 192)\lambda_i\lambda_j^3\lambda_i^2 + 12(96\beta - 192)\lambda_i^3\lambda_j\lambda_i^2 \\
&\quad + 4320\lambda_i^4\lambda_j^2 + 6(-288 + 144\beta)\lambda_i\lambda_j^4\lambda_i \\
&\quad + 6(-288 + 144\beta)\lambda_i^4\lambda_j\lambda_i + 6(-324\beta + 54\beta^2 + 432)\lambda_i^2\lambda_j^3\lambda_i \\
&\quad + 6(-324\beta + 54\beta^2 + 432)\lambda_i^3\lambda_j^2\lambda_i + 2(18\beta^2 + 18\beta - 108)\lambda_i^2\lambda_j^4 \\
&\quad + 2(-102\beta^2 - 240 + 276\beta + 12\beta^3)\lambda_i^3\lambda_j^3 + 2(18\beta^2 + 18\beta - 108)\lambda_i^4\lambda_j^2, \\
\xi_2^{(3)}(\lambda_i; \lambda_j; \lambda_i; \beta) &= 8640\lambda_j^4\lambda_i + 24(-216 + 108\beta)\lambda_i^2\lambda_j^2\lambda_i \\
&\quad + 24(96\beta - 192)\lambda_i\lambda_j^3\lambda_i + 24(96\beta - 192)\lambda_i^3\lambda_j\lambda_i \\
&\quad + 8640\lambda_i^4\lambda_j + 6(-288 + 144\beta)\lambda_i\lambda_j^4 \\
&\quad + 6(-288 + 144\beta)\lambda_i^4\lambda_j + 6(-324\beta + 54\beta^2 + 432)\lambda_i^2\lambda_j^3 \\
&\quad + 6(-324\beta + 54\beta^2 + 432)\lambda_i^3\lambda_j^2, \\
\xi_2^{(4)}(\lambda_i; \lambda_j; \lambda_i; \beta) &= 8640\lambda_j^4 + 24(-216 + 108\beta)\lambda_i^2\lambda_j^2 \\
&\quad + 24(96\beta - 192)\lambda_i\lambda_j^3 + 24(96\beta - 192)\lambda_i^3\lambda_j \\
&\quad + 8640\lambda_i^4.
\end{aligned}$$

Define $\xi_{24}(\lambda_j; \lambda_i; \beta) = \xi_2^{(4)}(\lambda_i; \lambda_j; \lambda_i; \beta)$. We wish to show that ξ_{24} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{24} with respect to λ_j gives

$$\begin{aligned}
\xi_{24}(\lambda_j; \lambda_i; \beta) &= 8640\lambda_j^4 + 24(-216 + 108\beta)\lambda_i^2\lambda_j^2 \\
&\quad + 24(96\beta - 192)\lambda_i\lambda_j^3 + 24(96\beta - 192)\lambda_i^3\lambda_j + 8640\lambda_i^4, \\
\xi_{24}^{(1)}(\lambda_j; \lambda_i; \beta) &= 34560\lambda_j^3 + 48(-216 + 108\beta)\lambda_i^2\lambda_j + 72(96\beta - 192)\lambda_i\lambda_j^2 + 24(96\beta - 192)\lambda_i^3, \\
\xi_{24}^{(2)}(\lambda_j; \lambda_i; \beta) &= 103680\lambda_j^2 + 48(-216 + 108\beta)\lambda_i^2 + 144(96\beta - 192)\lambda_i\lambda_j, \\
\xi_{24}^{(3)}(\lambda_j; \lambda_i; \beta) &= 207360\lambda_j + 144(96\beta - 192)\lambda_i, \\
\xi_{24}^{(4)}(\lambda_j; \lambda_i; \beta) &= 207360 > 0.
\end{aligned}$$

Now

$$\begin{aligned}\xi_{24}(\lambda_i; \lambda_i; \beta) &= 1440\lambda_i^4(2 + 5\beta) \geq 0 \text{ for all } \beta \geq -\frac{2}{5}, \\ \xi_{24}^{(1)}(\lambda_i; \lambda_i; \beta) &= 2880\lambda_i^3(2 + 5\beta) \geq 0 \text{ for all } \beta \geq -\frac{2}{5}, \\ \xi_{24}^{(2)}(\lambda_i; \lambda_i; \beta) &= 1728\lambda_i^2(38 + 11\beta) \geq 0 \text{ for all } \beta \geq -3.4545, \\ \xi_{24}^{(3)}(\lambda_i; \lambda_i; \beta) &= 13824\lambda_i(13 + \beta) \geq 0 \text{ for all } \beta \geq -13.\end{aligned}$$

Hence, $\xi_2^{(4)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{23}(\lambda_j; \lambda_i; \beta) = \xi_2^{(3)}(\lambda_j; \lambda_j; \lambda_i; \beta)/\lambda_j$. We wish to show that ξ_{23} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{23} with respect to λ_j gives

$$\begin{aligned}\xi_{23}(\lambda_j; \lambda_i; \beta) &= 8640\lambda_j^4 + (3168\beta - 6336)\lambda_i\lambda_j^3 \\ &\quad + (-2592 + 648\beta + 324\beta^2)\lambda_i^2\lambda_j^2 + (-2016 + 360\beta + 324\beta^2)\lambda_i^3\lambda_j \\ &\quad + (6912 + 864\beta)\lambda_i^4, \\ \xi_{23}^{(1)}(\lambda_j; \lambda_i; \beta) &= 34560\lambda_j^3 + 3(3168\beta - 6336)\lambda_i\lambda_j^2 \\ &\quad + 2(-2592 + 648\beta + 324\beta^2)\lambda_i^2\lambda_j + (-2016 + 360\beta + 324\beta^2)\lambda_i^3, \\ \xi_{23}^{(2)}(\lambda_j; \lambda_i; \beta) &= 103680\lambda_j^2 + 6(3168\beta - 6336)\lambda_i\lambda_j \\ &\quad + 2(-2592 + 648\beta + 324\beta^2)\lambda_i^2, \\ \xi_{23}^{(3)}(\lambda_j; \lambda_i; \beta) &= 207360\lambda_j + 6(3168\beta - 6336)\lambda_i, \\ \xi_{23}^{(4)}(\lambda_j; \lambda_i; \beta) &= 207360 > 0.\end{aligned}$$

Now

$$\begin{aligned}\xi_{23}(\lambda_i; \lambda_i; \beta) &= 72\lambda_i^4(64 + 70\beta + 9\beta^2) \geq 0 \text{ for all } \beta \geq -1.0583, \\ \xi_{23}^{(1)}(\lambda_i; \lambda_i; \beta) &= 36\lambda_i^3(232 + 310\beta + 27\beta^2) \geq 0 \text{ for all } \beta \geq -0.8048, \\ \xi_{23}^{(2)}(\lambda_i; \lambda_i; \beta) &= 216\lambda_i^2(\beta + 28)(3\beta + 10) \geq 0 \text{ for all } \beta \geq -\frac{10}{3}, \\ \xi_{23}^{(3)}(\lambda_i; \lambda_i; \beta) &= 1728\lambda_i(98 + 11\beta) \geq 0 \text{ for all } \beta \geq -\frac{98}{11}.\end{aligned}$$

Hence, $\xi_2^{(3)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{22}(\lambda_j; \lambda_i; \beta) = \xi_2^{(2)}(\lambda_j; \lambda_j; \lambda_i; \beta)/\lambda_j^2$. We wish to show that ξ_{22} is non-negative

for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{22} with respect to λ_j gives

$$\begin{aligned}\xi_{22}(\lambda_j; \lambda_i; \beta) &= 4320\lambda_j^4 + (2016\beta - 4032)\lambda_i\lambda_j^3 \\ &\quad + (-216 - 612\beta + 360\beta^2)\lambda_i^2\lambda_j^2 + (24\beta^3 - 192 + 120\beta^2 - 240\beta)\lambda_i^3\lambda_j \\ &\quad + (2376 + 900\beta + 36\beta^2)\lambda_i^4, \\ \xi_{22}^{(1)}(\lambda_j; \lambda_i; \beta) &= 17280\lambda_j^3 + 3(2016\beta - 4032)\lambda_i\lambda_j^2 \\ &\quad + 2(-216 - 612\beta + 360\beta^2)\lambda_i^2\lambda_j + (24\beta^3 - 192 + 120\beta^2 - 240\beta)\lambda_i^3, \\ \xi_{22}^{(2)}(\lambda_j; \lambda_i; \beta) &= 51840\lambda_j^2 + 6(2016\beta - 4032)\lambda_i\lambda_j \\ &\quad + 2(-216 - 612\beta + 360\beta^2)\lambda_i^2, \\ \xi_{22}^{(3)}(\lambda_j; \lambda_i; \beta) &= 103680\lambda_j + 6(2016\beta - 4032)\lambda_i, \\ \xi_{22}^{(4)}(\lambda_j; \lambda_i; \beta) &= 103680 > 0.\end{aligned}$$

Now

$$\begin{aligned}\xi_{22}(\lambda_i; \lambda_i; \beta) &= 12\lambda_i^4(\beta + 2)(2\beta^2 + 39\beta + 94) \geq 0 \text{ for all } \beta \geq -2, \\ \xi_{22}^{(1)}(\lambda_i; \lambda_i; \beta) &= 24\lambda_i^3(190 + 191\beta + 35\beta^2 + \beta^3) \geq 0 \text{ for all } \beta \geq -1.2872, \\ \xi_{22}^{(2)}(\lambda_i; \lambda_i; \beta) &= 72\lambda_i^2(378 + 151\beta + 10\beta^2) \geq 0 \text{ for all } \beta \geq -3.1679, \\ \xi_{22}^{(3)}(\lambda_i; \lambda_i; \beta) &= 1728\lambda_i(46 + 7\beta) \geq 0 \text{ for all } \beta \geq -\frac{46}{7}.\end{aligned}$$

Hence, $\xi_{22}^{(2)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{21}(\lambda_j; \lambda_i; \beta) = \xi_2^{(1)}(\lambda_j; \lambda_j; \lambda_i; \beta)/\lambda_j^3$. We wish to show that ξ_{21} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{21} with respect to λ_j gives

$$\begin{aligned}\xi_{21}(\lambda_j; \lambda_i; \beta) &= 1440\lambda_j^4 + (816\beta - 1632)\lambda_i\lambda_j^3 \\ &\quad + (216 - 504\beta + 198\beta^2)\lambda_i^2\lambda_j^2 + (24\beta^3 + 48 - 18\beta^2 - 84\beta)\lambda_i^3\lambda_j \\ &\quad + (360 + 420\beta + 60\beta^2)\lambda_i^4, \\ \xi_{21}^{(1)}(\lambda_j; \lambda_i; \beta) &= 5760\lambda_j^3 + 3(816\beta - 1632)\lambda_i\lambda_j^2 \\ &\quad + 2(216 - 504\beta + 198\beta^2)\lambda_i^2\lambda_j + (24\beta^3 + 48 - 18\beta^2 - 84\beta)\lambda_i^3, \\ \xi_{21}^{(2)}(\lambda_j; \lambda_i; \beta) &= 17280\lambda_j^2 + 6(816\beta - 1632)\lambda_i\lambda_j \\ &\quad + 2(216 - 504\beta + 198\beta^2)\lambda_i^2, \\ \xi_{21}^{(3)}(\lambda_j; \lambda_i; \beta) &= 34560\lambda_j + 6(816\beta - 1632)\lambda_i, \\ \xi_{21}^{(4)}(\lambda_j; \lambda_i; \beta) &= 34560 > 0.\end{aligned}$$

Now

$$\begin{aligned}\xi_{21}(\lambda_i; \lambda_i; \beta) &= 24\lambda_i^4(6 + \beta)(\beta + 3)(\beta + 1) \geq 0 \text{ for all } \beta \geq -1, \\ \xi_{21}^{(1)}(\lambda_i; \lambda_i; \beta) &= 6\lambda_i^3(4\beta + 7)(\beta^2 + 14\beta + 32) \geq 0 \text{ for all } \beta \geq -\frac{7}{4}, \\ \xi_{21}^{(2)}(\lambda_i; \lambda_i; \beta) &= 36\lambda_i^2(220 + 108\beta + 11\beta^2) \geq 0 \text{ for all } \beta \geq -2.8844, \\ \xi_{21}^{(3)}(\lambda_i; \lambda_i; \beta) &= 288\lambda_i(86 + 17\beta) \geq 0 \text{ for all } \beta \geq -\frac{86}{17}.\end{aligned}$$

Hence, $\xi_2^{(1)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{20}(\lambda_j; \lambda_i; \beta) = \xi_2(\lambda_j; \lambda_j; \lambda_i; \beta)/\lambda_j^4$. We wish to show that ξ_{20} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{20} with respect to λ_j gives

$$\begin{aligned}\xi_{20}(\lambda_j; \lambda_i; \beta) &= 360\lambda_j^4 + (240\beta - 480)\lambda_i\lambda_j^3 \\ &\quad + (108 - 198\beta + 72\beta^2)\lambda_i^2\lambda_j^2 + (-24\beta^2 + 12\beta^3)\lambda_i^3\lambda_j \\ &\quad + (192 + 180\beta + 42\beta^2)\lambda_i^4, \\ \xi_{20}^{(1)}(\lambda_j; \lambda_i; \beta) &= 1440\lambda_j^3 + 3(240\beta - 480)\lambda_i\lambda_j^2 \\ &\quad + 2(108 - 198\beta + 72\beta^2)\lambda_i^2\lambda_j + (-24\beta^2 + 12\beta^3)\lambda_i^3, \\ \xi_{20}^{(2)}(\lambda_j; \lambda_i; \beta) &= 4320\lambda_j^2 + 6(240\beta - 480)\lambda_i\lambda_j \\ &\quad + 2(108 - 198\beta + 72\beta^2)\lambda_i^2, \\ \xi_{20}^{(3)}(\lambda_j; \lambda_i; \beta) &= 8640\lambda_j + 6(240\beta - 480)\lambda_i, \\ \xi_{20}^{(4)}(\lambda_j; \lambda_i; \beta) &= 8640 > 0.\end{aligned}$$

Now

$$\begin{aligned}\xi_{20}(\lambda_i; \lambda_i; \beta) &= 6\lambda_i^4(2\beta + 5)(\beta + 3)(\beta + 2) > 0 \text{ for all } \beta > -2, \\ \xi_{20}^{(1)}(\lambda_i; \lambda_i; \beta) &= 12\lambda_i^3(6 + \beta)(\beta + 3)(\beta + 1) > 0 \text{ for all } \beta > -1, \\ \xi_{20}^{(2)}(\lambda_i; \lambda_i; \beta) &= 36\lambda_i^2(46 + 29\beta + 4\beta^2) > 0 \text{ for all } \beta > -2.3441, \\ \xi_{20}^{(3)}(\lambda_i; \lambda_i; \beta) &= 1440\lambda_i(\beta + 4) > 0 \text{ for all } \beta > -4.\end{aligned}$$

Hence, ξ_2 is positive for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$. This implies that $\xi(\lambda_l; \lambda_l; \lambda_j; \lambda_i; \beta)$ is positive for all $\lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Differentiating $\xi_3(\lambda_l; \lambda_j; \lambda_i; \beta)$ with respect to λ_l we obtain

$$\begin{aligned}\xi_3(\lambda_l; \lambda_j; \lambda_i; \beta) &= 720\lambda_j^4\lambda_l^4 + (-432 + 216\beta)\lambda_i^2\lambda_j^2\lambda_l^4 \\ &\quad + (-384 + 192\beta)\lambda_i\lambda_j^3\lambda_l^4 + (-384 + 192\beta)\lambda_i^3\lambda_j\lambda_l^4 \\ &\quad + 720\lambda_i^4\lambda_l^4 + (240\beta - 480)\lambda_i\lambda_j^4\lambda_l^3 \\ &\quad + (240\beta - 480)\lambda_i^4\lambda_j\lambda_l^3 + (90\beta^2 + 720 - 540\beta)\lambda_i^2\lambda_j^3\lambda_l^3 \\ &\quad + (90\beta^2 + 720 - 540\beta)\lambda_i^3\lambda_j^2\lambda_l^3 + (-288 + 108\beta + 18\beta^2)\lambda_i^2\lambda_j^4\lambda_l^2 \\ &\quad + (96\beta - 72\beta^2 + 12\beta^3)\lambda_i^3\lambda_j^3\lambda_l^2 + (-288 + 108\beta + 18\beta^2)\lambda_i^4\lambda_j^2\lambda_l^2 \\ &\quad + (84\beta - 240 + 18\beta^2)\lambda_i^3\lambda_j^4\lambda_l + (84\beta - 240 + 18\beta^2)\lambda_i^4\lambda_j^3\lambda_l \\ &\quad + (624 + 48\beta)\lambda_i^4\lambda_j^4,\end{aligned}$$

$$\begin{aligned}
\xi_3^{(1)}(\lambda_l; \lambda_j; \lambda_i; \beta) &= 2880\lambda_j^4\lambda_l^3 + 4(-432 + 216\beta)\lambda_i^2\lambda_j^2\lambda_l^3 \\
&\quad + 4(-384 + 192\beta)\lambda_i\lambda_j^3\lambda_l^3 + 4(-384 + 192\beta)\lambda_i^3\lambda_j\lambda_l^3 \\
&\quad + 2880\lambda_i^4\lambda_l^3 + 3(240\beta - 480)\lambda_i\lambda_j^4\lambda_l^2 \\
&\quad + 3(240\beta - 480)\lambda_i^4\lambda_j\lambda_l^2 + 3(90\beta^2 + 720 - 540\beta)\lambda_i^2\lambda_j^3\lambda_l^2 \\
&\quad + 3(90\beta^2 + 720 - 540\beta)\lambda_i^3\lambda_j^2\lambda_l^2 + 2(-288 + 108\beta + 18\beta^2)\lambda_i^2\lambda_j^4\lambda_l \\
&\quad + 2(96\beta - 72\beta^2 + 12\beta^3)\lambda_i^3\lambda_j^3\lambda_l + 2(-288 + 108\beta + 18\beta^2)\lambda_i^4\lambda_j^2\lambda_l \\
&\quad + (84\beta - 240 + 18\beta^2)\lambda_i^3\lambda_j^4 + (84\beta - 240 + 18\beta^2)\lambda_i^4\lambda_j^3, \\
\xi_3^{(2)}(\lambda_l; \lambda_j; \lambda_i; \beta) &= 8640\lambda_j^4\lambda_l^2 + 12(-432 + 216\beta)\lambda_i^2\lambda_j^2\lambda_l^2 \\
&\quad + 12(-384 + 192\beta)\lambda_i\lambda_j^3\lambda_l^2 + 12(-384 + 192\beta)\lambda_i^3\lambda_j\lambda_l^2 \\
&\quad + 8640\lambda_i^4\lambda_l^2 + 6(240\beta - 480)\lambda_i\lambda_j^4\lambda_l \\
&\quad + 6(240\beta - 480)\lambda_i^4\lambda_j\lambda_l + 6(90\beta^2 + 720 - 540\beta)\lambda_i^2\lambda_j^3\lambda_l \\
&\quad + 6(90\beta^2 + 720 - 540\beta)\lambda_i^3\lambda_j^2\lambda_l + 2(-288 + 108\beta + 18\beta^2)\lambda_i^2\lambda_j^4 \\
&\quad + 2(96\beta - 72\beta^2 + 12\beta^3)\lambda_i^3\lambda_j^3 + 2(-288 + 108\beta + 18\beta^2)\lambda_i^4\lambda_j^2, \\
\xi_3^{(3)}(\lambda_l; \lambda_j; \lambda_i; \beta) &= 17280\lambda_j^4\lambda_l + 24(-432 + 216\beta)\lambda_i^2\lambda_j^2\lambda_l \\
&\quad + 24(-384 + 192\beta)\lambda_i\lambda_j^3\lambda_l + 24(-384 + 192\beta)\lambda_i^3\lambda_j\lambda_l \\
&\quad + 17280\lambda_i^4\lambda_l + 6(240\beta - 480)\lambda_i\lambda_j^4 \\
&\quad + 6(240\beta - 480)\lambda_i^4\lambda_j + 6(90\beta^2 + 720 - 540\beta)\lambda_i^2\lambda_j^3 \\
&\quad + 6(90\beta^2 + 720 - 540\beta)\lambda_i^3\lambda_j^2, \\
\xi_3^{(4)}(\lambda_l; \lambda_j; \lambda_i; \beta) &= 17280\lambda_j^4 + 24(-432 + 216\beta)\lambda_i^2\lambda_j^2 \\
&\quad + 24(-384 + 192\beta)\lambda_i\lambda_j^3 + 24(-384 + 192\beta)\lambda_i^3\lambda_j \\
&\quad + 17280\lambda_i^4.
\end{aligned}$$

Define $\xi_{34}(\lambda_j; \lambda_i; \beta) = \xi_3^{(4)}(\lambda_l; \lambda_j; \lambda_i; \beta)$. We wish to show that ξ_{34} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{34} with respect to λ_j gives

$$\begin{aligned}
\xi_{34}(\lambda_j; \lambda_i; \beta) &= 17280\lambda_j^4 + 24(-432 + 216\beta)\lambda_i^2\lambda_j^2 \\
&\quad + 24(-384 + 192\beta)\lambda_i\lambda_j^3 + 24(-384 + 192\beta)\lambda_i^3\lambda_j + 17280\lambda_i^4, \\
\xi_{34}^{(1)}(\lambda_j; \lambda_i; \beta) &= 69120\lambda_j^3 + 48(-432 + 216\beta)\lambda_i^2\lambda_j \\
&\quad + 72(-384 + 192\beta)\lambda_i\lambda_j^2 + 24(-384 + 192\beta)\lambda_i^3, \\
\xi_{34}^{(2)}(\lambda_j; \lambda_i; \beta) &= 207360\lambda_j^2 + 48(-432 + 216\beta)\lambda_i^2 \\
&\quad + 144(-384 + 192\beta)\lambda_i\lambda_j, \\
\xi_{34}^{(3)}(\lambda_j; \lambda_i; \beta) &= 414720\lambda_j + 144(-384 + 192\beta)\lambda_i, \\
\xi_{34}^{(4)}(\lambda_j; \lambda_i; \beta) &= 414720 > 0.
\end{aligned}$$

Now

$$\begin{aligned}\xi_{34}(\lambda_i; \lambda_i; \beta) &= 2880\lambda_i^4(2 + 5\beta) \geq 0 \text{ for all } \beta \geq -\frac{2}{5}, \\ \xi_{34}^{(1)}(\lambda_i; \lambda_i; \beta) &= 5760\lambda_i^3(2 + 5\beta) \geq 0 \text{ for all } \beta \geq -\frac{2}{5}, \\ \xi_{34}^{(2)}(\lambda_i; \lambda_i; \beta) &= 3456\lambda_i^2(38 + 11\beta) \geq 0 \text{ for all } \beta \geq -\frac{38}{11}, \\ \xi_{34}^{(3)}(\lambda_i; \lambda_i; \beta) &= 27648\lambda_i(13 + \beta) \geq 0 \text{ for all } \beta \geq -13.\end{aligned}$$

Hence, $\xi_3^{(4)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{33}(\lambda_j; \lambda_i; \beta) = \xi_3^{(3)}(\lambda_j; \lambda_j; \lambda_i; \beta)/\lambda_j$. We wish to show that ξ_{33} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{33} with respect to λ_j gives

$$\begin{aligned}\xi_{33}(\lambda_j; \lambda_i; \beta) &= 17280\lambda_j^4 + (-12096 + 6048\beta)\lambda_i\lambda_j^3 \\ &\quad + (-6048 + 1944\beta + 540\beta^2)\lambda_i^2\lambda_j^2 + (1368\beta - 4896 + 540\beta^2)\lambda_i^3\lambda_j \\ &\quad + (14400 + 1440\beta)\lambda_i^4, \\ \xi_{33}^{(1)}(\lambda_j; \lambda_i; \beta) &= 69120\lambda_j^3 + 3(-12096 + 6048\beta)\lambda_i\lambda_j^2 \\ &\quad + 2(-6048 + 1944\beta + 540\beta^2)\lambda_i^2\lambda_j + (1368\beta - 4896 + 540\beta^2)\lambda_i^3, \\ \xi_{33}^{(2)}(\lambda_j; \lambda_i; \beta) &= 207360\lambda_j^2 + 6(-12096 + 6048\beta)\lambda_i\lambda_j \\ &\quad + 2(-6048 + 1944\beta + 540\beta^2)\lambda_i^2, \\ \xi_{33}^{(3)}(\lambda_j; \lambda_i; \beta) &= 414720\lambda_j + 6(-12096 + 6048\beta)\lambda_i, \\ \xi_{33}^{(4)}(\lambda_j; \lambda_i; \beta) &= 414720 > 0.\end{aligned}$$

Now

$$\begin{aligned}\xi_{33}(\lambda_i; \lambda_i; \beta) &= 1080\lambda_i^4(8 + 10\beta + \beta^2) \geq 0 \text{ for all } \beta \geq -0.8769, \\ \xi_{33}^{(1)}(\lambda_i; \lambda_i; \beta) &= 180\lambda_i^3(88 + 130\beta + 9\beta^2) \geq 0 \text{ for all } \beta \geq -0.7120, \\ \xi_{33}^{(2)}(\lambda_i; \lambda_i; \beta) &= 216\lambda_i^2(568 + 186\beta + 5\beta^2) \geq 0 \text{ for all } \beta \geq -3.3566, \\ \xi_{33}^{(3)}(\lambda_i; \lambda_i; \beta) &= 5184\lambda_i(66 + 7\beta) \geq 0 \text{ for all } \beta \geq -\frac{66}{7}.\end{aligned}$$

Hence, $\xi_3^{(3)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{32}(\lambda_j; \lambda_i; \beta) = \xi_3^{(2)}(\lambda_j; \lambda_j; \lambda_i; \beta)/\lambda_j^2$. We wish to show that ξ_{32} is non-negative

for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{32} with respect to λ_j gives

$$\begin{aligned}
\xi_{32}(\lambda_j; \lambda_i; \beta) &= 8640\lambda_j^4 + (3744\beta - 7488)\lambda_i\lambda_j^3 \\
&\quad + (-1440 - 432\beta + 576\beta^2)\lambda_i^2\lambda_j^2 + (24\beta^3 - 288 + 396\beta^2 - 744\beta)\lambda_i^3\lambda_j \\
&\quad + (5184 + 1656\beta + 36\beta^2)\lambda_i^4, \\
\xi_{32}^{(1)}(\lambda_j; \lambda_i; \beta) &= 34560\lambda_j^3 + 3(3744\beta - 7488)\lambda_i\lambda_j^2 \\
&\quad + 2(-1440 - 432\beta + 576\beta^2)\lambda_i^2\lambda_j + (24\beta^3 - 288 + 396\beta^2 - 744\beta)\lambda_i^3, \\
\xi_{32}^{(2)}(\lambda_j; \lambda_i; \beta) &= 103680\lambda_j^2 + 6(3744\beta - 7488)\lambda_i\lambda_j \\
&\quad + 2(-1440 - 432\beta + 576\beta^2)\lambda_i^2, \\
\xi_{32}^{(3)}(\lambda_j; \lambda_i; \beta) &= 207360\lambda_j + 6(3744\beta - 7488)\lambda_i, \\
\xi_{32}^{(4)}(\lambda_j; \lambda_i; \beta) &= 207360 > 0.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{32}(\lambda_i; \lambda_i; \beta) &= 24\lambda_i^4(\beta + 2)(\beta^2 + 40\beta + 96) \geq 0 \text{ for all } \beta \geq -2, \\
\xi_{32}^{(1)}(\lambda_i; \lambda_i; \beta) &= 12\lambda_i^3(744 + 802\beta + 129\beta^2 + 2\beta^3) \geq 0 \text{ for all } \beta \geq -1.1292, \\
\xi_{32}^{(2)}(\lambda_i; \lambda_i; \beta) &= 288\lambda_i^2(194 + 75\beta + 4\beta^2) \geq 0 \text{ for all } \beta \geq -3.0988, \\
\xi_{32}^{(3)}(\lambda_i; \lambda_i; \beta) &= 1728\lambda_i(94 + 13\beta) \geq 0 \text{ for all } \beta \geq -\frac{94}{13}.
\end{aligned}$$

Hence, $\xi_3^{(2)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{31}(\lambda_j; \lambda_i; \beta) = \xi_3^{(1)}(\lambda_j; \lambda_j; \lambda_i; \beta)/\lambda_j^3$. We wish to show that ξ_{31} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{31} with respect to λ_j gives

$$\begin{aligned}
\xi_{31}(\lambda_j; \lambda_i; \beta) &= 2880\lambda_j^4 + (1488\beta - 2976)\lambda_i\lambda_j^3 \\
&\quad + (-144 - 540\beta + 306\beta^2)\lambda_i^2\lambda_j^2 + (24\beta^3 + 384 + 144\beta^2 - 576\beta)\lambda_i^3\lambda_j \\
&\quad + (624 + 1020\beta + 54\beta^2)\lambda_i^4, \\
\xi_{31}^{(1)}(\lambda_j; \lambda_i; \beta) &= 11520\lambda_j^3 + 3(1488\beta - 2976)\lambda_i\lambda_j^2 \\
&\quad + 2(-144 - 540\beta + 306\beta^2)\lambda_i^2\lambda_j + (24\beta^3 + 384 + 144\beta^2 - 576\beta)\lambda_i^3, \\
\xi_{31}^{(2)}(\lambda_j; \lambda_i; \beta) &= 34560\lambda_j^2 + 6(1488\beta - 2976)\lambda_i\lambda_j \\
&\quad + 2(-144 - 540\beta + 306\beta^2)\lambda_i^2, \\
\xi_{31}^{(3)}(\lambda_j; \lambda_i; \beta) &= 69120\lambda_j + 6(1488\beta - 2976)\lambda_i, \\
\xi_{31}^{(4)}(\lambda_j; \lambda_i; \beta) &= 69120 > 0.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{31}(\lambda_i; \lambda_i; \beta) &= 24\lambda_i^4(32 + 58\beta + 21\beta^2 + \beta^3) \geq 0 \text{ for all } \beta \geq -0.7461, \\
\xi_{31}^{(1)}(\lambda_i; \lambda_i; \beta) &= 12\lambda_i^3(224 + 234\beta + 63\beta^2 + 2\beta^3) \geq 0 \text{ for all } \beta \geq -1.6603, \\
\xi_{31}^{(2)}(\lambda_i; \lambda_i; \beta) &= 36\lambda_i^2(456 + 218\beta + 17\beta^2) \geq 0 \text{ for all } \beta \geq -2.6319, \\
\xi_{31}^{(3)}(\lambda_i; \lambda_i; \beta) &= 288\lambda_i(178 + 31\beta) \geq 0 \text{ for all } \beta \geq -\frac{178}{31}.
\end{aligned}$$

Hence, $\xi_3^{(1)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{30}(\lambda_j; \lambda_i; \beta) = \xi_3(\lambda_j; \lambda_j; \lambda_i; \beta)/\lambda_j^4$. We wish to show that ξ_{30} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{30} with respect to λ_j gives

$$\begin{aligned}\xi_{30}(\lambda_j; \lambda_i; \beta) &= 720\lambda_j^4 + (-864 + 432\beta)\lambda_i\lambda_j^3 \\ &\quad + (-216\beta + 108\beta^2)\lambda_i^2\lambda_j^2 + (96 - 168\beta + 36\beta^2 + 12\beta^3)\lambda_i^3\lambda_j \\ &\quad + (336 + 480\beta + 36\beta^2)\lambda_i^4, \\ \xi_{30}^{(1)}(\lambda_j; \lambda_i; \beta) &= 2880\lambda_j^3 + 3(-864 + 432\beta)\lambda_i\lambda_j^2 \\ &\quad + 2(-216\beta + 108\beta^2)\lambda_i^2\lambda_j + (96 - 168\beta + 36\beta^2 + 12\beta^3)\lambda_i^3, \\ \xi_{30}^{(2)}(\lambda_j; \lambda_i; \beta) &= 8640\lambda_j^2 + 6(-864 + 432\beta)\lambda_i\lambda_j \\ &\quad + 2(-216\beta + 108\beta^2)\lambda_i^2, \\ \xi_{30}^{(3)}(\lambda_j; \lambda_i; \beta) &= 17280\lambda_j + 6(-864 + 432\beta)\lambda_i, \\ \xi_{30}^{(4)}(\lambda_j; \lambda_i; \beta) &= 17280 > 0.\end{aligned}$$

Now

$$\begin{aligned}\xi_{30}(\lambda_i; \lambda_i; \beta) &= 12\lambda_i^4(\beta + 3)(\beta^2 + 12\beta + 8) > 0 \text{ for all } \beta > -0.7085, \\ \xi_{30}^{(1)}(\lambda_i; \lambda_i; \beta) &= 12\lambda_i^3(32 + 58\beta + 21\beta^2 + \beta^3) > 0 \text{ for all } \beta > -0.7461, \\ \xi_{30}^{(2)}(\lambda_i; \lambda_i; \beta) &= 216\lambda_i^2(8 + \beta)(\beta + 2) > 0 \text{ for all } \beta > -2, \\ \xi_{30}^{(3)}(\lambda_i; \lambda_i; \beta) &= 864\lambda_i(14 + 3\beta) > 0 \text{ for all } \beta > -\frac{14}{3}.\end{aligned}$$

Hence, ξ_3 is positive for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$. This implies that $\xi(\lambda_l; \lambda_l; \lambda_j; \lambda_i; \beta)$ is positive for all $\lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Differentiating $\xi_4(\lambda_l; \lambda_j; \lambda_i; \beta)$ with respect to λ_l we obtain

$$\begin{aligned}\xi_4(\lambda_l; \lambda_j; \lambda_i; \beta) &= 720\lambda_j^4\lambda_l^4 + (-432 + 216\beta)\lambda_i^2\lambda_j^2\lambda_l^4 \\ &\quad + (-384 + 192\beta)\lambda_i\lambda_j^3\lambda_l^4 + (-384 + 192\beta)\lambda_i^3\lambda_j\lambda_l^4 \\ &\quad + 720\lambda_i^4\lambda_l^4 + (-384 + 192\beta)\lambda_i\lambda_j^4\lambda_l^3 \\ &\quad + (-384 + 192\beta)\lambda_i^4\lambda_j\lambda_l^3 + (576 - 432\beta + 72\beta^2)\lambda_i^2\lambda_j^3\lambda_l^3 \\ &\quad + (576 - 432\beta + 72\beta^2)\lambda_i^3\lambda_j^2\lambda_l^3 + (-432 + 216\beta)\lambda_i^2\lambda_j^4\lambda_l^2 \\ &\quad + (576 - 432\beta + 72\beta^2)\lambda_i^3\lambda_j^3\lambda_l^2 + (-432 + 216\beta)\lambda_i^4\lambda_j^2\lambda_l^2 \\ &\quad + (-384 + 192\beta)\lambda_i^3\lambda_j^4\lambda_l + (-384 + 192\beta)\lambda_i^4\lambda_j^3\lambda_l \\ &\quad + 720\lambda_i^4\lambda_j^4,\end{aligned}$$

$$\begin{aligned}
\xi_4^{(1)}(\lambda_l; \lambda_j; \lambda_i; \beta) &= 2880\lambda_j^4\lambda_l^3 + 4(-432 + 216\beta)\lambda_i^2\lambda_j^2\lambda_l^3 \\
&\quad + 4(-384 + 192\beta)\lambda_i\lambda_j^3\lambda_l^3 + 4(-384 + 192\beta)\lambda_i^3\lambda_j\lambda_l^3 \\
&\quad + 2880\lambda_i^4\lambda_l^3 + 3(-384 + 192\beta)\lambda_i\lambda_j^4\lambda_l^2 \\
&\quad + 3(-384 + 192\beta)\lambda_i^4\lambda_j\lambda_l^2 + 3(576 - 432\beta + 72\beta^2)\lambda_i^2\lambda_j^3\lambda_l^2 \\
&\quad + 3(576 - 432\beta + 72\beta^2)\lambda_i^3\lambda_j^2\lambda_l^2 + 2(-432 + 216\beta)\lambda_i^2\lambda_j^4\lambda_l \\
&\quad + 2(576 - 432\beta + 72\beta^2)\lambda_i^3\lambda_j^3\lambda_l + 2(-432 + 216\beta)\lambda_i^4\lambda_j^2\lambda_l \\
&\quad + (-384 + 192\beta)\lambda_i^3\lambda_j^4 + (-384 + 192\beta)\lambda_i^4\lambda_j^3, \\
\xi_4^{(2)}(\lambda_l; \lambda_j; \lambda_i; \beta) &= 8640\lambda_j^4\lambda_l^2 + 12(-432 + 216\beta)\lambda_i^2\lambda_j^2\lambda_l^2 \\
&\quad + 12(-384 + 192\beta)\lambda_i\lambda_j^3\lambda_l^2 + 12(-384 + 192\beta)\lambda_i^3\lambda_j\lambda_l^2 \\
&\quad + 8640\lambda_i^4\lambda_l^2 + 6(-384 + 192\beta)\lambda_i\lambda_j^4\lambda_l \\
&\quad + 6(-384 + 192\beta)\lambda_i^4\lambda_j\lambda_l + 6(576 - 432\beta + 72\beta^2)\lambda_i^2\lambda_j^3\lambda_l \\
&\quad + 6(576 - 432\beta + 72\beta^2)\lambda_i^3\lambda_j^2\lambda_l + 2(-432 + 216\beta)\lambda_i^2\lambda_j^4 \\
&\quad + 2(576 - 432\beta + 72\beta^2)\lambda_i^3\lambda_j^3 + 2(-432 + 216\beta)\lambda_i^4\lambda_j^2, \\
\xi_4^{(3)}(\lambda_l; \lambda_j; \lambda_i; \beta) &= 17280\lambda_j^4\lambda_l + 24(-432 + 216\beta)\lambda_i^2\lambda_j^2\lambda_l \\
&\quad + 24(-384 + 192\beta)\lambda_i\lambda_j^3\lambda_l + 24(-384 + 192\beta)\lambda_i^3\lambda_j\lambda_l \\
&\quad + 17280\lambda_i^4\lambda_l + 6(-384 + 192\beta)\lambda_i\lambda_j^4 \\
&\quad + 6(-384 + 192\beta)\lambda_i^4\lambda_j + 6(576 - 432\beta + 72\beta^2)\lambda_i^2\lambda_j^3 \\
&\quad + 6(576 - 432\beta + 72\beta^2)\lambda_i^3\lambda_j^2, \\
\xi_4^{(4)}(\lambda_l; \lambda_j; \lambda_i; \beta) &= 17280\lambda_j^4 + 24(-432 + 216\beta)\lambda_i^2\lambda_j^2 \\
&\quad + 24(-384 + 192\beta)\lambda_i\lambda_j^3 + 24(-384 + 192\beta)\lambda_i^3\lambda_j \\
&\quad + 17280\lambda_i^4.
\end{aligned}$$

Define $\xi_{44}(\lambda_j; \lambda_i; \beta) = \xi_4^{(4)}(\lambda_l; \lambda_j; \lambda_i; \beta)$. We wish to show that ξ_{44} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{44} with respect to λ_j gives

$$\begin{aligned}
\xi_{44}(\lambda_j; \lambda_i; \beta) &= 17280\lambda_j^4 + (-9216 + 4608\beta)\lambda_i\lambda_j^3 \\
&\quad + (-10368 + 5184\beta)\lambda_i^2\lambda_j^2 + (-9216 + 4608\beta)\lambda_i^3\lambda_j \\
&\quad + 17280\lambda_i^4, \\
\xi_{44}^{(1)}(\lambda_j; \lambda_i; \beta) &= 69120\lambda_j^3 + 3(-9216 + 4608\beta)\lambda_i\lambda_j^2 \\
&\quad + 2(-10368 + 5184\beta)\lambda_i^2\lambda_j + (-9216 + 4608\beta)\lambda_i^3, \\
\xi_{44}^{(2)}(\lambda_j; \lambda_i; \beta) &= 207360\lambda_j^2 + 6(-9216 + 4608\beta)\lambda_i\lambda_j \\
&\quad + 2(-10368 + 5184\beta)\lambda_i^2, \\
\xi_{44}^{(3)}(\lambda_j; \lambda_i; \beta) &= 414720\lambda_j + 6(-9216 + 4608\beta)\lambda_i, \\
\xi_{44}^{(4)}(\lambda_j; \lambda_i; \beta) &= 414720 > 0.
\end{aligned}$$

Now

$$\begin{aligned}\xi_{44}(\lambda_i; \lambda_i; \beta) &= 2880\lambda_i^4(2 + 5\beta) \geq 0 \text{ for all } \beta \geq -\frac{2}{5}, \\ \xi_{44}^{(1)}(\lambda_i; \lambda_i; \beta) &= 5760\lambda_i^3(2 + 5\beta) \geq 0 \text{ for all } \beta \geq -\frac{2}{5}, \\ \xi_{44}^{(2)}(\lambda_i; \lambda_i; \beta) &= 3456\lambda_i^2(38 + 11\beta) \geq 0 \text{ for all } \beta \geq -\frac{38}{11}, \\ \xi_{44}^{(3)}(\lambda_i; \lambda_i; \beta) &= 27648\lambda_i(13 + \beta) \geq 0 \text{ for all } \beta \geq -13.\end{aligned}$$

Hence, $\xi_4^{(4)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{43}(\lambda_j; \lambda_i; \beta) = \xi_4^{(3)}(\lambda_j; \lambda_j; \lambda_i; \beta)/\lambda_j$. We wish to show that ξ_{43} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{43} with respect to λ_j gives

$$\begin{aligned}\xi_{43}(\lambda_j; \lambda_i; \beta) &= 17280\lambda_j^4 + (-11520 + 5760\beta)\lambda_i\lambda_j^3 \\ &\quad + (-6912 + 2592\beta + 432\beta^2)\lambda_i^2\lambda_j^2 + (2016\beta - 5760 + 432\beta^2)\lambda_i^3\lambda_j \\ &\quad + (14976 + 1152\beta)\lambda_i^4, \\ \xi_{43}^{(1)}(\lambda_j; \lambda_i; \beta) &= 69120\lambda_j^3 + 3(-11520 + 5760\beta)\lambda_i\lambda_j^2 \\ &\quad + 2(-6912 + 2592\beta + 432\beta^2)\lambda_i^2\lambda_j + (2016\beta - 5760 + 432\beta^2)\lambda_i^3, \\ \xi_{43}^{(2)}(\lambda_j; \lambda_i; \beta) &= 207360\lambda_j^2 + 6(-11520 + 5760\beta)\lambda_i\lambda_j \\ &\quad + 2(-6912 + 2592\beta + 432\beta^2)\lambda_i^2, \\ \xi_{43}^{(3)}(\lambda_j; \lambda_i; \beta) &= 414720\lambda_j + 6(-11520 + 5760\beta)\lambda_i, \\ \xi_{43}^{(4)}(\lambda_j; \lambda_i; \beta) &= 414720 > 0.\end{aligned}$$

Now

$$\begin{aligned}\xi_{43}(\lambda_i; \lambda_i; \beta) &= 288\lambda_i^4(28 + 40\beta + 3\beta^2) \geq 0 \text{ for all } \beta \geq -0.7412, \\ \xi_{43}^{(1)}(\lambda_i; \lambda_i; \beta) &= 144\lambda_i^3(104 + 170\beta + 9\beta^2) \geq 0 \text{ for all } \beta \geq -0.6330, \\ \xi_{43}^{(2)}(\lambda_i; \lambda_i; \beta) &= 864\lambda_i^2(144 + 46\beta + \beta^2) \geq 0 \text{ for all } \beta \geq -3.3786, \\ \xi_{43}^{(3)}(\lambda_i; \lambda_i; \beta) &= 34560\lambda_i(\beta + 10) \geq 0 \text{ for all } \beta \geq -10.\end{aligned}$$

Hence, $\xi_4^{(3)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{42}(\lambda_j; \lambda_i; \beta) = \xi_4^{(2)}(\lambda_j; \lambda_j; \lambda_i; \beta)/\lambda_j^2$. We wish to show that ξ_{42} is non-negative

for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{42} with respect to λ_j gives

$$\begin{aligned}\xi_{42}(\lambda_j; \lambda_i; \beta) &= 8640\lambda_j^4 + (-6912 + 3456\beta)\lambda_i\lambda_j^3 \\ &\quad + (-2592 + 432\beta + 432\beta^2)\lambda_i^2\lambda_j^2 + (-1152\beta + 576\beta^2)\lambda_i^3\lambda_j \\ &\quad + (5472 + 1584\beta)\lambda_i^4, \\ \xi_{42}^{(1)}(\lambda_j; \lambda_i; \beta) &= 34560\lambda_j^3 + 3(-6912 + 3456\beta)\lambda_i\lambda_j^2 + 2(-2592 + 432\beta \\ &\quad + 432\beta^2)\lambda_i^2\lambda_j + (-1152\beta + 576\beta^2)\lambda_i^3, \\ \xi_{42}^{(2)}(\lambda_j; \lambda_i; \beta) &= 103680\lambda_j^2 + 6(-6912 + 3456\beta)\lambda_i\lambda_j + 2(-2592 + 432\beta \\ &\quad + 432\beta^2)\lambda_i^2, \\ \xi_{42}^{(3)}(\lambda_j; \lambda_i; \beta) &= 207360\lambda_j + 6(-6912 + 3456\beta)\lambda_i, \\ \xi_{42}^{(4)}(\lambda_j; \lambda_i; \beta) &= 207360 > 0.\end{aligned}$$

Now

$$\begin{aligned}\xi_{42}(\lambda_i; \lambda_i; \beta) &= 144\lambda_i^4(7\beta + 16)(\beta + 2) \geq 0 \text{ for all } \beta \geq -2, \\ \xi_{42}^{(1)}(\lambda_i; \lambda_i; \beta) &= 1440\lambda_i^3(6 + \beta)(\beta + 1) \geq 0 \text{ for all } \beta \geq -1, \\ \xi_{42}^{(2)}(\lambda_i; \lambda_i; \beta) &= 864\lambda_i^2(\beta + 22)(\beta + 3) \geq 0 \text{ for all } \beta \geq -3, \\ \xi_{42}^{(3)}(\lambda_i; \lambda_i; \beta) &= 20736\lambda_i(8 + \beta) \geq 0 \text{ for all } \beta \geq -8.\end{aligned}$$

Hence, $\xi_4^{(2)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{41}(\lambda_j; \lambda_i; \beta) = \xi_4^{(1)}(\lambda_j; \lambda_j; \lambda_i; \beta)/\lambda_j^3$. We wish to show that ξ_{41} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{41} with respect to λ_j gives

$$\begin{aligned}\xi_{41}(\lambda_j; \lambda_i; \beta) &= 2880\lambda_j^4 + (-2688 + 1344\beta)\lambda_i\lambda_j^3 \\ &\quad + (-864 + 216\beta^2)\lambda_i^2\lambda_j^2 + (-1200\beta + 960 + 360\beta^2)\lambda_i^3\lambda_j \\ &\quad + (480 + 1200\beta)\lambda_i^4, \\ \xi_{41}^{(1)}(\lambda_j; \lambda_i; \beta) &= 11520\lambda_j^3 + 3(-2688 + 1344\beta)\lambda_i\lambda_j^2 \\ &\quad + 2(-864 + 216\beta^2)\lambda_i^2\lambda_j + (-1200\beta + 960 + 360\beta^2)\lambda_i^3, \\ \xi_{41}^{(2)}(\lambda_j; \lambda_i; \beta) &= 34560\lambda_j^2 + 6(-2688 + 1344\beta)\lambda_i\lambda_j \\ &\quad + 2(-864 + 216\beta^2)\lambda_i^2, \\ \xi_{41}^{(3)}(\lambda_j; \lambda_i; \beta) &= 69120\lambda_j + 6(-2688 + 1344\beta)\lambda_i, \\ \xi_{41}^{(4)}(\lambda_j; \lambda_i; \beta) &= 69120 > 0.\end{aligned}$$

Now

$$\begin{aligned}\xi_{41}(\lambda_i; \lambda_i; \beta) &= 192\lambda_i^4(3\beta + 4)(\beta + 1) \geq 0 \text{ for all } \beta \geq -1, \\ \xi_{41}^{(1)}(\lambda_i; \lambda_i; \beta) &= 24\lambda_i^3(112 + 118\beta + 33\beta^2) \geq 0 \text{ for all } \beta, \\ \xi_{41}^{(2)}(\lambda_i; \lambda_i; \beta) &= 144\lambda_i^2(116 + 56\beta + 3\beta^2) \geq 0 \text{ for all } \beta \geq -2.3731, \\ \xi_{41}^{(3)}(\lambda_i; \lambda_i; \beta) &= 1152\lambda_i(46 + 7\beta) \geq 0 \text{ for all } \beta \geq -\frac{46}{7}.\end{aligned}$$

Hence, $\xi_4^{(1)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{40}(\lambda_j; \lambda_i; \beta) = \xi_4(\lambda_j; \lambda_j; \lambda_i; \beta)/\lambda_j^4$. We wish to show that ξ_{40} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{40} with respect to λ_j gives

$$\begin{aligned}\xi_{40}(\lambda_j; \lambda_i; \beta) &= 720\lambda_j^4 + (-768 + 384\beta)\lambda_i\lambda_j^3 \\ &\quad + (-288 + 72\beta^2)\lambda_i^2\lambda_j^2 + (-480\beta + 384 + 144\beta^2)\lambda_i^3\lambda_j \\ &\quad + (240 + 600\beta)\lambda_i^4, \\ \xi_{40}^{(1)}(\lambda_j; \lambda_i; \beta) &= 2880\lambda_j^3 + 3(-768 + 384\beta)\lambda_i\lambda_j^2 \\ &\quad + 2(-288 + 72\beta^2)\lambda_i^2\lambda_j + (-480\beta + 384 + 144\beta^2)\lambda_i^3, \\ \xi_{40}^{(2)}(\lambda_j; \lambda_i; \beta) &= 8640\lambda_j^2 + 6(-768 + 384\beta)\lambda_i\lambda_j \\ &\quad + 2(-288 + 72\beta^2)\lambda_i^2, \\ \xi_{40}^{(3)}(\lambda_j; \lambda_i; \beta) &= 17280\lambda_j + 6(-768 + 384\beta)\lambda_i, \\ \xi_{40}^{(4)}(\lambda_j; \lambda_i; \beta) &= 17280 > 0.\end{aligned}$$

Now

$$\begin{aligned}\xi_{40}(\lambda_i; \lambda_i; \beta) &= 72\lambda_i^4(3\beta + 4)(\beta + 1) > 0 \text{ for all } \beta > -1, \\ \xi_{40}^{(1)}(\lambda_i; \lambda_i; \beta) &= 96\lambda_i^3(3\beta + 4)(\beta + 1) > 0 \text{ for all } \beta > -1, \\ \xi_{40}^{(2)}(\lambda_i; \lambda_i; \beta) &= 144\lambda_i^2(24 + 16\beta + \beta^2) > 0 \text{ for all } \beta > -1.6754, \\ \xi_{40}^{(3)}(\lambda_i; \lambda_i; \beta) &= 1152\lambda_i(11 + 2\beta) > 0 \text{ for all } \beta > -\frac{11}{2}.\end{aligned}$$

Hence, ξ_4 is positive for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$. This implies that $\xi(\lambda_l; \lambda_l; \lambda_j; \lambda_i; \beta)$ is positive for all $\lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$. □

Lemma B.12 Let

$$\begin{aligned}\xi(\lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta) &= (56\beta + 12\beta^2 - 160)t(1, 3, 3, 4) \\ &\quad + (-144\beta + 36\beta^2 + 144)t(2, 2, 3, 4) \\ &\quad + (208 + 16\beta)t(0, 3, 4, 4) \\ &\quad + (84\beta - 168)t(1, 2, 4, 4) \\ &\quad + (-72\beta^2 + 96\beta + 12\beta^3)s(2, 3, 3, 3),\end{aligned}$$

where

$$t(a, b, c, d) = \sum_{\substack{(n_1, n_2, n_3, n_4) \\ \in \text{perm}(a, b, c, d)}} \lambda_i^{n_1} \lambda_j^{n_2} \lambda_l^{n_3} \lambda_k^{n_4}.$$

The expression $\xi(\lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta)$ is positive for all $\lambda_k \geq \lambda_l \geq \lambda_j \geq \lambda_i$, and $\beta \geq -\frac{2}{5}$.

Proof.

Let

$$\hat{s}(a, b, c) = \sum_{\substack{(n_1, n_2, n_3) \\ \in \text{perm}(a, b, c)}} \lambda_i^{n_1} \lambda_j^{n_2} \lambda_l^{n_3}.$$

Differentiating $\xi(\lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta)$ with respect to λ_k we obtain

$$\begin{aligned} \xi^{(1)}(\lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta) &= (208 + 16\beta)\lambda_k^3 \hat{s}(0, 3, 4) \\ &\quad + (56\beta + 12\beta^2 - 160)\lambda_k^3 \hat{s}(1, 3, 3) \\ &\quad + (-144\beta + 36\beta^2 + 144)\lambda_k^3 \hat{s}(2, 2, 3) \\ &\quad + (-168 + 84\beta)\lambda_k^3 \hat{s}(1, 2, 4) \\ &\quad + (156 + 12\beta)\lambda_k^2 \hat{s}(0, 4, 4) \\ &\quad + (9\beta^2 + 42\beta - 120)\lambda_k^2 \hat{s}(1, 3, 4) \\ &\quad + (72\beta + 9\beta^3 - 54\beta^2)\lambda_k^2 \hat{s}(2, 3, 3) \\ &\quad + (-108\beta + 27\beta^2 + 108)\lambda_k^2 \hat{s}(2, 2, 4) \\ &\quad + (-36\beta^2 + 48\beta + 6\beta^3)\lambda_k \lambda_k \hat{s}(3, 3, 3) \\ &\quad + (18\beta^2 - 72\beta + 72)\lambda_k \hat{s}(2, 3, 4) \\ &\quad + (-84 + 42\beta)\lambda_k \hat{s}(1, 4, 4) \\ &\quad + (14\beta + 3\beta^2 - 40)\hat{s}(3, 3, 4) \\ &\quad + (-42 + 21\beta)\hat{s}(2, 4, 4), \\ \xi^{(2)}(\lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta) &= (-480 + 36\beta^2 + 168\beta)\lambda_k^2 \hat{s}(1, 3, 3) \\ &\quad + (432 + 108\beta^2 - 432\beta)\lambda_k^2 \hat{s}(2, 2, 3) \\ &\quad + (252\beta - 504)\lambda_k^2 \hat{s}(1, 2, 4) \\ &\quad + (624 + 48\beta)\lambda_k^2 \hat{s}(0, 3, 4) \\ &\quad + (-240 + 84\beta + 18\beta^2)\lambda_k \hat{s}(1, 3, 4) \\ &\quad + (-216\beta + 216 + 54\beta^2)\lambda_k \hat{s}(2, 2, 4) \\ &\quad + (-108\beta^2 + 18\beta^3 + 144\beta)\lambda_k \hat{s}(2, 3, 3) \\ &\quad + (24\beta + 312)\lambda_k \hat{s}(0, 4, 4) \\ &\quad + (-36\beta^2 + 48\beta + 6\beta^3)\hat{s}(3, 3, 3) \\ &\quad + (18\beta^2 - 72\beta + 72)\hat{s}(2, 3, 4) \\ &\quad + (-84 + 42\beta)\hat{s}(1, 4, 4), \end{aligned}$$

$$\begin{aligned}
\xi^{(3)}(\lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta) &= (72\beta^2 - 960 + 336\beta)\lambda_k \hat{s}(1, 3, 3) \\
&+ (864 - 864\beta + 216\beta^2)\lambda_k \hat{s}(2, 2, 3) \\
&+ (-1008 + 504\beta)\lambda_k \hat{s}(1, 2, 4) \\
&+ (96\beta + 1248)\lambda_k \hat{s}(0, 3, 4) \\
&+ (-240 + 84\beta + 18\beta^2)\hat{s}(1, 3, 4) \\
&+ (-216\beta + 216 + 54\beta^2)\hat{s}(2, 2, 4) \\
&+ (-108\beta^2 + 18\beta^3 + 144\beta)\hat{s}(2, 3, 3) \\
&+ (24\beta + 312)\hat{s}(0, 4, 4), \\
\xi^{(4)}(\lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta) &= (864 - 864\beta + 216\beta^2)\hat{s}(2, 2, 3) \\
&+ (72\beta^2 - 960 + 336\beta)\hat{s}(1, 3, 3) \\
&+ (-1008 + 504\beta)\hat{s}(1, 2, 4) \\
&+ (96\beta + 1248)\hat{s}(0, 3, 4).
\end{aligned}$$

Define

$$\begin{aligned}
\xi_0(\lambda_l; \lambda_j; \lambda_i; \beta) &\stackrel{\text{def}}{=} \xi(\lambda_l; \lambda_l; \lambda_j; \lambda_i; \beta)/\lambda_l^3, \\
\xi_1(\lambda_l; \lambda_j; \lambda_i; \beta) &\stackrel{\text{def}}{=} \xi^{(1)}(\lambda_l; \lambda_l; \lambda_j; \lambda_i; \beta)/\lambda_l^2, \\
\xi_2(\lambda_l; \lambda_j; \lambda_i; \beta) &\stackrel{\text{def}}{=} \xi^{(2)}(\lambda_l; \lambda_l; \lambda_j; \lambda_i; \beta)/\lambda_l, \\
\xi_3(\lambda_l; \lambda_j; \lambda_i; \beta) &\stackrel{\text{def}}{=} \xi^{(3)}(\lambda_l; \lambda_l; \lambda_j; \lambda_i; \beta), \\
\xi_4(\lambda_l; \lambda_j; \lambda_i; \beta) &\stackrel{\text{def}}{=} \xi^{(4)}(\lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta).
\end{aligned}$$

We will prove that the above equations are non-negative for all $\lambda_k \geq \lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$ and, hence, deduce that $\xi(\lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta)$ is positive for all $\lambda_k \geq \lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Differentiating $\xi_0(\lambda_i; \lambda_j; \lambda_i; \beta)$ with respect to λ_i we obtain

$$\begin{aligned}
\xi_0(\lambda_i; \lambda_j; \lambda_i; \beta) &= (4\beta + 52)\lambda_i^5(\lambda_j^3 + \lambda_i^3) \\
&\quad + (-42 + 21\beta)\lambda_i^5(\lambda_i^2\lambda_j + \lambda_i\lambda_j^2) \\
&\quad + (28\beta - 80 + 6\beta^2)\lambda_i^4(\lambda_i\lambda_j^3 + \lambda_i^3\lambda_j) \\
&\quad + (8\beta + 104)\lambda_i^4(\lambda_j^4 + \lambda_i^4) \\
&\quad + (-72\beta + 72 + 18\beta^2)\lambda_i^2\lambda_j^2\lambda_i^4 \\
&\quad + (-124 + 56\beta + 3\beta^2)\lambda_i^3(\lambda_i\lambda_j^4 + \lambda_i^4\lambda_j) \\
&\quad + (-48\beta + 72 + 3\beta^3)\lambda_i^3(\lambda_i^2\lambda_j^3 + \lambda_i^3\lambda_j^2) \\
&\quad + (-80 - 30\beta^2 + 76\beta + 6\beta^3)\lambda_i^2\lambda_i^3\lambda_j^3 \\
&\quad + (18\beta^2 - 12 - 30\beta)\lambda_i^2(\lambda_i^2\lambda_j^4 + \lambda_i^4\lambda_j^2) \\
&\quad + (60 + 15\beta^2)\lambda_i(\lambda_i^3\lambda_j^4 + \lambda_i^4\lambda_j^3) \\
&\quad + (20 + 50\beta)\lambda_i^4\lambda_j^4, \\
\xi_0^{(1)}(\lambda_i; \lambda_j; \lambda_i; \beta) &= (105\beta - 210)\lambda_i^4(\lambda_i^2\lambda_j + \lambda_i\lambda_j^2) \\
&\quad + (20\beta + 260)\lambda_i^4(\lambda_j^3 + \lambda_i^3) \\
&\quad + (24\beta^2 - 320 + 112\beta)\lambda_i^3(\lambda_i\lambda_j^3 + \lambda_i^3\lambda_j) \\
&\quad + (416 + 32\beta)\lambda_i^3(\lambda_j^4 + \lambda_i^4) \\
&\quad + (288 + 72\beta^2 - 288\beta)\lambda_i^3\lambda_i^2\lambda_j^2 \\
&\quad + (216 - 144\beta + 9\beta^3)\lambda_i^2(\lambda_i^2\lambda_j^3 + \lambda_i^3\lambda_j^2) \\
&\quad + (9\beta^2 - 372 + 168\beta)\lambda_i^2(\lambda_i\lambda_j^4 + \lambda_i^4\lambda_j) \\
&\quad + (-60\beta - 24 + 36\beta^2)\lambda_i(\lambda_i^2\lambda_j^4 + \lambda_i^4\lambda_j^2) \\
&\quad + (-160 + 152\beta - 60\beta^2 + 12\beta^3)\lambda_i\lambda_i^3\lambda_j^3 \\
&\quad + (60 + 15\beta^2)(\lambda_i^3\lambda_j^4 + \lambda_i^4\lambda_j^3), \\
\xi_0^{(2)}(\lambda_i; \lambda_j; \lambda_i; \beta) &= (420\beta - 840)\lambda_i^3(\lambda_i^2\lambda_j + \lambda_i\lambda_j^2) \\
&\quad + (1040 + 80\beta)\lambda_i^3(\lambda_j^3 + \lambda_i^3) \\
&\quad + (72\beta^2 - 960 + 336\beta)\lambda_i^2(\lambda_i\lambda_j^3 + \lambda_i^3\lambda_j) \\
&\quad + (96\beta + 1248)(\lambda_i^4 + \lambda_j^4)\lambda_i^2 \\
&\quad + (864 - 864\beta + 216\beta^2)\lambda_i^2\lambda_i^2\lambda_j^2 \\
&\quad + (432 - 288\beta + 18\beta^3)\lambda_i(\lambda_i^2\lambda_j^3 + \lambda_i^3\lambda_j^2) \\
&\quad + (336\beta - 744 + 18\beta^2)\lambda_i(\lambda_i\lambda_j^4 + \lambda_i^4\lambda_j) \\
&\quad + (-160 + 152\beta - 60\beta^2 + 12\beta^3)\lambda_i^3\lambda_j^3 \\
&\quad + (-60\beta - 24 + 36\beta^2)(\lambda_i^2\lambda_j^4 + \lambda_i^4\lambda_j^2),
\end{aligned}$$

$$\begin{aligned}
\xi_0^{(3)}(\lambda_l; \lambda_j; \lambda_i; \beta) &= (-2520 + 1260\beta)(\lambda_i^2\lambda_j + \lambda_i\lambda_j^2)\lambda_l^2 \\
&\quad + (240\beta + 3120)(\lambda_j^3 + \lambda_i^3)\lambda_l^2 \\
&\quad + (-1920 + 144\beta^2 + 672\beta)(\lambda_i\lambda_j^3 + \lambda_i^3\lambda_j)\lambda_l \\
&\quad + (192\beta + 2496)(\lambda_j^4 + \lambda_i^4)\lambda_l \\
&\quad + (-1728\beta + 1728 + 432\beta^2)\lambda_l\lambda_i^2\lambda_j^2 \\
&\quad + (432 - 288\beta + 18\beta^3)(\lambda_i^2\lambda_j^3 + \lambda_i^3\lambda_j^2) \\
&\quad + (336\beta - 744 + 18\beta^2)(\lambda_i^4\lambda_j + \lambda_i\lambda_j^4), \\
\xi_0^{(4)}(\lambda_l; \lambda_j; \lambda_i; \beta) &= (-5040 + 2520\beta)\lambda_l(\lambda_i\lambda_j^2 + \lambda_i^2\lambda_j) \\
&\quad + (480\beta + 6240)(\lambda_j^3 + \lambda_i^3)\lambda_l \\
&\quad + (-1728\beta + 1728 + 432\beta^2)\lambda_i^2\lambda_j^2 \\
&\quad + (192\beta + 2496)(\lambda_j^4 + \lambda_i^4) \\
&\quad + (-1920 + 144\beta^2 + 672\beta)(\lambda_i\lambda_j^3 + \lambda_i^3\lambda_j), \\
\xi_0^{(5)}(\lambda_l; \lambda_j; \lambda_i; \beta) &= (480\beta + 6240)(\lambda_j^3 + \lambda_i^3) \\
&\quad + (-5040 + 2520\beta)(\lambda_i\lambda_j^2 + \lambda_i^2\lambda_j).
\end{aligned}$$

Define $\xi_{05}(\lambda_j; \lambda_i; \beta) = \xi_0^{(5)}(\lambda_l; \lambda_j; \lambda_i; \beta)$. We wish to show that ξ_{05} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{05} with respect to λ_j gives

$$\begin{aligned}
\xi_{05}(\lambda_j; \lambda_i; \beta) &= (480\beta + 6240)\lambda_j^3 + (-5040 + 2520\beta)\lambda_i\lambda_j^2 \\
&\quad + (-5040 + 2520\beta)\lambda_i^2\lambda_j + (480\beta + 6240)\lambda_i^3, \\
\xi_{05}^{(1)}(\lambda_j; \lambda_i; \beta) &= (-5040 + 2520\beta)\lambda_j^2 + 2(-5040 + 2520\beta)\lambda_i\lambda_j \\
&\quad + 3(480\beta + 6240)\lambda_i^2, \\
\xi_{05}^{(2)}(\lambda_j; \lambda_i; \beta) &= 2(-5040 + 2520\beta)\lambda_j + 6(480\beta + 6240)\lambda_i, \\
\xi_{05}^{(3)}(\lambda_j; \lambda_i; \beta) &= 2(-5040 + 2520\beta) \geq 0 \text{ for all } \beta \geq -13.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{05}(\lambda_i; \lambda_i; \beta) &= 1200\lambda_i^3(5\beta + 2) \geq 0 \text{ for all } \beta \geq -\frac{2}{5}, \\
\xi_{05}^{(1)}(\lambda_i; \lambda_i; \beta) &= 1800\lambda_i^2(5\beta + 2) \geq 0 \text{ for all } \beta \geq -\frac{2}{5}, \\
\xi_{05}^{(2)}(\lambda_i; \lambda_i; \beta) &= 720\lambda_i(38 + 11\beta) \geq 0 \text{ for all } \beta \geq -3.4545.
\end{aligned}$$

Hence, $\xi_0^{(5)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{04}(\lambda_j; \lambda_i; \beta) = \xi_0^{(4)}(\lambda_j; \lambda_j; \lambda_i; \beta)$. We wish to show that ξ_{04} is non-negative for

all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{04} with respect to λ_j gives

$$\begin{aligned}\xi_{04}(\lambda_j; \lambda_i; \beta) &= (672\beta + 8736)\lambda_j^4 + (-6960 + 144\beta^2 + 3192\beta)\lambda_i\lambda_j^3 \\ &\quad + (-3312 + 792\beta + 432\beta^2)\lambda_i^2\lambda_j^2 + (1152\beta + 4320 + 144\beta^2)\lambda_i^3\lambda_j \\ &\quad + (192\beta + 2496)\lambda_i^4, \\ \xi_{04}^{(1)}(\lambda_j; \lambda_i; \beta) &= (2688\beta + 34944)\lambda_j^3 + (-20880 + 432\beta^2 + 9576\beta)\lambda_i\lambda_j^2 \\ &\quad + (-6624 + 1584\beta + 864\beta^2)\lambda_i^2\lambda_j + (1152\beta + 4320 + 144\beta^2)\lambda_i^3, \\ \xi_{04}^{(2)}(\lambda_j; \lambda_i; \beta) &= (8064\beta + 104832)\lambda_j^2 + (-41760 + 864\beta^2 + 19152\beta)\lambda_i\lambda_j \\ &\quad + (-6624 + 1584\beta + 864\beta^2)\lambda_i^2, \\ \xi_{04}^{(3)}(\lambda_j; \lambda_i; \beta) &= (16128\beta + 209664)\lambda_j + (-41760 + 864\beta^2 + 19152\beta)\lambda_i, \\ \xi_{04}^{(4)}(\lambda_j; \lambda_i; \beta) &= 16128\beta + 209664 \geq 0 \text{ for all } \beta \geq -13.\end{aligned}$$

Now

$$\begin{aligned}\xi_{04}(\lambda_i; \lambda_i; \beta) &= 240\lambda_i^4(3\beta + 22)(\beta + 1) \geq 0 \text{ for all } \beta \geq -1, \\ \xi_{04}^{(1)}(\lambda_i; \lambda_i; \beta) &= 120\lambda_i^3(125\beta + 98 + 12\beta^2) \geq 0 \text{ for all } \beta \geq -0.8540, \\ \xi_{04}^{(2)}(\lambda_i; \lambda_i; \beta) &= 576\lambda_i^2(50\beta + 98 + 3\beta^2) \geq 0 \text{ for all } \beta \geq -2.2689, \\ \xi_{04}^{(3)}(\lambda_i; \lambda_i; \beta) &= 144\lambda_i(3\beta + 106)(2\beta + 11) \geq 0 \text{ for all } \beta \geq -5.5.\end{aligned}$$

Hence, $\xi_0^{(4)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{03}(\lambda_j; \lambda_i; \beta) = \xi_0^{(3)}(\lambda_j; \lambda_j; \lambda_i; \beta)/\lambda_j$. We wish to show that ξ_{03} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{03} with respect to λ_j gives

$$\begin{aligned}\xi_{03}(\lambda_j; \lambda_i; \beta) &= (432\beta + 5616)\lambda_j^4 + (-5184 + 2268\beta + 162\beta^2)\lambda_i\lambda_j^3 \\ &\quad + (-360 - 756\beta + 432\beta^2 + 18\beta^3)\lambda_i^2\lambda_j^2 + (624\beta + 1632 + 144\beta^2 + 18\beta^3)\lambda_i^3\lambda_j \\ &\quad + (528\beta + 1752 + 18\beta^2)\lambda_i^4, \\ \xi_{03}^{(1)}(\lambda_j; \lambda_i; \beta) &= (1728\beta + 22464)\lambda_j^3 + (-15552 + 6804\beta + 486\beta^2)\lambda_i\lambda_j^2 \\ &\quad + (-720 - 1512\beta + 864\beta^2 + 36\beta^3)\lambda_i^2\lambda_j + (624\beta + 1632 + 144\beta^2 + 18\beta^3)\lambda_i^3, \\ \xi_{03}^{(2)}(\lambda_j; \lambda_i; \beta) &= (5184\beta + 67392)\lambda_j^2 + (-31104 + 13608\beta + 972\beta^2)\lambda_i\lambda_j \\ &\quad + (-720 - 1512\beta + 864\beta^2 + 36\beta^3)\lambda_i^2, \\ \xi_{03}^{(3)}(\lambda_j; \lambda_i; \beta) &= (10368\beta + 134784)\lambda_j + (-31104 + 13608\beta + 972\beta^2)\lambda_i, \\ \xi_{03}^{(4)}(\lambda_j; \lambda_i; \beta) &= 10368\beta + 134784 \geq 0 \text{ for all } \beta \geq -13.\end{aligned}$$

Now

$$\begin{aligned}\xi_{03}(\lambda_i; \lambda_i; \beta) &= 36\lambda_i^4(\beta + 3)(\beta + 2)(\beta + 16) \geq 0 \text{ for all } \beta \geq -2, \\ \xi_{03}^{(1)}(\lambda_i; \lambda_i; \beta) &= 6\lambda_i^3(1274\beta + 1304 + 249\beta^2 + 9\beta^3) \geq 0 \text{ for all } \beta \geq -1.3744, \\ \xi_{03}^{(2)}(\lambda_i; \lambda_i; \beta) &= 36\lambda_i^2(480\beta + 988 + 51\beta^2 + \beta^3) \geq 0 \text{ for all } \beta \geq -2.9025, \\ \xi_{03}^{(3)}(\lambda_i; \lambda_i; \beta) &= 324\lambda_i(74\beta + 320 + 3\beta^2) \geq 0 \text{ for all } \beta \geq -5.5921.\end{aligned}$$

Hence, $\xi_0^{(3)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{02}(\lambda_j; \lambda_i; \beta) = \xi_0^{(2)}(\lambda_j; \lambda_j; \lambda_i; \beta)/\lambda_j^2$. We wish to show that ξ_{02} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{02} with respect to λ_j gives

$$\begin{aligned}\xi_{02}(\lambda_j; \lambda_i; \beta) &= (2288 + 176\beta)\lambda_j^4 + (-2544 + 1092\beta + 90\beta^2)\lambda_i\lambda_j^3 \\ &\quad + (432 - 792\beta + 252\beta^2 + 18\beta^3)\lambda_i^2\lambda_j^2 + (352 + 280\beta + 12\beta^2 + 30\beta^3)\lambda_i^3\lambda_j \\ &\quad + (372\beta + 480 + 54\beta^2)\lambda_i^4, \\ \xi_{02}^{(1)}(\lambda_j; \lambda_i; \beta) &= (9152 + 704\beta)\lambda_j^3 + (-7632 + 3276\beta + 270\beta^2)\lambda_i\lambda_j^2 \\ &\quad + (864 - 1584\beta + 504\beta^2 + 36\beta^3)\lambda_i^2\lambda_j + (352 + 280\beta + 12\beta^2 + 30\beta^3)\lambda_i^3, \\ \xi_{02}^{(2)}(\lambda_j; \lambda_i; \beta) &= (27456 + 2112\beta)\lambda_j^2 + (-15264 + 6552\beta + 540\beta^2)\lambda_i\lambda_j \\ &\quad + (864 - 1584\beta + 504\beta^2 + 36\beta^3)\lambda_i^2, \\ \xi_{02}^{(3)}(\lambda_j; \lambda_i; \beta) &= (54912 + 4224\beta)\lambda_j + (-15264 + 6552\beta + 540\beta^2)\lambda_i, \\ \xi_{02}^{(4)}(\lambda_j; \lambda_i; \beta) &= (54912 + 4224\beta) \geq 0 \text{ for all } \beta \geq -13.\end{aligned}$$

Now

$$\begin{aligned}\xi_{02}(\lambda_i; \lambda_i; \beta) &= 24\lambda_i^4(\beta + 3)(\beta + 2)(2\beta + 7) \geq 0 \text{ for all } \beta \geq -2, \\ \xi_{02}^{(1)}(\lambda_i; \lambda_i; \beta) &= 6\lambda_i^3(\beta + 3)(\beta + 2)(11\beta + 76) \geq 0 \text{ for all } \beta \geq -2, \\ \xi_{02}^{(2)}(\lambda_i; \lambda_i; \beta) &= 12\lambda_i^2(1088 + 590\beta + 87\beta^2 + 3\beta^3) \geq 0 \text{ for all } \beta \geq -3.1423, \\ \xi_{02}^{(3)}(\lambda_i; \lambda_i; \beta) &= 12\lambda_i(3304 + 898\beta + 45\beta^2) \geq 0 \text{ for all } \beta \geq -4.8657.\end{aligned}$$

Hence, $\xi_0^{(2)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{01}(\lambda_j; \lambda_i; \beta) = \xi_0^{(1)}(\lambda_j; \lambda_j; \lambda_i; \beta)/\lambda_j^3$. We wish to show that ξ_{01} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{01} with respect to λ_j gives

$$\begin{aligned}\xi_{01}(\lambda_j; \lambda_i; \beta) &= (676 + 52\beta)\lambda_j^4 + (385\beta - 902 + 33\beta^2)\lambda_i\lambda_j^3 \\ &\quad + (270 + 108\beta^2 - 387\beta + 9\beta^3)\lambda_i^2\lambda_j^2 + (56 - 21\beta^2 + 21\beta^3 + 140\beta)\lambda_i^3\lambda_j \\ &\quad + (80 + 140\beta + 60\beta^2)\lambda_i^4, \\ \xi_{01}^{(1)}(\lambda_j; \lambda_i; \beta) &= (2704 + 208\beta)\lambda_j^3 + (1155\beta - 2706 + 99\beta^2)\lambda_i\lambda_j^2 \\ &\quad + (540 + 216\beta^2 - 774\beta + 18\beta^3)\lambda_i^2\lambda_j + (56 - 21\beta^2 + 21\beta^3 + 140\beta)\lambda_i^3, \\ \xi_{01}^{(2)}(\lambda_j; \lambda_i; \beta) &= (8112 + 624\beta)\lambda_j^2 + (2310\beta - 5412 + 198\beta^2)\lambda_i\lambda_j \\ &\quad + (540 + 216\beta^2 - 774\beta + 18\beta^3)\lambda_i^2, \\ \xi_{01}^{(3)}(\lambda_j; \lambda_i; \beta) &= (16224 + 1248\beta)\lambda_j + (2310\beta - 5412 + 198\beta^2)\lambda_i, \\ \xi_{01}^{(4)}(\lambda_j; \lambda_i; \beta) &= 16224 + 1248\beta \geq 0 \text{ for all } \beta \geq -13.\end{aligned}$$

Now

$$\begin{aligned}\xi_{01}(\lambda_i; \lambda_i; \beta) &= 30\lambda_i^4(\beta + 3)(\beta + 2)(\beta + 1) \geq 0 \text{ for all } \beta \geq -1, \\ \xi_{01}^{(1)}(\lambda_i; \lambda_i; \beta) &= 3\lambda_i^3(\beta + 3)(\beta + 2)(13\beta + 33) \geq 0 \text{ for all } \beta \geq -2, \\ \xi_{01}^{(2)}(\lambda_i; \lambda_i; \beta) &= 18\lambda_i^2(\beta + 3)(\beta^2 + 20\beta + 60) \geq 0 \text{ for all } \beta \geq -3, \\ \xi_{01}^{(3)}(\lambda_i; \lambda_i; \beta) &= 6\lambda_i(1802 + 593\beta + 33\beta^2) \geq 0 \text{ for all } \beta \geq -4.\end{aligned}$$

Hence, $\xi_0^{(1)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{00}(\lambda_j; \lambda_i; \beta) = \xi_0(\lambda_j; \lambda_j; \lambda_i; \beta)/\lambda_j^4$. We wish to show that ξ_{00} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{00} with respect to λ_j gives

$$\begin{aligned}\xi_{00}(\lambda_j; \lambda_i; \beta) &= (156 + 12\beta)\lambda_j^4 + (105\beta - 246 + 9\beta^2)\lambda_i\lambda_j^3 \\ &\quad + (90 + 36\beta^2 - 129\beta + 3\beta^3)\lambda_i^2\lambda_j^2 + (24 - 9\beta^2 + 9\beta^3 + 60\beta)\lambda_i^3\lambda_j \\ &\quad + (48 + 84\beta + 36\beta^2)\lambda_i^4, \\ \xi_{00}^{(1)}(\lambda_j; \lambda_i; \beta) &= (624 + 48\beta)\lambda_j^3 + (315\beta - 738 + 27\beta^2)\lambda_i\lambda_j^2 \\ &\quad + (180 + 72\beta^2 - 258\beta + 6\beta^3)\lambda_i^2\lambda_j + (24 - 9\beta^2 + 9\beta^3 + 60\beta)\lambda_i^3, \\ \xi_{00}^{(2)}(\lambda_j; \lambda_i; \beta) &= (1872 + 144\beta)\lambda_j^2 + (630\beta - 1476 + 54\beta^2)\lambda_i\lambda_j \\ &\quad + (180 + 72\beta^2 - 258\beta + 6\beta^3)\lambda_i^2, \\ \xi_{00}^{(3)}(\lambda_j; \lambda_i; \beta) &= (3744 + 288\beta)\lambda_j + (630\beta - 1476 + 54\beta^2)\lambda_i, \\ \xi_{00}^{(4)}(\lambda_j; \lambda_i; \beta) &= 3744 + 288\beta \geq 0 \text{ for all } \beta \geq -13.\end{aligned}$$

Now

$$\begin{aligned}\xi_{00}(\lambda_i; \lambda_i; \beta) &= 12\lambda_i^4(\beta + 3)(\beta + 2)(\beta + 1) > 0 \text{ for all } \beta > -1, \\ \xi_{00}^{(1)}(\lambda_i; \lambda_i; \beta) &= 15\lambda_i^3(\beta + 3)(\beta + 2)(\beta + 1) > 0 \text{ for all } \beta > -1, \\ \xi_{00}^{(2)}(\lambda_i; \lambda_i; \beta) &= 6\lambda_i^2(\beta + 3)(\beta + 2)(\beta + 16) > 0 \text{ for all } \beta > -2, \\ \xi_{00}^{(3)}(\lambda_i; \lambda_i; \beta) &= 54\lambda_i(\beta + 14)(\beta + 3) > 0 \text{ for all } \beta > -3.\end{aligned}$$

Hence, ξ_0 is positive for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$. This implies that $\xi(\lambda_l; \lambda_l; \lambda_j; \lambda_i; \beta)$ is positive for all $\lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Differentiating $\xi_1(\lambda_l; \lambda_j; \lambda_i; \beta)$ with respect to λ_l we obtain

$$\begin{aligned}\xi_1(\lambda_l; \lambda_j; \lambda_i; \beta) &= (-168 + 84\beta)(\lambda_i^2\lambda_j + \lambda_i\lambda_j^2)\lambda_l^5 \\ &\quad + (16\beta + 208)(\lambda_j^3 + \lambda_i^3)\lambda_l^5 \\ &\quad + (21\beta^2 + 98\beta - 280)(\lambda_i\lambda_j^3 + \lambda_i^3\lambda_j)\lambda_l^4 \\ &\quad + (364 + 28\beta)(\lambda_j^4 + \lambda_i^4)\lambda_l^4 \\ &\quad + (63\beta^2 - 252\beta + 252)\lambda_i^2\lambda_j^2\lambda_l^4 \\ &\quad + (216 - 144\beta + 9\beta^3)(\lambda_i^2\lambda_j^3 + \lambda_i^3\lambda_j^2)\lambda_l^3 \\ &\quad + (9\beta^2 + 168\beta - 372)(\lambda_i\lambda_j^4 + \lambda_i^4\lambda_j)\lambda_l^3 \\ &\quad + (45\beta^2 - 75\beta - 30)(\lambda_i^2\lambda_j^4 + \lambda_i^4\lambda_j^2)\lambda_l^2 \\ &\quad + (190\beta + 15\beta^3 - 75\beta^2 - 200)\lambda_i^3\lambda_j^3\lambda_l^2 \\ &\quad + (30\beta^2 + 120)(\lambda_i^3\lambda_j^4 + \lambda_i^4\lambda_j^3)\lambda_l \\ &\quad + (30 + 75\beta)\lambda_i^4\lambda_j^4,\end{aligned}$$

$$\begin{aligned}
\xi_1^{(1)}(\lambda_l; \lambda_j; \lambda_i; \beta) &= (420\beta - 840)(\lambda_i^2 \lambda_j + \lambda_i \lambda_j^2) \lambda_l^4 \\
&\quad + (1040 + 80\beta)(\lambda_j^3 + \lambda_i^3) \lambda_l^4 \\
&\quad + (392\beta - 1120 + 84\beta^2)(\lambda_i \lambda_j^3 + \lambda_i^3 \lambda_j) \lambda_l^3 \\
&\quad + (1456 + 112\beta)(\lambda_j^4 + \lambda_i^4) \lambda_l^3 \\
&\quad + (1008 - 1008\beta + 252\beta^2) \lambda_i^2 \lambda_j^2 \lambda_l^3 \\
&\quad + (648 - 432\beta + 27\beta^3)(\lambda_i^2 \lambda_j^3 + \lambda_i^3 \lambda_j^2) \lambda_l^2 \\
&\quad + (504\beta - 1116 + 27\beta^2)(\lambda_i \lambda_j^4 + \lambda_i^4 \lambda_j) \lambda_l^2 \\
&\quad + (90\beta^2 - 60 - 150\beta)(\lambda_i^2 \lambda_j^4 + \lambda_i^4 \lambda_j^2) \lambda_l \\
&\quad + (-400 + 380\beta + 30\beta^3 - 150\beta^2) \lambda_i^3 \lambda_j^3 \lambda_l \\
&\quad + (30\beta^2 + 120)(\lambda_i^3 \lambda_j^4 + \lambda_i^4 \lambda_j^3), \\
\xi_1^{(2)}(\lambda_l; \lambda_j; \lambda_i; \beta) &= (1680\beta - 3360)(\lambda_i^2 \lambda_j + \lambda_i \lambda_j^2) \lambda_l^3 \\
&\quad + (320\beta + 4160)(\lambda_j^3 + \lambda_i^3) \lambda_l^3 \\
&\quad + (-3360 + 252\beta^2 + 1176\beta)(\lambda_i \lambda_j^3 + \lambda_i^3 \lambda_j) \lambda_l^2 \\
&\quad + (4368 + 336\beta)(\lambda_j^4 + \lambda_i^4) \lambda_l^2 \\
&\quad + (3024 + 756\beta^2 - 3024\beta) \lambda_i^2 \lambda_j^2 \lambda_l^2 \\
&\quad + (54\beta^3 - 864\beta + 1296)(\lambda_i^2 \lambda_j^3 + \lambda_i^3 \lambda_j^2) \lambda_l \\
&\quad + (54\beta^2 - 2232 + 1008\beta)(\lambda_i \lambda_j^4 + \lambda_i^4 \lambda_j) \lambda_l \\
&\quad + (-400 + 380\beta + 30\beta^3 - 150\beta^2) \lambda_i^3 \lambda_j^3 \\
&\quad + (90\beta^2 - 60 - 150\beta)(\lambda_i^2 \lambda_j^4 + \lambda_i^4 \lambda_j^2), \\
\xi_1^{(3)}(\lambda_l; \lambda_j; \lambda_i; \beta) &= (-10080 + 5040\beta)(\lambda_i^2 \lambda_j + \lambda_i \lambda_j^2) \lambda_l^2 \\
&\quad + (12480 + 960\beta)(\lambda_j^3 + \lambda_i^3) \lambda_l^2 \\
&\quad + (672\beta + 8736)(\lambda_j^4 + \lambda_i^4) \lambda_l \\
&\quad + (2352\beta + 504\beta^2 - 6720)(\lambda_i \lambda_j^3 + \lambda_i^3 \lambda_j) \lambda_l \\
&\quad + (6048 - 6048\beta + 1512\beta^2) \lambda_i^2 \lambda_j^2 \lambda_l \\
&\quad + (54\beta^2 - 2232 + 1008\beta)(\lambda_i \lambda_j^4 + \lambda_i^4 \lambda_j) \\
&\quad + (54\beta^3 - 864\beta + 1296)(\lambda_i^3 \lambda_j^2 + \lambda_i^2 \lambda_j^3), \\
\xi_1^{(4)}(\lambda_l; \lambda_j; \lambda_i; \beta) &= (-20160 + 10080\beta)(\lambda_i \lambda_j^2 + \lambda_i^2 \lambda_j) \lambda_l \\
&\quad + (24960 + 1920\beta)(\lambda_j^3 + \lambda_i^3) \lambda_l \\
&\quad + (6048 - 6048\beta + 1512\beta^2) \lambda_i^2 \lambda_j^2 \\
&\quad + (672\beta + 8736)(\lambda_j^4 + \lambda_i^4) \\
&\quad + (2352\beta + 504\beta^2 - 6720)(\lambda_i \lambda_j^3 + \lambda_i^3 \lambda_j), \\
\xi_1^{(5)}(\lambda_l; \lambda_j; \lambda_i; \beta) &= (24960 + 1920\beta)(\lambda_j^3 + \lambda_i^3) \\
&\quad + (-20160 + 10080\beta)(\lambda_i \lambda_j^2 + \lambda_i^2 \lambda_j).
\end{aligned}$$

Define $\xi_{15}(\lambda_j; \lambda_i; \beta) = \xi_1^{(5)}(\lambda_i; \lambda_j; \lambda_i; \beta)$. We wish to show that ξ_{15} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{15} with respect to λ_j gives

$$\begin{aligned}\xi_{15}(\lambda_j; \lambda_i; \beta) &= (24960 + 1920\beta)\lambda_j^3 + (-20160 + 10080\beta)\lambda_i\lambda_j^2 \\ &\quad + (-20160 + 10080\beta)\lambda_i^2\lambda_j + (24960 + 1920\beta)\lambda_i^3, \\ \xi_{15}^{(1)}(\lambda_j; \lambda_i; \beta) &= (74880 + 5760\beta)\lambda_j^2 + (-40320 + 20160\beta)\lambda_i\lambda_j \\ &\quad + (-20160 + 10080\beta)\lambda_i^2, \\ \xi_{15}^{(2)}(\lambda_j; \lambda_i; \beta) &= (149760 + 11520\beta)\lambda_j + (-40320 + 20160\beta)\lambda_i, \\ \xi_{15}^{(3)}(\lambda_j; \lambda_i; \beta) &= 149760 + 11520\beta \geq 0 \text{ for all } \beta \geq -\frac{2}{5}.\end{aligned}$$

Now

$$\begin{aligned}\xi_{15}(\lambda_i; \lambda_i; \beta) &= 4800\lambda_i^3(2 + 5\beta) \geq 0 \text{ for all } \beta \geq -\frac{2}{5}, \\ \xi_{15}^{(1)}(\lambda_i; \lambda_i; \beta) &= 7200\lambda_i^2(2 + 5\beta) \geq 0 \text{ for all } \beta \geq -\frac{2}{5}, \\ \xi_{15}^{(2)}(\lambda_i; \lambda_i; \beta) &= 149760 + 11520\beta \geq 0 \text{ for all } \beta \geq -13.\end{aligned}$$

Hence, $\xi_1^{(5)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{14}(\lambda_j; \lambda_i; \beta) = \xi_1^{(4)}(\lambda_j; \lambda_j; \lambda_i; \beta)/\lambda_j$. We wish to show that ξ_{14} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{14} with respect to λ_j gives

$$\begin{aligned}\xi_{14}(\lambda_j; \lambda_i; \beta) &= (33696 + 2592\beta)\lambda_j^4 + (12432\beta + 504\beta^2 - 26880)\lambda_i\lambda_j^3 \\ &\quad + (-14112 + 4032\beta + 1512\beta^2)\lambda_i^2\lambda_j^2 + (18240 + 4272\beta + 504\beta^2)\lambda_i^3\lambda_j \\ &\quad + (672\beta + 8736)\lambda_i^4, \\ \xi_{14}^{(1)}(\lambda_j; \lambda_i; \beta) &= (134784 + 10368\beta)\lambda_j^3 + (37296\beta + 1512\beta^2 - 80640)\lambda_i\lambda_j^2 \\ &\quad + (-28224 + 8064\beta + 3024\beta^2)\lambda_i^2\lambda_j + (18240 + 4272\beta + 504\beta^2)\lambda_i^3, \\ \xi_{14}^{(2)}(\lambda_j; \lambda_i; \beta) &= (404352 + 31104\beta)\lambda_j^2 + (74592\beta + 3024\beta^2 - 161280)\lambda_i\lambda_j \\ &\quad + (-28224 + 8064\beta + 3024\beta^2)\lambda_i^2, \\ \xi_{14}^{(3)}(\lambda_j; \lambda_i; \beta) &= (808704 + 62208\beta)\lambda_j + (74592\beta + 3024\beta^2 - 161280)\lambda_i, \\ \xi_{14}^{(4)}(\lambda_j; \lambda_i; \beta) &= 808704 + 62208\beta \geq 0 \text{ for all } \beta \geq -13.\end{aligned}$$

Now

$$\begin{aligned}\xi_{14}(\lambda_i; \lambda_i; \beta) &= 120\lambda_i^4(164 + 200\beta + 21\beta^2) \geq 0 \text{ for all } \beta \geq -0.9062, \\ \xi_{14}^{(1)}(\lambda_i; \lambda_i; \beta) &= 240\lambda_i^3(184 + 250\beta + 21\beta^2) \geq 0 \text{ for all } \beta \geq -0.7882, \\ \xi_{14}^{(2)}(\lambda_i; \lambda_i; \beta) &= 288\lambda_i^2(746 + 395\beta + 21\beta^2) \geq 0 \text{ for all } \beta \geq -2.1298, \\ \xi_{14}^{(3)}(\lambda_i; \lambda_i; \beta) &= 144\lambda_i(4496 + 950\beta + 21\beta^2) \geq 0 \text{ for all } \beta \geq -5.3701.\end{aligned}$$

Hence, $\xi_1^{(4)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{13}(\lambda_j; \lambda_i; \beta) = \xi_1^{(3)}(\lambda_j; \lambda_j; \lambda_i; \beta)/\lambda_j$. We wish to show that ξ_{13} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{13} with respect to λ_j gives

$$\begin{aligned}\xi_{13}(\lambda_j; \lambda_i; \beta) &= (21216 + 1632\beta)\lambda_j^4 + (-19032 + 8400\beta + 558\beta^2)\lambda_i\lambda_j^3 \\ &\quad + (-2736 - 1872\beta + 1512\beta^2 + 54\beta^3)\lambda_i^2\lambda_j^2 \\ &\quad + (7056 + 2448\beta + 504\beta^2 + 54\beta^3)\lambda_i^3\lambda_j \\ &\quad + (1680\beta + 6504 + 54\beta^2)\lambda_i^4, \\ \xi_{13}^{(1)}(\lambda_j; \lambda_i; \beta) &= (84864 + 6528\beta)\lambda_j^3 + (-57096 + 25200\beta + 1674\beta^2)\lambda_i\lambda_j^2 \\ &\quad + (-5472 - 3744\beta + 3024\beta^2 + 108\beta^3)\lambda_i^2\lambda_j \\ &\quad + (7056 + 2448\beta + 504\beta^2 + 54\beta^3)\lambda_i^3, \\ \xi_{13}^{(2)}(\lambda_j; \lambda_i; \beta) &= (254592 + 19584\beta)\lambda_j^2 + (-114192 + 50400\beta + 3348\beta^2)\lambda_i\lambda_j \\ &\quad + (-5472 - 3744\beta + 3024\beta^2 + 108\beta^3)\lambda_i^2, \\ \xi_{13}^{(3)}(\lambda_j; \lambda_i; \beta) &= (509184 + 39168\beta)\lambda_j + (-114192 + 50400\beta + 3348\beta^2)\lambda_i, \\ \xi_{13}^{(4)}(\lambda_j; \lambda_i; \beta) &= 509184 + 39168\beta \geq 0 \text{ for all } \beta \geq -13.\end{aligned}$$

Now

$$\begin{aligned}\xi_{13}(\lambda_i; \lambda_i; \beta) &= 12\lambda_i^4(1084 + 1024\beta + 219\beta^2 + 9\beta^3) \geq 0 \text{ for all } \beta \geq -1.5245, \\ \xi_{13}^{(1)}(\lambda_i; \lambda_i; \beta) &= 6\lambda_i^3(4892 + 5072\beta + 867\beta^2 + 27\beta^3) \geq 0 \text{ for all } \beta \geq -1.2024, \\ \xi_{13}^{(2)}(\lambda_i; \lambda_i; \beta) &= 36\lambda_i^2(3748 + 1840\beta + 177\beta^2 + 3\beta^3) \geq 0 \text{ for all } \beta \geq -2.7119, \\ \xi_{13}^{(3)}(\lambda_i; \lambda_i; \beta) &= 36\lambda_i(10972 + 2488\beta + 93\beta^2) \geq 0 \text{ for all } \beta \geq -5.5694.\end{aligned}$$

Hence, $\xi_1^{(3)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{12}(\lambda_j; \lambda_i; \beta) = \xi_1^{(2)}(\lambda_j; \lambda_j; \lambda_i; \beta)/\lambda_j^2$. We wish to show that ξ_{12} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{12} with respect to λ_j gives

$$\begin{aligned}\xi_{12}(\lambda_j; \lambda_i; \beta) &= (8528 + 656\beta)\lambda_j^4 + (3864\beta + 306\beta^2 - 8952)\lambda_i\lambda_j^3 \\ &\quad + (900 - 2358\beta + 846\beta^2 + 54\beta^3)\lambda_i^2\lambda_j^2 \\ &\quad + (1696 + 1012\beta + 102\beta^2 + 84\beta^3)\lambda_i^3\lambda_j \\ &\quad + (1194\beta + 2076 + 144\beta^2)\lambda_i^4, \\ \xi_{12}^{(1)}(\lambda_j; \lambda_i; \beta) &= (34112 + 2624\beta)\lambda_j^3 + (11592\beta + 918\beta^2 - 26856)\lambda_i\lambda_j^2 \\ &\quad + (1800 - 4716\beta + 1692\beta^2 + 108\beta^3)\lambda_i^2\lambda_j \\ &\quad + (1696 + 1012\beta + 102\beta^2 + 84\beta^3)\lambda_i^3, \\ \xi_{12}^{(2)}(\lambda_j; \lambda_i; \beta) &= (102336 + 7872\beta)\lambda_j^2 + (23184\beta + 1836\beta^2 - 53712)\lambda_i\lambda_j \\ &\quad + (1800 - 4716\beta + 1692\beta^2 + 108\beta^3)\lambda_i^2, \\ \xi_{12}^{(3)}(\lambda_j; \lambda_i; \beta) &= (204672 + 15744\beta)\lambda_j + (23184\beta + 1836\beta^2 - 53712)\lambda_i, \\ \xi_{12}^{(4)}(\lambda_j; \lambda_i; \beta) &= 204672 + 15744\beta \geq 0 \text{ for all } \beta \geq -13.\end{aligned}$$

Now

$$\begin{aligned}
\xi_{12}(\lambda_i; \lambda_i; \beta) &= 6\lambda_i^4(\beta + 3)(\beta + 2)(23\beta + 118) \geq 0 \text{ for all } \beta \geq -2, \\
\xi_{12}^{(1)}(\lambda_i; \lambda_i; \beta) &= 24\lambda_i^3(448 + 438\beta + 113\beta^2 + 8\beta^3) \geq 0 \text{ for all } \beta \geq -1.6273, \\
\xi_{12}^{(2)}(\lambda_i; \lambda_i; \beta) &= 12\lambda_i^2(4202 + 2195\beta + 294\beta^2 + 9\beta^3) \geq 0 \text{ for all } \beta \geq -3.0300, \\
\xi_{12}^{(3)}(\lambda_i; \lambda_i; \beta) &= 12\lambda_i(12580 + 3244\beta + 153\beta^2) \geq 0 \text{ for all } \beta \geq -5.1090.
\end{aligned}$$

Hence, $\xi_1^{(2)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{11}(\lambda_j; \lambda_i; \beta) = \xi_1^{(1)}(\lambda_j; \lambda_j; \lambda_i; \beta)/\lambda_j^3$. We wish to show that ξ_{11} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{11} with respect to λ_j gives

$$\begin{aligned}
\xi_{11}(\lambda_j; \lambda_i; \beta) &= (2496 + 192\beta)\lambda_j^4 + (111\beta^2 - 3076 + 1316\beta)\lambda_i\lambda_j^3 \\
&\quad + (756 - 1170\beta + 342\beta^2 + 27\beta^3)\lambda_i^2\lambda_j^2 \\
&\quad + (288 + 57\beta^3 + 420\beta - 36\beta^2)\lambda_i^3\lambda_j \\
&\quad + (466\beta + 400 + 147\beta^2)\lambda_i^4, \\
\xi_{11}^{(1)}(\lambda_j; \lambda_i; \beta) &= (9984 + 768\beta)\lambda_j^3 + (333\beta^2 - 9228 + 3948\beta)\lambda_i\lambda_j^2 \\
&\quad + (1512 - 2340\beta + 684\beta^2 + 54\beta^3)\lambda_i^2\lambda_j \\
&\quad + (288 + 57\beta^3 + 420\beta - 36\beta^2)\lambda_i^3, \\
\xi_{11}^{(2)}(\lambda_j; \lambda_i; \beta) &= (29952 + 2304\beta)\lambda_j^2 + (666\beta^2 - 18456 + 7896\beta)\lambda_i\lambda_j \\
&\quad + (1512 - 2340\beta + 684\beta^2 + 54\beta^3)\lambda_i^2, \\
\xi_{11}^{(3)}(\lambda_j; \lambda_i; \beta) &= (59904 + 4608\beta)\lambda_j + (666\beta^2 - 18456 + 7896\beta)\lambda_i, \\
\xi_{11}^{(4)}(\lambda_j; \lambda_i; \beta) &= 59904 + 4608\beta \geq 0 \text{ for all } \beta \geq -13.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{11}(\lambda_i; \lambda_i; \beta) &= 12\lambda_i^4(\beta + 3)(\beta + 2)(7\beta + 12) \geq 0 \text{ for all } \beta \geq -2, \\
\xi_{11}^{(1)}(\lambda_i; \lambda_i; \beta) &= 3\lambda_i^3(\beta + 3)(\beta + 2)(37\beta + 142) \geq 0 \text{ for all } \beta \geq -2, \\
\xi_{11}^{(2)}(\lambda_i; \lambda_i; \beta) &= 6\lambda_i^2(2168 + 1310\beta + 225\beta^2 + 9\beta^3) \geq 0 \text{ for all } \beta \geq -3.1065, \\
\xi_{11}^{(3)}(\lambda_i; \lambda_i; \beta) &= 6\lambda_i(6908 + 2084\beta + 111\beta^2) \geq 0 \text{ for all } \beta \geq -4.2993.
\end{aligned}$$

Hence, $\xi_1^{(1)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{10}(\lambda_j; \lambda_i; \beta) = \xi_1(\lambda_j; \lambda_j; \lambda_i; \beta)/\lambda_j^4$. We wish to show that ξ_{10} is non-negative for

all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{10} with respect to λ_j gives

$$\begin{aligned}\xi_{10}(\lambda_j; \lambda_i; \beta) &= (572 + 44\beta)\lambda_j^4 + (30\beta^2 - 820 + 350\beta)\lambda_i\lambda_j^3 \\ &\quad + (270 - 387\beta + 108\beta^2 + 9\beta^3)\lambda_i^2\lambda_j^2 + (64 + 24\beta^3 + 160\beta - 24\beta^2)\lambda_i^3\lambda_j \\ &\quad + (196\beta + 112 + 84\beta^2)\lambda_i^4, \\ \xi_{10}^{(1)}(\lambda_j; \lambda_i; \beta) &= (2288 + 176\beta)\lambda_j^3 + (90\beta^2 - 2460 + 1050\beta)\lambda_i\lambda_j^2 \\ &\quad + (540 - 774\beta + 216\beta^2 + 18\beta^3)\lambda_i^2\lambda_j + (64 + 24\beta^3 + 160\beta - 24\beta^2)\lambda_i^3, \\ \xi_{10}^{(2)}(\lambda_j; \lambda_i; \beta) &= (6864 + 528\beta)\lambda_j^2 + (180\beta^2 - 4920 + 2100\beta)\lambda_i\lambda_j \\ &\quad + (540 - 774\beta + 216\beta^2 + 18\beta^3)\lambda_i^2, \\ \xi_{10}^{(3)}(\lambda_j; \lambda_i; \beta) &= (13728 + 1056\beta)\lambda_j + (180\beta^2 - 4920 + 2100\beta)\lambda_i, \\ \xi_{10}^{(4)}(\lambda_j; \lambda_i; \beta) &= 13728 + 1056\beta > 0 \text{ for all } \beta > -13.\end{aligned}$$

Now

$$\begin{aligned}\xi_{10}(\lambda_i; \lambda_i; \beta) &= 33\lambda_i^4(\beta + 3)(\beta + 2)(\beta + 1) > 0 \text{ for all } \beta > -1, \\ \xi_{10}^{(1)}(\lambda_i; \lambda_i; \beta) &= 6\lambda_i^3(\beta + 3)(\beta + 2)(7\beta + 12) > 0 \text{ for all } \beta > -1.7143, \\ \xi_{10}^{(2)}(\lambda_i; \lambda_i; \beta) &= 18\lambda_i^2(\beta + 3)(\beta^2 + 19\beta + 46) > 0 \text{ for all } \beta > -2.8479, \\ \xi_{10}^{(3)}(\lambda_i; \lambda_i; \beta) &= 12\lambda_i(734 + 263\beta + 15\beta^2) > 0 \text{ for all } \beta > -3.4826.\end{aligned}$$

Hence, ξ_1 is positive for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$. This implies that $\xi(\lambda_l; \lambda_l; \lambda_j; \lambda_i; \beta)$ is positive for all $\lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Differentiating $\xi_2(\lambda_l; \lambda_j; \lambda_i; \beta)$ with respect to λ_l we obtain

$$\begin{aligned}\xi_2(\lambda_l; \lambda_j; \lambda_i; \beta) &= (624 + 48\beta)(\lambda_j^3 + \lambda_i^3)\lambda_l^5 \\ &\quad + (252\beta - 504)(\lambda_i^2\lambda_j + \lambda_i\lambda_j^2)\lambda_l^5 \\ &\quad + (252\beta + 54\beta^2 - 720)(\lambda_i\lambda_j^3 + \lambda_i^3\lambda_j)\lambda_l^4 \\ &\quad + (72\beta + 936)(\lambda_j^4 + \lambda_i^4)\lambda_l^4 \\ &\quad + (162\beta^2 - 648\beta + 648)\lambda_i^2\lambda_j^2\lambda_l^4 \\ &\quad + (18\beta^3 + 504 - 360\beta + 18\beta^2)(\lambda_i^2\lambda_j^3 + \lambda_i^3\lambda_j^2)\lambda_l^3 \\ &\quad + (18\beta^2 - 828 + 378\beta)(\lambda_i\lambda_j^4 + \lambda_i^4\lambda_j)\lambda_l^3 \\ &\quad + (-36\beta - 216 + 72\beta^2)(\lambda_i^2\lambda_j^4 + \lambda_i^4\lambda_j^2)\lambda_l^2 \\ &\quad + (360\beta - 108\beta^2 + 24\beta^3 - 480)\lambda_i^3\lambda_j^3\lambda_l^2 \\ &\quad + (60\beta + 456 + 36\beta^2)(\lambda_i^3\lambda_j^4 + \lambda_i^4\lambda_j^3)\lambda_l \\ &\quad + (66\beta + 228)\lambda_i^4\lambda_j^4,\end{aligned}$$

$$\begin{aligned}
\xi_2^{(1)}(\lambda_l; \lambda_j; \lambda_i; \beta) &= (1260\beta - 2520)(\lambda_i^2\lambda_j + \lambda_i\lambda_j^2)\lambda_l^4 \\
&\quad + (240\beta + 3120)(\lambda_j^3 + \lambda_i^3)\lambda_l^4 \\
&\quad + (1008\beta + 216\beta^2 - 2880)(\lambda_i\lambda_j^3 + \lambda_i^3\lambda_j)\lambda_l^3 \\
&\quad + (288\beta + 3744)(\lambda_j^4 + \lambda_i^4)\lambda_l^3 \\
&\quad + (-2592\beta + 648\beta^2 + 2592)\lambda_i^2\lambda_j^2\lambda_l^3 \\
&\quad + (1134\beta + 54\beta^2 - 2484)(\lambda_i\lambda_j^4 + \lambda_i^4\lambda_j)\lambda_l^2 \\
&\quad + (1512 - 1080\beta + 54\beta^3 + 54\beta^2)(\lambda_i^2\lambda_j^3 + \lambda_i^3\lambda_j^2)\lambda_l^2 \\
&\quad + (144\beta^2 - 72\beta - 432)(\lambda_i^2\lambda_j^4 + \lambda_i^4\lambda_j^2)\lambda_l \\
&\quad + (720\beta - 216\beta^2 - 960 + 48\beta^3)\lambda_i^3\lambda_j^3\lambda_l \\
&\quad + (60\beta + 456 + 36\beta^2)(\lambda_i^3\lambda_j^4 + \lambda_i^4\lambda_j^3), \\
\xi_2^{(2)}(\lambda_l; \lambda_j; \lambda_i; \beta) &= (-10080 + 5040\beta)(\lambda_i^2\lambda_j + \lambda_i\lambda_j^2)\lambda_l^3 \\
&\quad + (12480 + 960\beta)(\lambda_j^3 + \lambda_i^3)\lambda_l^3 \\
&\quad + (3024\beta - 8640 + 648\beta^2)(\lambda_i\lambda_j^3 + \lambda_i^3\lambda_j)\lambda_l^2 \\
&\quad + (864\beta + 11232)(\lambda_j^4 + \lambda_i^4)\lambda_l^2 \\
&\quad + (7776 + 1944\beta^2 - 7776\beta)\lambda_i^2\lambda_j^2\lambda_l^2 \\
&\quad + (108\beta^3 - 2160\beta + 3024 + 108\beta^2)(\lambda_i^2\lambda_j^3 + \lambda_i^3\lambda_j^2)\lambda_l \\
&\quad + (2268\beta + 108\beta^2 - 4968)(\lambda_i\lambda_j^4 + \lambda_i^4\lambda_j)\lambda_l \\
&\quad + (720\beta - 216\beta^2 - 960 + 48\beta^3)\lambda_i^3\lambda_j^3 \\
&\quad + (144\beta^2 - 72\beta - 432)(\lambda_i^2\lambda_j^4 + \lambda_i^4\lambda_j^2), \\
\xi_2^{(3)}(\lambda_l; \lambda_j; \lambda_i; \beta) &= (15120\beta - 30240)(\lambda_i^2\lambda_j + \lambda_i\lambda_j^2)\lambda_l^2 \\
&\quad + (37440 + 2880\beta)(\lambda_j^3 + \lambda_i^3)\lambda_l^2 \\
&\quad + (1728\beta + 22464)(\lambda_j^4 + \lambda_i^4)\lambda_l \\
&\quad + (1296\beta^2 - 17280 + 6048\beta)(\lambda_i\lambda_j^3 + \lambda_i^3\lambda_j)\lambda_l \\
&\quad + (-15552\beta + 15552 + 3888\beta^2)\lambda_i^2\lambda_j^2 \\
&\quad + (108\beta^3 - 2160\beta + 3024 + 108\beta^2)(\lambda_i^2\lambda_j^3 + \lambda_i^3\lambda_j^2) \\
&\quad + (2268\beta + 108\beta^2 - 4968)(\lambda_i\lambda_j^4 + \lambda_i^4\lambda_j), \\
\xi_2^{(4)}(\lambda_l; \lambda_j; \lambda_i; \beta) &= (30240\beta - 60480)(\lambda_i\lambda_j^2 + \lambda_i^2\lambda_j)\lambda_l \\
&\quad + (74880 + 5760\beta)(\lambda_j^3 + \lambda_i^3)\lambda_l \\
&\quad + (-15552\beta + 15552 + 3888\beta^2)\lambda_i^2\lambda_j^2 \\
&\quad + (1728\beta + 22464)(\lambda_j^4 + \lambda_i^4) \\
&\quad + (1296\beta^2 - 17280 + 6048\beta)(\lambda_i\lambda_j^3 + \lambda_i^3\lambda_j), \\
\xi_2^{(5)}(\lambda_l; \lambda_j; \lambda_i; \beta) &= (30240\beta - 60480)(\lambda_i\lambda_j^2 + \lambda_i^2\lambda_j) \\
&\quad + (74880 + 5760\beta)(\lambda_j^3 + \lambda_i^3).
\end{aligned}$$

Define $\xi_{25}(\lambda_j; \lambda_i; \beta) = \xi_2^{(5)}(\lambda_i; \lambda_j; \lambda_i; \beta)$. We wish to show that ξ_{25} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{25} with respect to λ_j gives

$$\begin{aligned}\xi_{25}(\lambda_j; \lambda_i; \beta) &= (74880 + 5760\beta)\lambda_j^3 + (30240\beta - 60480)\lambda_i\lambda_j^2 \\ &\quad + (30240\beta - 60480)\lambda_i^2\lambda_j + (74880 + 5760\beta)\lambda_i^3, \\ \xi_{25}^{(1)}(\lambda_j; \lambda_i; \beta) &= (224640 + 17280\beta)\lambda_j^2 + (60480\beta - 120960)\lambda_i\lambda_j \\ &\quad + (30240\beta - 60480)\lambda_i^2, \\ \xi_{25}^{(2)}(\lambda_j; \lambda_i; \beta) &= (449280 + 34560\beta)\lambda_j + (60480\beta - 120960)\lambda_i, \\ \xi_{25}^{(3)}(\lambda_j; \lambda_i; \beta) &= 449280 + 34560\beta \geq 0 \text{ for all } \beta \geq -13.\end{aligned}$$

Now

$$\begin{aligned}\xi_{25}(\lambda_i; \lambda_i; \beta) &= 14400\lambda_i^3(2 + 5\beta) \geq 0 \text{ for all } \beta \geq -\frac{2}{5}, \\ \xi_{25}^{(1)}(\lambda_i; \lambda_i; \beta) &= 21600\lambda_i^2(2 + 5\beta) \geq 0 \text{ for all } \beta \geq -\frac{2}{5}, \\ \xi_{25}^{(2)}(\lambda_i; \lambda_i; \beta) &= 8640\lambda_i(38 + 11\beta) \geq 0 \text{ for all } \beta \geq -3.4545.\end{aligned}$$

Hence, $\xi_2^{(5)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{24}(\lambda_j; \lambda_i; \beta) = \xi_2^{(4)}(\lambda_i; \lambda_j; \lambda_i; \beta)$. We wish to show that ξ_{24} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{24} with respect to λ_j gives

$$\begin{aligned}\xi_{24}(\lambda_j; \lambda_i; \beta) &= (97344 + 7488\beta)\lambda_j^4 + (1296\beta^2 - 77760 + 36288\beta)\lambda_i\lambda_j^3 \\ &\quad + (14688\beta - 44928 + 3888\beta^2)\lambda_i^2\lambda_j^2 + (57600 + 11808\beta + 1296\beta^2)\lambda_i^3\lambda_j \\ &\quad + (1728\beta + 22464)\lambda_i^4, \\ \xi_{24}^{(1)}(\lambda_j; \lambda_i; \beta) &= (389376 + 29952\beta)\lambda_j^3 + (3888\beta^2 - 233280 + 108864\beta)\lambda_i\lambda_j^2 \\ &\quad + (29376\beta - 89856 + 7776\beta^2)\lambda_i^2\lambda_j + (57600 + 11808\beta + 1296\beta^2)\lambda_i^3, \\ \xi_{24}^{(2)}(\lambda_j; \lambda_i; \beta) &= (1168128 + 89856\beta)\lambda_j^2 + (7776\beta^2 - 466560 + 217728\beta)\lambda_i\lambda_j \\ &\quad + (29376\beta - 89856 + 7776\beta^2)\lambda_i^2, \\ \xi_{24}^{(3)}(\lambda_j; \lambda_i; \beta) &= (2336256 + 179712\beta)\lambda_j + (7776\beta^2 - 466560 + 217728\beta)\lambda_i, \\ \xi_{24}^{(4)}(\lambda_j; \lambda_i; \beta) &= 2336256 + 179712\beta \geq 0 \text{ for all } \beta \geq -13.\end{aligned}$$

Now

$$\begin{aligned}\xi_{24}(\lambda_i; \lambda_i; \beta) &= 720\lambda_i^4(76 + 100\beta + 9\beta^2) \geq 0 \text{ for all } \beta \geq -0.8206, \\ \xi_{24}^{(1)}(\lambda_i; \lambda_i; \beta) &= 1440\lambda_i^3(86 + 125\beta + 9\beta^2) \geq 0 \text{ for all } \beta \geq -0.7259, \\ \xi_{24}^{(2)}(\lambda_i; \lambda_i; \beta) &= 5184\lambda_i^2(3\beta + 59)(\beta + 2) \geq 0 \text{ for all } \beta \geq -2, \\ \xi_{24}^{(3)}(\lambda_i; \lambda_i; \beta) &= 864\lambda_i(2164 + 460\beta + 9\beta^2) \geq 0 \text{ for all } \beta \geq -5.2420.\end{aligned}$$

Hence, $\xi_2^{(4)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{23}(\lambda_j; \lambda_i; \beta) = \xi_2^{(3)}(\lambda_j; \lambda_j; \lambda_i; \beta)/\lambda_j$. We wish to show that ξ_{23} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{23} with respect to λ_j gives

$$\begin{aligned}
\xi_{23}(\lambda_j; \lambda_i; \beta) &= (59904 + 4608\beta)\lambda_j^4 + (23436\beta - 52488 + 1404\beta^2)\lambda_i\lambda_j^3 \\
&\quad + (-2592\beta - 11664 + 3996\beta^2 + 108\beta^3)\lambda_i^2\lambda_j^2 \\
&\quad + (23184 + 6768\beta + 1404\beta^2 + 108\beta^3)\lambda_i^3\lambda_j \\
&\quad + (3996\beta + 17496 + 108\beta^2)\lambda_i^4, \\
\xi_{23}^{(1)}(\lambda_j; \lambda_i; \beta) &= (239616 + 18432\beta)\lambda_j^3 + (70308\beta - 157464 + 4212\beta^2)\lambda_i\lambda_j^2 \\
&\quad + (-5184\beta - 23328 + 7992\beta^2 + 216\beta^3)\lambda_i^2\lambda_j \\
&\quad + (23184 + 6768\beta + 1404\beta^2 + 108\beta^3)\lambda_i^3, \\
\xi_{23}^{(2)}(\lambda_j; \lambda_i; \beta) &= (718848 + 55296\beta)\lambda_j^2 + (140616\beta - 314928 + 8424\beta^2)\lambda_i\lambda_j \\
&\quad + (-5184\beta - 23328 + 7992\beta^2 + 216\beta^3)\lambda_i^2, \\
\xi_{23}^{(3)}(\lambda_j; \lambda_i; \beta) &= (1437696 + 110592\beta)\lambda_j + (140616\beta - 314928 + 8424\beta^2)\lambda_i, \\
\xi_{23}^{(4)}(\lambda_j; \lambda_i; \beta) &= 1437696 + 110592\beta \geq 0 \text{ for all } \beta \geq -13.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{23}(\lambda_i; \lambda_i; \beta) &= 72\lambda_i^4(506 + 503\beta + 96\beta^2 + 3\beta^3) \geq 0 \text{ for all } \beta \geq -1.3291, \\
\xi_{23}^{(1)}(\lambda_i; \lambda_i; \beta) &= 36\lambda_i^3(2278 + 2509\beta + 378\beta^2 + 9\beta^3) \geq 0 \text{ for all } \beta \geq -1.0787, \\
\xi_{23}^{(2)}(\lambda_i; \lambda_i; \beta) &= 216\lambda_i^2(1762 + 883\beta + 76\beta^2 + \beta^3) \geq 0 \text{ for all } \beta \geq -2.5267, \\
\xi_{23}^{(3)}(\lambda_i; \lambda_i; \beta) &= 216\lambda_i(5198 + 1163\beta + 39\beta^2) \geq 0 \text{ for all } \beta \geq -5.4745.
\end{aligned}$$

Hence, $\xi_2^{(3)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{22}(\lambda_j; \lambda_i; \beta) = \xi_2^{(2)}(\lambda_j; \lambda_j; \lambda_i; \beta)/\lambda_j^2$. We wish to show that ξ_{22} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{22} with respect to λ_j gives

$$\begin{aligned}
\xi_{22}(\lambda_j; \lambda_i; \beta) &= (23712 + 1824\beta)\lambda_j^4 + (10332\beta + 756\beta^2 - 23688)\lambda_i\lambda_j^3 \\
&\quad + (288 - 4968\beta + 2196\beta^2 + 108\beta^3)\lambda_i^2\lambda_j^2 \\
&\quad + (5904 + 2544\beta + 540\beta^2 + 156\beta^3)\lambda_i^3\lambda_j \\
&\quad + (3060\beta + 5832 + 252\beta^2)\lambda_i^4, \\
\xi_{22}^{(1)}(\lambda_j; \lambda_i; \beta) &= (94848 + 7296\beta)\lambda_j^3 + (30996\beta + 2268\beta^2 - 71064)\lambda_i\lambda_j^2 \\
&\quad + (576 - 9936\beta + 4392\beta^2 + 216\beta^3)\lambda_i^2\lambda_j \\
&\quad + (5904 + 2544\beta + 540\beta^2 + 156\beta^3)\lambda_i^3, \\
\xi_{22}^{(2)}(\lambda_j; \lambda_i; \beta) &= (284544 + 21888\beta)\lambda_j^2 + (61992\beta + 4536\beta^2 - 142128)\lambda_i\lambda_j \\
&\quad + (576 - 9936\beta + 4392\beta^2 + 216\beta^3)\lambda_i^2, \\
\xi_{22}^{(3)}(\lambda_j; \lambda_i; \beta) &= (569088 + 43776\beta)\lambda_j + (61992\beta + 4536\beta^2 - 142128)\lambda_i, \\
\xi_{22}^{(4)}(\lambda_j; \lambda_i; \beta) &= 569088 + 43776\beta \geq 0 \text{ for all } \beta \geq -13.
\end{aligned}$$

Now

$$\begin{aligned}\xi_{22}(\lambda_i; \lambda_i; \beta) &= 24\lambda_i^4(502 + 533\beta + 156\beta^2 + 11\beta^3) \geq 0 \text{ for all } \beta \geq -1.6304, \\ \xi_{22}^{(1)}(\lambda_i; \lambda_i; \beta) &= 12\lambda_i^3(2522 + 2575\beta + 600\beta^2 + 31\beta^3) \geq 0 \text{ for all } \beta \geq -1.4074, \\ \xi_{22}^{(2)}(\lambda_i; \lambda_i; \beta) &= 72\lambda_i^2(1986 + 1027\beta + 124\beta^2 + 3\beta^3) \geq 0 \text{ for all } \beta \geq -2.8419, \\ \xi_{22}^{(3)}(\lambda_i; \lambda_i; \beta) &= 72\lambda_i(5930 + 1469\beta + 63\beta^2) \geq 0 \text{ for all } \beta \geq -5.1935.\end{aligned}$$

Hence, $\xi_2^{(2)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{21}(\lambda_j; \lambda_i; \beta) = \xi_2^{(1)}(\lambda_j; \lambda_j; \lambda_i; \beta)/\lambda_j^3$. We wish to show that ξ_{21} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{21} with respect to λ_j gives

$$\begin{aligned}\xi_{21}(\lambda_j; \lambda_i; \beta) &= (6864 + 528\beta)\lambda_j^4 + (3402\beta + 270\beta^2 - 7884)\lambda_i\lambda_j^3 \\ &\quad + (1152 - 2484\beta + 846\beta^2 + 54\beta^3)\lambda_i^2\lambda_j^2 \\ &\quad + (1248 + 948\beta + 90\beta^2 + 102\beta^3)\lambda_i^3\lambda_j \\ &\quad + (1410\beta + 1284 + 234\beta^2)\lambda_i^4, \\ \xi_{21}^{(1)}(\lambda_j; \lambda_i; \beta) &= (27456 + 2112\beta)\lambda_j^3 + (10206\beta + 810\beta^2 - 23652)\lambda_i\lambda_j^2 \\ &\quad + (2304 - 4968\beta + 1692\beta^2 + 108\beta^3)\lambda_i^2\lambda_j \\ &\quad + (1248 + 948\beta + 90\beta^2 + 102\beta^3)\lambda_i^3, \\ \xi_{21}^{(2)}(\lambda_j; \lambda_i; \beta) &= (82368 + 6336\beta)\lambda_j^2 + (20412\beta + 1620\beta^2 - 47304)\lambda_i\lambda_j \\ &\quad + (2304 - 4968\beta + 1692\beta^2 + 108\beta^3)\lambda_i^2, \\ \xi_{21}^{(3)}(\lambda_j; \lambda_i; \beta) &= (164736 + 12672\beta)\lambda_j + (20412\beta + 1620\beta^2 - 47304)\lambda_i, \\ \xi_{21}^{(4)}(\lambda_j; \lambda_i; \beta) &= 164736 + 12672\beta \geq 0 \text{ for all } \beta \geq -13.\end{aligned}$$

Now

$$\begin{aligned}\xi_{21}(\lambda_i; \lambda_i; \beta) &= 12\lambda_i^4(\beta + 3)(13\beta^2 + 81\beta + 74) \geq 0 \text{ for all } \beta \geq -1.1121, \\ \xi_{21}^{(1)}(\lambda_i; \lambda_i; \beta) &= 6\lambda_i^3(1226 + 1383\beta + 432\beta^2 + 35\beta^3) \geq 0 \text{ for all } \beta \geq -1.5169, \\ \xi_{21}^{(2)}(\lambda_i; \lambda_i; \beta) &= 36\lambda_i^2(1038 + 605\beta + 92\beta^2 + 3\beta^3) \geq 0 \text{ for all } \beta \geq -2.7964, \\ \xi_{21}^{(3)}(\lambda_i; \lambda_i; \beta) &= 36\lambda_i(3262 + 919\beta + 45\beta^2) \geq 0 \text{ for all } \beta \geq -4.5739.\end{aligned}$$

Hence, $\xi_2^{(1)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{20}(\lambda_j; \lambda_i; \beta) = \xi_2(\lambda_j; \lambda_j; \lambda_i; \beta)/\lambda_j^4$. We wish to show that ξ_{20} is non-negative for

all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{20} with respect to λ_j gives

$$\begin{aligned}\xi_{20}(\lambda_j; \lambda_i; \beta) &= (1560 + 120\beta)\lambda_j^4 + (882\beta + 72\beta^2 - 2052)\lambda_i\lambda_j^3 \\ &\quad + (432 + 18\beta^3 - 792\beta + 252\beta^2)\lambda_i^2\lambda_j^2 + (384 + 360\beta + 42\beta^3)\lambda_i^3\lambda_j \\ &\quad + (540\beta + 126\beta^2 + 576)\lambda_i^4, \\ \xi_{20}^{(1)}(\lambda_j; \lambda_i; \beta) &= (6240 + 480\beta)\lambda_j^3 + (2646\beta + 216\beta^2 - 6156)\lambda_i\lambda_j^2 \\ &\quad + (864 + 36\beta^3 - 1584\beta + 504\beta^2)\lambda_i^2\lambda_j + (384 + 360\beta + 42\beta^3)\lambda_i^3, \\ \xi_{20}^{(2)}(\lambda_j; \lambda_i; \beta) &= (18720 + 1440\beta)\lambda_j^2 + (5292\beta + 432\beta^2 - 12312)\lambda_i\lambda_j \\ &\quad + (864 + 36\beta^3 - 1584\beta + 504\beta^2)\lambda_i^2, \\ \xi_{20}^{(3)}(\lambda_j; \lambda_i; \beta) &= (37440 + 2880\beta)\lambda_j + (5292\beta + 432\beta^2 - 12312)\lambda_i, \\ \xi_{20}^{(4)}(\lambda_j; \lambda_i; \beta) &= 37440 + 2880\beta > 0 \text{ for all } \beta > -13.\end{aligned}$$

Now

$$\begin{aligned}\xi_{20}(\lambda_i; \lambda_i; \beta) &= 30\lambda_i^4(2\beta + 5)(\beta + 3)(\beta + 2) > 0 \text{ for all } \beta > -2, \\ \xi_{20}^{(1)}(\lambda_i; \lambda_i; \beta) &= 6\lambda_i^3(\beta + 3)(13\beta^2 + 81\beta + 74) > 0 \text{ for all } \beta > -3, \\ \xi_{20}^{(2)}(\lambda_i; \lambda_i; \beta) &= 36\lambda_i^2(202 + 143\beta + 26\beta^2 + \beta^3) > 0 \text{ for all } \beta > -2.2618, \\ \xi_{20}^{(3)}(\lambda_i; \lambda_i; \beta) &= 36\lambda_i(698 + 227\beta + 12\beta^2) > 0 \text{ for all } \beta > -3.8643.\end{aligned}$$

Hence, ξ_2 is positive for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$. This implies that $\xi(\lambda_l; \lambda_l; \lambda_j; \lambda_i; \beta)$ is positive for all $\lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Differentiating $\xi_3(\lambda_l; \lambda_j; \lambda_i; \beta)$ with respect to λ_l we obtain

$$\begin{aligned}\xi_3(\lambda_l; \lambda_j; \lambda_i; \beta) &= (-1008 + 504\beta)(\lambda_i^2\lambda_j + \lambda_i\lambda_j^2)\lambda_l^5 \\ &\quad + (96\beta + 1248)(\lambda_j^3 + \lambda_i^3)\lambda_l^5 \\ &\quad + (-1200 + 90\beta^2 + 420\beta)(\lambda_i\lambda_j^3 + \lambda_i^3\lambda_j)\lambda_l^4 \\ &\quad + (120\beta + 1560)(\lambda_j^4 + \lambda_i^4)\lambda_l^4 \\ &\quad + (1080 + 270\beta^2 - 1080\beta)\lambda_i^2\lambda_j^2\lambda_l^4 \\ &\quad + (-720\beta + 108\beta^2 + 18\beta^3 + 864)(\lambda_i^2\lambda_j^3 + \lambda_i^3\lambda_j^2)\lambda_l^3 \\ &\quad + (18\beta^2 - 1248 + 588\beta)(\lambda_i\lambda_j^4 + \lambda_i^4\lambda_j)\lambda_l^3 \\ &\quad + (-792 + 54\beta^2 + 288\beta)(\lambda_i^2\lambda_j^4 + \lambda_i^4\lambda_j^2)\lambda_l^2 \\ &\quad + (-960 + 480\beta - 36\beta^2 + 18\beta^3)\lambda_i^3\lambda_j^3\lambda_l^2 \\ &\quad + (1008 + 18\beta^2 + 180\beta)(\lambda_i^3\lambda_j^4 + \lambda_i^4\lambda_j^3)\lambda_l \\ &\quad + (24\beta + 312)\lambda_i^4\lambda_j^4,\end{aligned}$$

$$\begin{aligned}
 \xi_3^{(1)}(\lambda_l; \lambda_j; \lambda_i; \beta) &= (-5040 + 2520\beta)(\lambda_i^2\lambda_j + \lambda_i\lambda_j^2)\lambda_l^4 \\
 &+ (6240 + 480\beta)(\lambda_j^3 + \lambda_i^3)\lambda_l^4 \\
 &+ (-4800 + 1680\beta + 360\beta^2)(\lambda_i\lambda_j^3 + \lambda_i^3\lambda_j)\lambda_l^3 \\
 &+ (6240 + 480\beta)(\lambda_j^4 + \lambda_i^4)\lambda_l^3 \\
 &+ (4320 - 4320\beta + 1080\beta^2)\lambda_i^2\lambda_j^2\lambda_l^3 \\
 &+ (54\beta^3 - 2160\beta + 2592 + 324\beta^2)(\lambda_i^2\lambda_j^3 + \lambda_i^3\lambda_j^2)\lambda_l^2 \\
 &+ (1764\beta + 54\beta^2 - 3744)(\lambda_i\lambda_j^4 + \lambda_i^4\lambda_j)\lambda_l^2 \\
 &+ (-1584 + 576\beta + 108\beta^2)(\lambda_i^2\lambda_j^4 + \lambda_i^4\lambda_j^2)\lambda_l \\
 &+ (36\beta^3 + 960\beta - 1920 - 72\beta^2)\lambda_i^3\lambda_j^3\lambda_l \\
 &+ (1008 + 18\beta^2 + 180\beta)(\lambda_i^3\lambda_j^4 + \lambda_i^4\lambda_j^3), \\
 \xi_3^{(2)}(\lambda_l; \lambda_j; \lambda_i; \beta) &= (-20160 + 10080\beta)(\lambda_i^2\lambda_j + \lambda_i\lambda_j^2)\lambda_l^3 \\
 &+ (24960 + 1920\beta)(\lambda_j^3 + \lambda_i^3)\lambda_l^3 \\
 &+ (5040\beta - 14400 + 1080\beta^2)(\lambda_i\lambda_j^3 + \lambda_i^3\lambda_j)\lambda_l^2 \\
 &+ (18720 + 1440\beta)(\lambda_j^4 + \lambda_i^4)\lambda_l^2 \\
 &+ (-12960\beta + 3240\beta^2 + 12960)\lambda_i^2\lambda_j^2\lambda_l^2 \\
 &+ (108\beta^3 - 4320\beta + 648\beta^2 + 5184)(\lambda_i^2\lambda_j^3 + \lambda_i^3\lambda_j^2)\lambda_l \\
 &+ (108\beta^2 - 7488 + 3528\beta)(\lambda_i\lambda_j^4 + \lambda_i^4\lambda_j)\lambda_l \\
 &+ (36\beta^3 + 960\beta - 1920 - 72\beta^2)\lambda_i^3\lambda_j^3 \\
 &+ (-1584 + 576\beta + 108\beta^2)(\lambda_i^2\lambda_j^4 + \lambda_i^4\lambda_j^2), \\
 \xi_3^{(3)}(\lambda_l; \lambda_j; \lambda_i; \beta) &= (-60480 + 30240\beta)(\lambda_i^2\lambda_j + \lambda_i\lambda_j^2)\lambda_l^2 \\
 &+ (5760\beta + 74880)(\lambda_j^3 + \lambda_i^3)\lambda_l^2 \\
 &+ (37440 + 2880\beta)(\lambda_j^4 + \lambda_i^4)\lambda_l \\
 &+ (10080\beta - 28800 + 2160\beta^2)(\lambda_i\lambda_j^3 + \lambda_i^3\lambda_j)\lambda_l \\
 &+ (6480\beta^2 - 25920\beta + 25920)\lambda_i^2\lambda_j^2\lambda_l \\
 &+ (108\beta^3 - 4320\beta + 648\beta^2 + 5184)(\lambda_i^2\lambda_j^3 + \lambda_i^3\lambda_j^2) \\
 &+ (108\beta^2 - 7488 + 3528\beta)(\lambda_i\lambda_j^4 + \lambda_i^4\lambda_j), \\
 \xi_3^{(4)}(\lambda_l; \lambda_j; \lambda_i; \beta) &= 2(-60480 + 30240\beta)(\lambda_i^2\lambda_j + \lambda_i\lambda_j^2)\lambda_l \\
 &+ 2(5760\beta + 74880)(\lambda_j^3 + \lambda_i^3)\lambda_l \\
 &+ (37440 + 2880\beta)(\lambda_j^4 + \lambda_i^4) \\
 &+ (10080\beta - 28800 + 2160\beta^2)(\lambda_i\lambda_j^3 + \lambda_i^3\lambda_j) \\
 &+ (6480\beta^2 - 25920\beta + 25920)\lambda_i^2\lambda_j^2, \\
 \xi_3^{(5)}(\lambda_l; \lambda_j; \lambda_i; \beta) &= 2(-60480 + 30240\beta)(\lambda_i^2\lambda_j + \lambda_i\lambda_j^2) \\
 &+ 2(5760\beta + 74880)(\lambda_j^3 + \lambda_i^3).
 \end{aligned}$$

Define $\xi_{35}(\lambda_j; \lambda_i; \beta) = \xi_3^{(5)}(\lambda_i; \lambda_j; \lambda_i; \beta)$. We wish to show that ξ_{35} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{35} with respect to λ_j gives

$$\begin{aligned}\xi_{35}(\lambda_j; \lambda_i; \beta) &= (11520\beta + 149760)\lambda_j^3 + (-120960 + 60480\beta)\lambda_i\lambda_j^2 \\ &\quad + (-120960 + 60480\beta)\lambda_i^2\lambda_j + (11520\beta + 149760)\lambda_i^3, \\ \xi_{35}^{(1)}(\lambda_j; \lambda_i; \beta) &= (34560\beta + 449280)\lambda_j^2 + (-241920 + 120960\beta)\lambda_i\lambda_j \\ &\quad + (-120960 + 60480\beta)\lambda_i^2, \\ \xi_{35}^{(2)}(\lambda_j; \lambda_i; \beta) &= (69120\beta + 898560)\lambda_j + (-241920 + 120960\beta)\lambda_i, \\ \xi_{35}^{(3)}(\lambda_j; \lambda_i; \beta) &= 69120\beta + 898560 \geq 0 \text{ for all } \beta \geq -13.\end{aligned}$$

Now

$$\begin{aligned}\xi_{35}(\lambda_i; \lambda_i; \beta) &= 28800\lambda_i^3(2 + 5\beta) \geq 0 \text{ for all } \beta \geq -\frac{2}{5}, \\ \xi_{35}^{(1)}(\lambda_i; \lambda_i; \beta) &= 43200\lambda_i^2(2 + 5\beta) \geq 0 \text{ for all } \beta \geq -\frac{2}{5}, \\ \xi_{35}^{(2)}(\lambda_i; \lambda_i; \beta) &= 17280\lambda_i(38 + 11\beta) \geq 0 \text{ for all } \beta \geq -3.4545.\end{aligned}$$

Hence, $\xi_3^{(5)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{34}(\lambda_j; \lambda_i; \beta) = \xi_3^{(4)}(\lambda_i; \lambda_j; \lambda_i; \beta)$. We wish to show that ξ_{34} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{34} with respect to λ_j gives

$$\begin{aligned}\xi_{34}(\lambda_j; \lambda_i; \beta) &= (14400\beta + 187200)\lambda_j^4 + (70560\beta - 149760 + 2160\beta^2)\lambda_i\lambda_j^3 \\ &\quad + (-95040 + 34560\beta + 6480\beta^2)\lambda_i^2\lambda_j^2 + (21600\beta + 120960 + 2160\beta^2)\lambda_i^3\lambda_j \\ &\quad + (37440 + 2880\beta)\lambda_i^4, \\ \xi_{34}^{(1)}(\lambda_j; \lambda_i; \beta) &= (57600\beta + 748800)\lambda_j^3 + (211680\beta - 449280 + 6480\beta^2)\lambda_i\lambda_j^2 \\ &\quad + (-190080 + 69120\beta + 12960\beta^2)\lambda_i^2\lambda_j + (21600\beta + 120960 + 2160\beta^2)\lambda_i^3, \\ \xi_{34}^{(2)}(\lambda_j; \lambda_i; \beta) &= (172800\beta + 2246400)\lambda_j^2 + (423360\beta - 898560 + 12960\beta^2)\lambda_i\lambda_j \\ &\quad + (-190080 + 69120\beta + 12960\beta^2)\lambda_i^2, \\ \xi_{34}^{(3)}(\lambda_j; \lambda_i; \beta) &= (345600\beta + 4492800)\lambda_j + (423360\beta - 898560 + 12960\beta^2)\lambda_i, \\ \xi_{34}^{(4)}(\lambda_j; \lambda_i; \beta) &= 345600\beta + 4492800 \geq 0 \text{ for all } \beta \geq -13.\end{aligned}$$

Now

$$\begin{aligned}\xi_{34}(\lambda_i; \lambda_i; \beta) &= 3600\lambda_i^4(28 + 40\beta + 3\beta^2) \geq 0 \text{ for all } \beta \geq -0.7412, \\ \xi_{34}^{(1)}(\lambda_i; \lambda_i; \beta) &= 7200\lambda_i^3(\beta + 16)(3\beta + 2) \geq 0 \text{ for all } \beta \geq -\frac{2}{3}, \\ \xi_{34}^{(2)}(\lambda_i; \lambda_i; \beta) &= 8640\lambda_i^2(77\beta + 134 + 3\beta^2) \geq 0 \text{ for all } \beta \geq -1.8776, \\ \xi_{34}^{(3)}(\lambda_i; \lambda_i; \beta) &= 4320\lambda_i(178\beta + 832 + 3\beta^2) \geq 0 \text{ for all } \beta \geq -5.1151.\end{aligned}$$

Hence, $\xi_3^{(4)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{33}(\lambda_j; \lambda_i; \beta) = \xi_3^{(3)}(\lambda_j; \lambda_j; \lambda_i; \beta)/\lambda_j$. We wish to show that ξ_{33} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{33} with respect to λ_j gives

$$\begin{aligned} \xi_{33}(\lambda_j; \lambda_i; \beta) &= (8640\beta + 112320)\lambda_j^4 + (-96768 + 43848\beta + 2268\beta^2)\lambda_i\lambda_j^3 \\ &\quad + (-29376 + 7128\beta^2 + 108\beta^3)\lambda_i^2\lambda_j^2 \\ &\quad + (11520\beta + 51264 + 2808\beta^2 + 108\beta^3)\lambda_i^3\lambda_j \\ &\quad + (29952 + 6408\beta + 108\beta^2)\lambda_i^4, \\ \xi_{33}^{(1)}(\lambda_j; \lambda_i; \beta) &= (34560\beta + 449280)\lambda_j^3 + (-290304 + 131544\beta + 6804\beta^2)\lambda_i\lambda_j^2 \\ &\quad + (-58752 + 14256\beta^2 + 216\beta^3)\lambda_i^2\lambda_j \\ &\quad + (11520\beta + 51264 + 2808\beta^2 + 108\beta^3)\lambda_i^3, \\ \xi_{33}^{(2)}(\lambda_j; \lambda_i; \beta) &= (103680\beta + 1347840)\lambda_j^2 + (-580608 + 263088\beta + 13608\beta^2)\lambda_i\lambda_j \\ &\quad + (-58752 + 14256\beta^2 + 216\beta^3)\lambda_i^2, \\ \xi_{33}^{(3)}(\lambda_j; \lambda_i; \beta) &= (207360\beta + 2695680)\lambda_j + (-580608 + 263088\beta + 13608\beta^2)\lambda_i, \\ \xi_{33}^{(4)}(\lambda_j; \lambda_i; \beta) &= 207360\beta + 2695680 \geq 0 \text{ for all } \beta \geq -13. \end{aligned}$$

Now

$$\begin{aligned} \xi_{33}(\lambda_i; \lambda_i; \beta) &= 216\lambda_i^4(326\beta + 312 + 57\beta^2 + \beta^3) \geq 0 \text{ for all } \beta \geq -1.2060, \\ \xi_{33}^{(1)}(\lambda_i; \lambda_i; \beta) &= 36\lambda_i^3(4934\beta + 4208 + 663\beta^2 + 9\beta^3) \geq 0 \text{ for all } \beta \geq -0.9803, \\ \xi_{33}^{(2)}(\lambda_i; \lambda_i; \beta) &= 216\lambda_i^2(1698\beta + 3280 + 129\beta^2 + \beta^3) \geq 0 \text{ for all } \beta \geq -2.34, \\ \xi_{33}^{(3)}(\lambda_i; \lambda_i; \beta) &= 1944\lambda_i(242\beta + 1088 + 7\beta^2) \geq 0 \text{ for all } \beta \geq -5.3121. \end{aligned}$$

Hence, $\xi_3^{(3)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{32}(\lambda_j; \lambda_i; \beta) = \xi_3^{(2)}(\lambda_j; \lambda_j; \lambda_i; \beta)/\lambda_j^2$. We wish to show that ξ_{32} is non-negative for all $\lambda_j \geq \lambda_i \geq 0$ for all $\beta \geq$ Differentiating ξ_{32} with respect to λ_j gives

$$\begin{aligned} \xi_{32}(\lambda_j; \lambda_i; \beta) &= (43680 + 3360\beta)\lambda_j^4 + (1188\beta^2 - 42048 + 18648\beta)\lambda_i\lambda_j^3 \\ &\quad + (-3600 - 6624\beta + 108\beta^3 + 3996\beta^2)\lambda_i^2\lambda_j^2 \\ &\quad + (13824 + 1656\beta^2 + 144\beta^3 + 3600\beta)\lambda_i^3\lambda_j \\ &\quad + (5544\beta + 9648 + 216\beta^2)\lambda_i^4, \\ \xi_{32}^{(1)}(\lambda_j; \lambda_i; \beta) &= (174720 + 13440\beta)\lambda_j^3 + (3564\beta^2 - 126144 + 55944\beta)\lambda_i\lambda_j^2 \\ &\quad + (-7200 - 13248\beta + 216\beta^3 + 7992\beta^2)\lambda_i^2\lambda_j \\ &\quad + (13824 + 1656\beta^2 + 144\beta^3 + 3600\beta)\lambda_i^3, \\ \xi_{32}^{(2)}(\lambda_j; \lambda_i; \beta) &= (524160 + 40320\beta)\lambda_j^2 + (7128\beta^2 - 252288 + 111888\beta)\lambda_i\lambda_j \\ &\quad + (-7200 - 13248\beta + 216\beta^3 + 7992\beta^2)\lambda_i^2, \\ \xi_{32}^{(3)}(\lambda_j; \lambda_i; \beta) &= (1048320 + 80640\beta)\lambda_j + (7128\beta^2 - 252288 + 111888\beta)\lambda_i, \\ \xi_{32}^{(4)}(\lambda_j; \lambda_i; \beta) &= 1048320 + 80640\beta \geq 0 \text{ for all } \beta \geq -13. \end{aligned}$$

Now

$$\begin{aligned}
\xi_{32}(\lambda_i; \lambda_i; \beta) &= 84\lambda_i^4(256 + 292\beta + 84\beta^2 + 3\beta^3) \geq 0 \text{ for all } \beta \geq -1.4510, \\
\xi_{32}^{(1)}(\lambda_i; \lambda_i; \beta) &= 12\lambda_i^3(4600 + 4978\beta + 1101\beta^2 + 30\beta^3) \geq 0 \text{ for all } \beta \geq -1.2667, \\
\xi_{32}^{(2)}(\lambda_i; \lambda_i; \beta) &= 72\lambda_i^2(3676 + 1930\beta + 210\beta^2 + 3\beta^3) \geq 0 \text{ for all } \beta \geq -2.6278, \\
\xi_{32}^{(3)}(\lambda_i; \lambda_i; \beta) &= 72\lambda_i(11056 + 2674\beta + 99\beta^2) \geq 0 \text{ for all } \beta \geq -5.0961.
\end{aligned}$$

Hence, $\xi_3^{(2)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{31}(\lambda_j; \lambda_i; \beta) = \xi_3^{(1)}(\lambda_j; \lambda_j; \lambda_i; \beta)/\lambda_j^3$. We wish to show that ξ_{31} is non-negative for all $\lambda_j \geq \lambda_i \geq 0$ for all $\beta \geq$ Differentiating ξ_{31} with respect to λ_j gives

$$\begin{aligned}
\xi_{31}(\lambda_j; \lambda_i; \beta) &= (12480 + 960\beta)\lambda_j^4 + (5964\beta + 414\beta^2 - 13584)\lambda_i\lambda_j^3 \\
&\quad + (288 + 54\beta^3 - 3384\beta + 1512\beta^2)\lambda_i^2\lambda_j^2 \\
&\quad + (3120 + 630\beta^2 + 90\beta^3 + 1140\beta)\lambda_i^3\lambda_j \\
&\quad + (3000\beta + 1920 + 180\beta^2)\lambda_i^4, \\
\xi_{31}^{(1)}(\lambda_j; \lambda_i; \beta) &= (49920 + 3840\beta)\lambda_j^3 + (17892\beta + 1242\beta^2 - 40752)\lambda_i\lambda_j^2 \\
&\quad + (576 + 108\beta^3 - 6768\beta + 3024\beta^2)\lambda_i^2\lambda_j \\
&\quad + (3120 + 630\beta^2 + 90\beta^3 + 1140\beta)\lambda_i^3, \\
\xi_{31}^{(2)}(\lambda_j; \lambda_i; \beta) &= (149760 + 11520\beta)\lambda_j^2 + (35784\beta + 2484\beta^2 - 81504)\lambda_i\lambda_j \\
&\quad + (576 + 108\beta^3 - 6768\beta + 3024\beta^2)\lambda_i^2, \\
\xi_{31}^{(3)}(\lambda_j; \lambda_i; \beta) &= (299520 + 23040\beta)\lambda_j + (35784\beta + 2484\beta^2 - 81504)\lambda_i, \\
\xi_{31}^{(4)}(\lambda_j; \lambda_i; \beta) &= 299520 + 23040\beta \geq 0 \text{ for all } \beta \geq -13.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{31}(\lambda_i; \lambda_i; \beta) &= 48\lambda_i^4(88 + 160\beta + 57\beta^2 + 3\beta^3) \geq 0 \text{ for all } \beta \geq -0.7350, \\
\xi_{31}^{(1)}(\lambda_i; \lambda_i; \beta) &= 6\lambda_i^3(2144 + 2684\beta + 816\beta^2 + 33\beta^3) \geq 0 \text{ for all } \beta \geq -1.2494, \\
\xi_{31}^{(2)}(\lambda_i; \lambda_i; \beta) &= 36\lambda_i^2(1912 + 1126\beta + 153\beta^2 + 3\beta^3) \geq 0 \text{ for all } \beta \geq -2.5154, \\
\xi_{31}^{(3)}(\lambda_i; \lambda_i; \beta) &= 36\lambda_i(6056 + 1634\beta + 69\beta^2) \geq 0 \text{ for all } \beta \geq -4.5996.
\end{aligned}$$

Hence, $\xi_3^{(1)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{30}(\lambda_j; \lambda_i; \beta) = \xi_3(\lambda_j; \lambda_j; \lambda_i; \beta)/\lambda_j^4$. We wish to show that ξ_{30} is non-negative for

all $\lambda_j \geq \lambda_i \geq 0$ for all $\beta \geq$ Differentiating ξ_{30} with respect to λ_j gives

$$\begin{aligned} \xi_{30}(\lambda_j; \lambda_i; \beta) &= (12480 + 960\beta)\lambda_j^4 + (5964\beta + 414\beta^2 - 13584)\lambda_i\lambda_j^3 \\ &\quad + (288 + 54\beta^3 - 3384\beta + 1512\beta^2)\lambda_i^2\lambda_j^2 \\ &\quad + (3120 + 630\beta^2 + 90\beta^3 + 1140\beta)\lambda_i^3\lambda_j \\ &\quad + (3000\beta + 1920 + 180\beta^2)\lambda_i^4, \\ \xi_{30}^{(1)}(\lambda_j; \lambda_i; \beta) &= (49920 + 3840\beta)\lambda_j^3 + (17892\beta + 1242\beta^2 - 40752)\lambda_i\lambda_j^2 \\ &\quad + (576 + 108\beta^3 - 6768\beta + 3024\beta^2)\lambda_i^2\lambda_j \\ &\quad + (3120 + 630\beta^2 + 90\beta^3 + 1140\beta)\lambda_i^3, \\ \xi_{30}^{(2)}(\lambda_j; \lambda_i; \beta) &= (149760 + 11520\beta)\lambda_j^2 + (35784\beta + 2484\beta^2 - 81504)\lambda_i\lambda_j \\ &\quad + (576 + 108\beta^3 - 6768\beta + 3024\beta^2)\lambda_i^2, \\ \xi_{30}^{(3)}(\lambda_j; \lambda_i; \beta) &= (299520 + 23040\beta)\lambda_j + (35784\beta + 2484\beta^2 - 81504)\lambda_i, \\ \xi_{30}^{(4)}(\lambda_j; \lambda_i; \beta) &= 299520 + 23040\beta \geq 0 \text{ for all } \beta \geq -13. \end{aligned}$$

Now

$$\begin{aligned} \xi_{30}(\lambda_i; \lambda_i; \beta) &= 48\lambda_i^4(88 + 160\beta + 57\beta^2 + 3\beta^3) > 0 \text{ for all } \beta > -0.7350, \\ \xi_{30}^{(1)}(\lambda_i; \lambda_i; \beta) &= 6\lambda_i^3(2144 + 2684\beta + 816\beta^2 + 33\beta^3) > 0 \text{ for all } \beta > -1.2494, \\ \xi_{30}^{(2)}(\lambda_i; \lambda_i; \beta) &= 36\lambda_i^2(1912 + 1126\beta + 153\beta^2 + 3\beta^3) > 0 \text{ for all } \beta > -2.5154, \\ \xi_{30}^{(3)}(\lambda_i; \lambda_i; \beta) &= 36\lambda_i(6056 + 1634\beta + 69\beta^2) > 0 \text{ for all } \beta > -4.5996. \end{aligned}$$

Hence, ξ_3 is positive for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$. This implies that $\xi(\lambda_l; \lambda_l; \lambda_j; \lambda_i; \beta)$ is positive for all $\lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Differentiating $\xi_4(\lambda_l; \lambda_j; \lambda_i; \beta)$ with respect to λ_l we obtain

$$\begin{aligned} \xi_4(\lambda_l; \lambda_j; \lambda_i; \beta) &= (-1008 + 504\beta)(\lambda_i^2\lambda_j + \lambda_i\lambda_j^2)\lambda_l^4 \\ &\quad + (96\beta + 1248)(\lambda_j^3 + \lambda_i^3)\lambda_l^4 \\ &\quad + (72\beta^2 - 960 + 336\beta)(\lambda_i\lambda_j^3 + \lambda_i^3\lambda_j)\lambda_l^3 \\ &\quad + (96\beta + 1248)(\lambda_i^4 + \lambda_j^4)\lambda_l^3 \\ &\quad + (864 - 864\beta + 216\beta^2)\lambda_i^2\lambda_j^2\lambda_l^3 \\ &\quad + (-1008 + 504\beta)(\lambda_i\lambda_j^4 + \lambda_i^4\lambda_j)\lambda_l^2 \\ &\quad + (864 - 864\beta + 216\beta^2)(\lambda_i^2\lambda_j^3 + \lambda_i^3\lambda_j^2)\lambda_l^2 \\ &\quad + (72\beta^2 - 960 + 336\beta)\lambda_i^3\lambda_j^3\lambda_l \\ &\quad + (-1008 + 504\beta)(\lambda_i^2\lambda_j^4 + \lambda_i^4\lambda_j^2)\lambda_l \\ &\quad + (96\beta + 1248)(\lambda_i^3\lambda_j^4 + \lambda_i^4\lambda_j^3), \end{aligned}$$

$$\begin{aligned}
\xi_4^{(1)}(\lambda_l; \lambda_j; \lambda_i; \beta) &= 4(-1008 + 504\beta)(\lambda_i^2\lambda_j + \lambda_i\lambda_j^2)\lambda_l^3 \\
&\quad + 4(96\beta + 1248)(\lambda_j^3 + \lambda_i^3)\lambda_l^3 \\
&\quad + 3(72\beta^2 - 960 + 336\beta)(\lambda_i\lambda_j^3 + \lambda_i^3\lambda_j)\lambda_l^2 \\
&\quad + 3(96\beta + 1248)(\lambda_i^4 + \lambda_j^4)\lambda_l^2 \\
&\quad + 3(864 - 864\beta + 216\beta^2)\lambda_i^2\lambda_j^2\lambda_l^2 \\
&\quad + 2(-1008 + 504\beta)(\lambda_i\lambda_j^4 + \lambda_i^4\lambda_j)\lambda_l \\
&\quad + 2(864 - 864\beta + 216\beta^2)(\lambda_i^2\lambda_j^3 + \lambda_i^3\lambda_j^2)\lambda_l \\
&\quad + (72\beta^2 - 960 + 336\beta)\lambda_i^3\lambda_j^3 \\
&\quad + (-1008 + 504\beta)(\lambda_i^2\lambda_j^4 + \lambda_i^4\lambda_j^2), \\
\xi_4^{(2)}(\lambda_l; \lambda_j; \lambda_i; \beta) &= 12(-1008 + 504\beta)(\lambda_i^2\lambda_j + \lambda_i\lambda_j^2)\lambda_l^2 \\
&\quad + 12(96\beta + 1248)(\lambda_j^3 + \lambda_i^3)\lambda_l^2 \\
&\quad + 6(72\beta^2 - 960 + 336\beta)(\lambda_i\lambda_j^3 + \lambda_i^3\lambda_j)\lambda_l \\
&\quad + 6(96\beta + 1248)(\lambda_i^4 + \lambda_j^4)\lambda_l \\
&\quad + 6(864 - 864\beta + 216\beta^2)\lambda_i^2\lambda_j^2\lambda_l \\
&\quad + 2(-1008 + 504\beta)(\lambda_i\lambda_j^4 + \lambda_i^4\lambda_j) \\
&\quad + 2(864 - 864\beta + 216\beta^2)(\lambda_i^2\lambda_j^3 + \lambda_i^3\lambda_j^2), \\
\xi_4^{(3)}(\lambda_l; \lambda_j; \lambda_i; \beta) &= 24(-1008 + 504\beta)(\lambda_i^2\lambda_j + \lambda_i\lambda_j^2)\lambda_l \\
&\quad + 24(96\beta + 1248)(\lambda_j^3 + \lambda_i^3)\lambda_l \\
&\quad + 6(72\beta^2 - 960 + 336\beta)(\lambda_i\lambda_j^3 + \lambda_i^3\lambda_j) \\
&\quad + 6(96\beta + 1248)(\lambda_i^4 + \lambda_j^4) \\
&\quad + 6(864 - 864\beta + 216\beta^2)\lambda_i^2\lambda_j^2, \\
\xi_4^{(4)}(\lambda_l; \lambda_j; \lambda_i; \beta) &= 24(-1008 + 504\beta)(\lambda_i^2\lambda_j + \lambda_i\lambda_j^2) \\
&\quad + 24(96\beta + 1248)(\lambda_j^3 + \lambda_i^3).
\end{aligned}$$

Define $\xi_{44}(\lambda_j; \lambda_i; \beta) = \xi_4^{(4)}(\lambda_l; \lambda_j; \lambda_i; \beta)$. We wish to show that ξ_{44} is non-negative for all $\lambda_j \geq \lambda_i \geq 0$ for all $\beta \geq 13$. Differentiating ξ_{44} with respect to λ_j gives

$$\begin{aligned}
\xi_{44}(\lambda_j; \lambda_i; \beta) &= (2304\beta + 29952)\lambda_j^3 + (-24192 + 12096\beta)\lambda_i\lambda_j^2 \\
&\quad + (-24192 + 12096\beta)\lambda_i^2\lambda_j + (2304\beta + 29952)\lambda_i^3, \\
\xi_{44}^{(1)}(\lambda_j; \lambda_i; \beta) &= (6912\beta + 89856)\lambda_j^2 + (-48384 + 24192\beta)\lambda_i\lambda_j \\
&\quad + (-24192 + 12096\beta)\lambda_i^2, \\
\xi_{44}^{(2)}(\lambda_j; \lambda_i; \beta) &= (13824\beta + 179712)\lambda_j + (-48384 + 24192\beta)\lambda_i, \\
\xi_{44}^{(3)}(\lambda_j; \lambda_i; \beta) &= 13824\beta + 179712 \geq 0 \text{ for all } \beta \geq 13.
\end{aligned}$$

Now

$$\begin{aligned}\xi_{44}(\lambda_i; \lambda_i; \beta) &= 5760\lambda_i^3(2 + 5\beta) \geq 0 \text{ for all } \beta \geq -\frac{2}{5}, \\ \xi_{44}^{(1)}(\lambda_i; \lambda_i; \beta) &= 8640\lambda_i^2(2 + 5\beta) \geq 0 \text{ for all } \beta \geq -\frac{2}{5}, \\ \xi_{44}^{(2)}(\lambda_i; \lambda_i; \beta) &= 3456\lambda_i(38 + 11\beta) \geq 0 \text{ for all } \beta \geq -3.4545.\end{aligned}$$

Hence, $\xi_4^{(4)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{43}(\lambda_j; \lambda_i; \beta) = \xi_4^{(3)}(\lambda_j; \lambda_j; \lambda_i; \beta)$. We wish to show that ξ_{43} is non-negative for all $\lambda_j \geq \lambda_i \geq 0$ for all $\beta \geq$ Differentiating ξ_{43} with respect to λ_j gives

$$\begin{aligned}\xi_{43}(\lambda_j; \lambda_i; \beta) &= (2880\beta + 37440)\lambda_j^4 + (14112\beta - 29952 + 432\beta^2)\lambda_i\lambda_j^3 \\ &\quad + (-19008 + 6912\beta + 1296\beta^2)\lambda_i^2\lambda_j^2 + (4320\beta + 24192 + 432\beta^2)\lambda_i^3\lambda_j \\ &\quad + (576\beta + 7488)\lambda_i^4, \\ \xi_{43}^{(1)}(\lambda_j; \lambda_i; \beta) &= (149760 + 11520\beta)\lambda_j^3 + (42336\beta - 89856 + 1296\beta^2)\lambda_i\lambda_j^2 \\ &\quad + (-38016 + 13824\beta + 2592\beta^2)\lambda_i^2\lambda_j + (4320\beta + 24192 + 432\beta^2)\lambda_i^3, \\ \xi_{43}^{(2)}(\lambda_j; \lambda_i; \beta) &= (449280 + 34560\beta)\lambda_j^2 + (84672\beta - 179712 + 2592\beta^2)\lambda_i\lambda_j \\ &\quad + (-38016 + 13824\beta + 2592\beta^2)\lambda_i^2, \\ \xi_{43}^{(3)}(\lambda_j; \lambda_i; \beta) &= (898560 + 69120\beta)\lambda_j + (84672\beta - 179712 + 2592\beta^2)\lambda_i, \\ \xi_{43}^{(4)}(\lambda_j; \lambda_i; \beta) &= 898560 + 69120\beta \geq 0 \text{ for all } \beta \geq -13.\end{aligned}$$

Now

$$\begin{aligned}\xi_{43}(\lambda_i; \lambda_i; \beta) &= 720\lambda_i^4(28 + 40\beta + 3\beta^2) \geq 0 \text{ for all } \beta \geq -0.7412, \\ \xi_{43}^{(1)}(\lambda_i; \lambda_i; \beta) &= 1440\lambda_i^3(\beta + 16)(3\beta + 2) \geq 0 \text{ for all } \beta \geq -\frac{2}{3}, \\ \xi_{43}^{(2)}(\lambda_i; \lambda_i; \beta) &= 1728\lambda_i^2(134 + 77\beta + 3\beta^2) \geq 0 \text{ for all } \beta \geq -1.8776, \\ \xi_{43}^{(3)}(\lambda_i; \lambda_i; \beta) &= 864\lambda_i(832 + 178\beta + 3\beta^2) \geq 0 \text{ for all } \beta \geq -5.1151.\end{aligned}$$

Hence, $\xi_4^{(3)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{42}(\lambda_j; \lambda_i; \beta) = \xi_4^{(2)}(\lambda_j; \lambda_j; \lambda_i; \beta)/\lambda_j$. We wish to show that ξ_{42} is non-negative for all $\lambda_j \geq \lambda_i \geq 0$ for all $\beta \geq$ Differentiating ξ_{42} with respect to λ_j gives

$$\begin{aligned}\xi_{42}(\lambda_j; \lambda_i; \beta) &= (1728\beta + 22464)\lambda_j^4 + (-19872 + 9072\beta + 432\beta^2)\lambda_i\lambda_j^3 \\ &\quad + (-5184 - 864\beta + 1728\beta^2)\lambda_i^2\lambda_j^2 + (1440\beta + 10944 + 864\beta^2)\lambda_i^3\lambda_j \\ &\quad + (1584\beta + 5472)\lambda_i^4, \\ \xi_{42}^{(1)}(\lambda_j; \lambda_i; \beta) &= (6912\beta + 89856)\lambda_j^3 + (-59616 + 27216\beta + 1296\beta^2)\lambda_i\lambda_j^2 \\ &\quad + (-10368 - 1728\beta + 3456\beta^2)\lambda_i^2\lambda_j + (1440\beta + 10944 + 864\beta^2)\lambda_i^3, \\ \xi_{42}^{(2)}(\lambda_j; \lambda_i; \beta) &= (20736\beta + 269568)\lambda_j^2 + (-119232 + 54432\beta + 2592\beta^2)\lambda_i\lambda_j \\ &\quad + (-10368 - 1728\beta + 3456\beta^2)\lambda_i^2, \\ \xi_{42}^{(3)}(\lambda_j; \lambda_i; \beta) &= (41472\beta + 539136)\lambda_j + (-119232 + 54432\beta + 2592\beta^2)\lambda_i, \\ \xi_{42}^{(4)}(\lambda_j; \lambda_i; \beta) &= 41472\beta + 539136 \geq 0 \text{ for all } \beta \geq -13.\end{aligned}$$

Now

$$\begin{aligned}\xi_{42}(\lambda_i; \lambda_i; \beta) &= 432\lambda_i^4(7\beta + 16)(\beta + 2) \geq 0 \text{ for all } \beta \geq -2, \\ \xi_{42}^{(1)}(\lambda_i; \lambda_i; \beta) &= 144\lambda_i^3(235\beta + 214 + 39\beta^2) \geq 0 \text{ for all } \beta \geq -1.1181, \\ \xi_{42}^{(2)}(\lambda_i; \lambda_i; \beta) &= 864\lambda_i^2(85\beta + 162 + 7\beta^2) \geq 0 \text{ for all } \beta \geq -2.3675, \\ \xi_{42}^{(3)}(\lambda_i; \lambda_i; \beta) &= 2592\lambda_i(37\beta + 162 + \beta^2) \geq 0 \text{ for all } \beta \geq -5.0743.\end{aligned}$$

Hence, $\xi_4^{(2)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{41}(\lambda_j; \lambda_i; \beta) = \xi_4^{(1)}(\lambda_j; \lambda_j; \lambda_i; \beta)/\lambda_j^2$. We wish to show that ξ_{41} is non-negative for all $\lambda_j \geq \lambda_i \geq 0$ for all $\beta \geq$ Differentiating ξ_{41} with respect to λ_j gives

$$\begin{aligned}\xi_{41}(\lambda_j; \lambda_i; \beta) &= (8736 + 672\beta)\lambda_j^4 + (-8928 + 4032\beta + 216\beta^2)\lambda_i\lambda_j^3 \\ &\quad + (-720 - 1800\beta + 1080\beta^2)\lambda_i^2\lambda_j^2 + (2880 + 720\beta^2)\lambda_i^3\lambda_j \\ &\quad + (720 + 1800\beta)\lambda_i^4, \\ \xi_{41}^{(1)}(\lambda_j; \lambda_i; \beta) &= (34944 + 2688\beta)\lambda_j^3 + (-26784 + 12096\beta + 648\beta^2)\lambda_i\lambda_j^2 \\ &\quad + (-1440 - 3600\beta + 2160\beta^2)\lambda_i^2\lambda_j + (2880 + 720\beta^2)\lambda_i^3, \\ \xi_{41}^{(2)}(\lambda_j; \lambda_i; \beta) &= (104832 + 8064\beta)\lambda_j^2 + (-53568 + 24192\beta + 1296\beta^2)\lambda_i\lambda_j \\ &\quad + (-1440 - 3600\beta + 2160\beta^2)\lambda_i^2, \\ \xi_{41}^{(3)}(\lambda_j; \lambda_i; \beta) &= (209664 + 16128\beta)\lambda_j + (-53568 + 24192\beta + 1296\beta^2)\lambda_i, \\ \xi_{41}^{(4)}(\lambda_j; \lambda_i; \beta) &= 209664 + 16128\beta \geq 0 \text{ for all } \beta \geq -13.\end{aligned}$$

Now

$$\begin{aligned}\xi_{41}(\lambda_i; \lambda_i; \beta) &= 672\lambda_i^4(3\beta + 4)(\beta + 1) \geq 0 \text{ for all } \beta \geq -1, \\ \xi_{41}^{(1)}(\lambda_i; \lambda_i; \beta) &= 24\lambda_i^3(400 + 466\beta + 147\beta^2) \geq 0 \text{ for all } \beta, \\ \xi_{41}^{(2)}(\lambda_i; \lambda_i; \beta) &= 144\lambda_i^2(346 + 199\beta + 24\beta^2) \geq 0 \text{ for all } \beta \geq -2.4811, \\ \xi_{41}^{(3)}(\lambda_i; \lambda_i; \beta) &= 144\lambda_i(1084 + 280\beta + 9\beta^2) \geq 0 \text{ for all } \beta \geq -4.5315.\end{aligned}$$

Hence, $\xi_4^{(1)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{40}(\lambda_j; \lambda_i; \beta) = \xi_4(\lambda_j; \lambda_j; \lambda_i; \beta)/\lambda_j^3$. We wish to show that ξ_{40} is non-negative for all $\lambda_j \geq \lambda_i \geq 0$ for all $\beta \geq$ Differentiating ξ_{40} with respect to λ_j gives

$$\begin{aligned}\xi_{40}(\lambda_j; \lambda_i; \beta) &= (192\beta + 2496)\lambda_j^4 + (-2976 + 1344\beta + 72\beta^2)\lambda_i\lambda_j^3 \\ &\quad + (-288 - 720\beta + 432\beta^2)\lambda_i^2\lambda_j^2 + (1440 + 360\beta^2)\lambda_i^3\lambda_j \\ &\quad + (1200\beta + 480)\lambda_i^4, \\ \xi_{40}^{(1)}(\lambda_j; \lambda_i; \beta) &= (768\beta + 9984)\lambda_j^3 + (-8928 + 4032\beta + 216\beta^2)\lambda_i\lambda_j^2 \\ &\quad + (-576 - 1440\beta + 864\beta^2)\lambda_i^2\lambda_j + (1440 + 360\beta^2)\lambda_i^3, \\ \xi_{40}^{(2)}(\lambda_j; \lambda_i; \beta) &= (2304\beta + 29952)\lambda_j^2 + (-17856 + 8064\beta + 432\beta^2)\lambda_i\lambda_j \\ &\quad + (-576 - 1440\beta + 864\beta^2)\lambda_i^2, \\ \xi_{40}^{(3)}(\lambda_j; \lambda_i; \beta) &= (4608\beta + 59904)\lambda_j + (-17856 + 8064\beta + 432\beta^2)\lambda_i, \\ \xi_{40}^{(4)}(\lambda_j; \lambda_i; \beta) &= 4608\beta + 59904 \geq 0 \text{ for all } \beta \geq -13.\end{aligned}$$

Now

$$\begin{aligned}\xi_{40}(\lambda_i; \lambda_i; \beta) &= 288\lambda_i^4(3\beta + 4)(\beta + 1) > 0 \text{ for all } \beta > -1, \\ \xi_{40}^{(1)}(\lambda_i; \lambda_i; \beta) &= 480\lambda_i^3(3\beta + 4)(\beta + 1) > 0 \text{ for all } \beta > -1, \\ \xi_{40}^{(2)}(\lambda_i; \lambda_i; \beta) &= 144\lambda_i^2(62\beta + 80 + 9\beta^2) > 0 \text{ for all } \beta > -1.7195, \\ \xi_{40}^{(3)}(\lambda_i; \lambda_i; \beta) &= 144\lambda_i(88\beta + 292 + 3\beta^2) > 0 \text{ for all } \beta > -3.8141.\end{aligned}$$

Hence, ξ_4 is positive for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$. This implies that $\xi(\lambda_l; \lambda_l; \lambda_j; \lambda_i; \beta)$ is positive for all $\lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

□

Lemma B.13 Let

$$\begin{aligned}\xi(\lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta) &= (-1296\beta + 324\beta^2 + 1296)t(2, 2, 2, 4) \\ &+ (336\beta - 672)t(1, 1, 4, 4) \\ &+ (-864 + 108\beta^2 + 216\beta)t(1, 2, 3, 4) \\ &+ (12\beta^2 + 1504 + 184\beta)t(0, 3, 3, 4) \\ &+ (132\beta + 456)t(0, 2, 4, 4) \\ &+ (24\beta^3 + 864\beta - 1920)t(1, 3, 3, 3) \\ &+ (1152 + 72\beta^3 - 144\beta^2 - 576\beta)t(2, 2, 3, 3),\end{aligned}$$

where

$$t(a, b, c, d) = \sum_{\substack{(n_1, n_2, n_3, n_4) \\ \in \text{perm}(a, b, c, d)}} \lambda_i^{n_1} \lambda_j^{n_2} \lambda_l^{n_3} \lambda_k^{n_4}.$$

The expression $\xi(\lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta)$ is positive for all $\lambda_k \geq \lambda_l \geq \lambda_j \geq \lambda_i$, and $\beta \geq -\frac{2}{5}$.

Proof.

Let

$$\hat{s}(a, b, c) = \sum_{\substack{(n_1, n_2, n_3) \\ \in \text{perm}(a, b, c)}} \lambda_i^{n_1} \lambda_j^{n_2} \lambda_l^{n_3}.$$

Differentiating $\xi(\lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta)$ with respect to λ_k we obtain

$$\begin{aligned}
\xi^{(1)}(\lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta) &= 4(-1296\beta + 324\beta^2 + 1296)\lambda_k^3\hat{s}(2, 2, 2) \\
&+ 4(336\beta - 672)\lambda_k^3\hat{s}(1, 1, 4) \\
&+ 4(-864 + 108\beta^2 + 216\beta)\lambda_k^3\hat{s}(1, 2, 3) \\
&+ 4(12\beta^2 + 1504 + 184\beta)\lambda_k^3\hat{s}(0, 3, 3) \\
&+ 4(132\beta + 456)\lambda_k^3\hat{s}(0, 2, 4) \\
&+ 3(-576\beta + 1152 + 72\beta^3 - 144\beta^2)\lambda_k^2\hat{s}(2, 2, 3) \\
&+ 3(-864 + 108\beta^2 + 216\beta)\lambda_k^2\hat{s}(1, 2, 4) \\
&+ 3(24\beta^3 + 864\beta - 1920)\lambda_k^2\hat{s}(1, 3, 3) \\
&+ 3(12\beta^2 + 1504 + 184\beta)\lambda_k^2\hat{s}(0, 3, 4) \\
&+ 2(132\beta + 456)\lambda_k\hat{s}(0, 4, 4) \\
&+ 2(-576\beta + 1152 + 72\beta^3 - 144\beta^2)\lambda_k\hat{s}(2, 3, 3) \\
&+ 2(-1296\beta + 324\beta^2 + 1296)\lambda_k\hat{s}(2, 2, 4) \\
&+ 2(-864 + 108\beta^2 + 216\beta)\lambda_k\hat{s}(1, 3, 4) \\
&+ (336\beta - 672)\hat{s}(1, 4, 4) \\
&+ (24\beta^3 + 864\beta - 1920)\hat{s}(3, 3, 3) \\
&+ (-864 + 108\beta^2 + 216\beta)\hat{s}(2, 3, 4), \\
\xi^{(2)}(\lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta) &= (3888\beta^2 - 15552\beta + 15552)\lambda_k^2\hat{s}(2, 2, 2) \\
&+ (2592\beta + 1296\beta^2 - 10368)\lambda_k^2\hat{s}(1, 2, 3) \\
&+ (-8064 + 4032\beta)\lambda_k^2\hat{s}(1, 1, 4) \\
&+ (18048 + 144\beta^2 + 2208\beta)\lambda_k^2\hat{s}(0, 3, 3) \\
&+ (5472 + 1584\beta)\lambda_k^2\hat{s}(0, 2, 4) \\
&+ (432\beta^3 - 3456\beta - 864\beta^2 + 6912)\lambda_k\hat{s}(2, 2, 3) \\
&+ (9024 + 1104\beta + 72\beta^2)\lambda_k\hat{s}(0, 3, 4) \\
&+ (-5184 + 1296\beta + 648\beta^2)\lambda_k\hat{s}(1, 2, 4) \\
&+ (5184\beta - 11520 + 144\beta^3)\lambda_k\hat{s}(1, 3, 3) \\
&+ (648\beta^2 + 2592 - 2592\beta)\hat{s}(2, 4, 4) \\
&+ (-1728 + 216\beta^2 + 432\beta)\hat{s}(1, 3, 4) \\
&+ (2304 + 144\beta^3 - 1152\beta - 288\beta^2)\hat{s}(2, 3, 3) \\
&+ (264\beta + 912)\hat{s}(0, 4, 4),
\end{aligned}$$

$$\begin{aligned}
\xi^{(3)}(\lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta) &= (7776\beta^2 + 31104 - 31104\beta)\lambda_k\hat{s}(2, 2, 2) \\
&\quad + (10944 + 3168\beta)\lambda_k\hat{s}(0, 2, 4) \\
&\quad + (8064\beta - 16128)\lambda_k\hat{s}(1, 1, 4) \\
&\quad + (5184\beta + 2592\beta^2 - 20736)\lambda_k\hat{s}(1, 2, 3) \\
&\quad + (4416\beta + 288\beta^2 + 36096)\lambda_k\hat{s}(0, 3, 3) \\
&\quad + (432\beta^3 - 3456\beta - 864\beta^2 + 6912)\hat{s}(2, 2, 3) \\
&\quad + (9024 + 1104\beta + 72\beta^2)\hat{s}(0, 3, 4) \\
&\quad + (-5184 + 1296\beta + 648\beta^2)\hat{s}(1, 2, 4) \\
&\quad + (5184\beta - 11520 + 144\beta^3)\hat{s}(1, 3, 3), \\
\xi^{(4)}(\lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta) &= (7776\beta^2 + 31104 - 31104\beta)\hat{s}(2, 2, 2) \\
&\quad + (10944 + 3168\beta)\hat{s}(0, 2, 4) \\
&\quad + (8064\beta - 16128)\hat{s}(1, 1, 4) \\
&\quad + (5184\beta + 2592\beta^2 - 20736)\hat{s}(1, 2, 3) \\
&\quad + (4416\beta + 288\beta^2 + 36096)\hat{s}(0, 3, 3).
\end{aligned}$$

Define

$$\begin{aligned}
\xi_0(\lambda_l; \lambda_j; \lambda_i; \beta) &\stackrel{\text{def}}{=} \xi(\lambda_l; \lambda_l; \lambda_j; \lambda_i; \beta)/\lambda_l^2, \\
\xi_1(\lambda_l; \lambda_j; \lambda_i; \beta) &\stackrel{\text{def}}{=} \xi^{(1)}(\lambda_l; \lambda_l; \lambda_j; \lambda_i; \beta)/\lambda_l, \\
\xi_2(\lambda_l; \lambda_j; \lambda_i; \beta) &\stackrel{\text{def}}{=} \xi^{(2)}(\lambda_l; \lambda_l; \lambda_j; \lambda_i; \beta), \\
\xi_3(\lambda_l; \lambda_j; \lambda_i; \beta) &\stackrel{\text{def}}{=} \xi^{(3)}(\lambda_l; \lambda_l; \lambda_j; \lambda_i; \beta), \\
\xi_4(\lambda_l; \lambda_j; \lambda_i; \beta) &\stackrel{\text{def}}{=} \xi^{(4)}(\lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta).
\end{aligned}$$

We will prove that the above equations are non-negative for all $\lambda_k \geq \lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$ and, hence, deduce that $\xi(\lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta)$ is positive for all $\lambda_k \geq \lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Now

$$\begin{aligned}
\xi_0(\lambda_l; \lambda_j; \lambda_i; \beta) = & (132\beta + 456)(\lambda_j^2 + \lambda_i^2)\lambda_l^6 + (336\beta - 672)\lambda_i\lambda_j\lambda_l^6 \\
& + (-1728 + 216\beta^2 + 432\beta)(\lambda_i\lambda_j^2 + \lambda_i^2\lambda_j)\lambda_l^5 \\
& + (3008 + 368\beta + 24\beta^2)(\lambda_j^3 + \lambda_i^3)\lambda_l^5 \\
& + (24\beta^3 - 3648 + 216\beta^2 + 1296\beta)(\lambda_i\lambda_j^3 + \lambda_i^3\lambda_j)\lambda_l^4 \\
& + (2416 + 448\beta + 12\beta^2)(\lambda_j^4 + \lambda_i^4)\lambda_l^4 \\
& + (-3168\beta + 72\beta^3 + 3744 + 504\beta^2)\lambda_i^2\lambda_j^2\lambda_l^4 \\
& + (-720\beta + 144\beta^3 - 72\beta^2 + 576)(\lambda_i^2\lambda_j^3 + \lambda_i^3\lambda_j^2)\lambda_l^3 \\
& + (1104\beta + 216\beta^2 - 3072)(\lambda_i\lambda_j^4 + \lambda_i^4\lambda_j)\lambda_l^3 \\
& + (540\beta^2 + 480 - 600\beta)(\lambda_i^2\lambda_j^4 + \lambda_i^4\lambda_j^2)\lambda_l^2 \\
& + (120\beta^3 - 120\beta^2 + 320 + 1520\beta)\lambda_i^3\lambda_j^3\lambda_l^2 \\
& + (1280 + 240\beta^2 + 800\beta)(\lambda_i^3\lambda_j^4 + \lambda_i^4\lambda_j^3)\lambda_l \\
& + (600\beta + 240)\lambda_i^4\lambda_j^4,
\end{aligned}$$

Define $\xi_{06}(\lambda_j; \lambda_i; \beta) = \xi_0^{(6)}(\lambda_l; \lambda_j; \lambda_i; \beta)$, where all derivatives are with respect to λ_l . We wish to show that ξ_{06} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{06} with respect to λ_j gives

$$\begin{aligned}
\xi_{06}(\lambda_j; \lambda_i; \beta) &= (95040\beta + 328320)\lambda_j^2 + (-483840 + 241920\beta)\lambda_i\lambda_j \\
&\quad + (95040\beta + 328320)\lambda_i^2, \\
\xi_{06}^{(1)}(\lambda_j; \lambda_i; \beta) &= (190080\beta + 656640)\lambda_j + (-483840 + 241920\beta)\lambda_i, \\
\xi_{06}^{(2)}(\lambda_j; \lambda_i; \beta) &= 190080\beta + 656640 \geq 0 \text{ for all } \beta \geq -3.4545.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{06}(\lambda_i; \lambda_i; \beta) &= 86400\lambda_i^2(2 + 5\beta) \geq 0 \text{ for all } \beta \geq -\frac{2}{5}, \\
\xi_{06}^{(1)}(\lambda_i; \lambda_i; \beta) &= 86400\lambda_i(2 + 5\beta) \geq 0 \text{ for all } \beta \geq -\frac{2}{5}.
\end{aligned}$$

Hence, $\xi_0^{(6)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{05}(\lambda_j; \lambda_i; \beta) = \xi_0^{(5)}(\lambda_l; \lambda_j; \lambda_i; \beta)$, where all derivatives are with respect to λ_l . We wish to show that ξ_{05} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{05} with

respect to λ_j gives

$$\begin{aligned}\xi_{05}(\lambda_j; \lambda_i; \beta) &= (139200\beta + 689280 + 2880\beta^2)\lambda_j^3 \\ &\quad + (-691200 + 293760\beta + 25920\beta^2)\lambda_i\lambda_j^2 \\ &\quad + (146880\beta + 120960 + 25920\beta^2)\lambda_i^2\lambda_j \\ &\quad + (2880\beta^2 + 44160\beta + 360960)\lambda_i^3, \\ \xi_{05}^{(1)}(\lambda_j; \lambda_i; \beta) &= (417600\beta + 2067840 + 8640\beta^2)\lambda_j^2 \\ &\quad + (-1382400 + 587520\beta + 51840\beta^2)\lambda_i\lambda_j \\ &\quad + (146880\beta + 120960 + 25920\beta^2)\lambda_i^2, \\ \xi_{05}^{(2)}(\lambda_j; \lambda_i; \beta) &= (835200\beta + 4135680 + 17280\beta^2)\lambda_j \\ &\quad + (-1382400 + 587520\beta + 51840\beta^2)\lambda_i, \\ \xi_{05}^{(3)}(\lambda_j; \lambda_i; \beta) &= 835200\beta + 4135680 + 17280\beta^2 \geq 0 \text{ for all } \beta \geq -5.6007.\end{aligned}$$

Now

$$\begin{aligned}\xi_{05}(\lambda_i; \lambda_i; \beta) &= 9600\lambda_i^3(\beta + 10)(6\beta + 5) \geq 0 \text{ for all } \beta \geq -\frac{5}{6}, \\ \xi_{05}^{(1)}(\lambda_i; \lambda_i; \beta) &= 28800\lambda_i^2(28 + 40\beta + 3\beta^2) \geq 0 \text{ for all } \beta \geq -0.7412, \\ \xi_{05}^{(2)}(\lambda_i; \lambda_i; \beta) &= 5760\lambda_i(247\beta + 478 + 12\beta^2) \geq 0 \text{ for all } \beta \geq -2.1624.\end{aligned}$$

Hence, $\xi_0^{(5)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{04}(\lambda_j; \lambda_i; \beta) = \xi_0^{(4)}(\lambda_j; \lambda_j; \lambda_i; \beta)$, where all derivatives are with respect to λ_l . We wish to show that ξ_{04} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{04} with respect to λ_j gives

$$\begin{aligned}\xi_{04}(\lambda_j; \lambda_i; \beta) &= (102432\beta + 583104 + 3168\beta^2)\lambda_j^4 \\ &\quad + (203904\beta + 31104\beta^2 - 536832 + 576\beta^3)\lambda_i\lambda_j^3 \\ &\quad + (23328\beta + 46656 + 38016\beta^2 + 1728\beta^3)\lambda_i^2\lambda_j^2 \\ &\quad + (8064\beta^2 + 75264\beta + 273408 + 576\beta^3)\lambda_i^3\lambda_j \\ &\quad + (57984 + 288\beta^2 + 10752\beta)\lambda_i^4, \\ \xi_{04}^{(1)}(\lambda_j; \lambda_i; \beta) &= (203904\beta + 31104\beta^2 - 536832 + 576\beta^3)\lambda_j^3 \\ &\quad + 2(23328\beta + 46656 + 38016\beta^2 + 1728\beta^3)\lambda_i\lambda_j^2 \\ &\quad + 3(8064\beta^2 + 75264\beta + 273408 + 576\beta^3)\lambda_i^2\lambda_j \\ &\quad + 4(57984 + 10752\beta + 288\beta^2)\lambda_i^3, \\ \xi_{04}^{(2)}(\lambda_j; \lambda_i; \beta) &= (1229184\beta + 6997248 + 38016\beta^2)\lambda_j^2 \\ &\quad + (1223424\beta + 186624\beta^2 - 3220992 + 3456\beta^3)\lambda_i\lambda_j \\ &\quad + (46656\beta + 93312 + 76032\beta^2 + 3456\beta^3)\lambda_i^2, \\ \xi_{04}^{(3)}(\lambda_j; \lambda_i; \beta) &= (2458368\beta + 13994496 + 76032\beta^2)\lambda_j \\ &\quad + (1223424\beta + 186624\beta^2 - 3220992 + 3456\beta^3)\lambda_i, \\ \xi_{04}^{(4)}(\lambda_j; \lambda_i; \beta) &= 2458368\beta + 13994496 + 76032\beta^2 \geq 0 \text{ for all } \beta \geq -7.3746.\end{aligned}$$

Now

$$\begin{aligned}\xi_{04}(\lambda_i; \lambda_i; \beta) &= 960\lambda_i^4(433\beta + 442 + 84\beta^2 + 3\beta^3) \geq 0 \text{ for all } \beta \geq -1.3643, \\ \xi_{04}^{(1)}(\lambda_i; \lambda_i; \beta) &= 2880\lambda_i^3(397\beta + 378 + 66\beta^2 + 2\beta^3) \geq 0 \text{ for all } \beta \geq -1.1726, \\ \xi_{04}^{(2)}(\lambda_i; \lambda_i; \beta) &= 576\lambda_i^2(4339\beta + 6718 + 522\beta^2 + 12\beta^3) \geq 0 \text{ for all } \beta \geq -2.0134, \\ \xi_{04}^{(3)}(\lambda_i; \lambda_i; \beta) &= 1152\lambda_i(3196\beta + 9352 + 228\beta^2 + 3\beta^3) \geq 0 \text{ for all } \beta \geq -4.0159.\end{aligned}$$

Hence, $\xi_0^{(4)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{03}(\lambda_j; \lambda_i; \beta) = \xi_0^{(3)}(\lambda_j; \lambda_j; \lambda_i; \beta)/\lambda_j$, where all derivatives are with respect to λ_i . We wish to show that ξ_{03} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{03} with respect to λ_j gives

$$\begin{aligned}\xi_{03}(\lambda_j; \lambda_i; \beta) &= (48672\beta + 293184 + 1728\beta^2)\lambda_j^4 \\ &\quad + (103968\beta - 290304 + 19440\beta^2 + 576\beta^3)\lambda_i\lambda_j^3 \\ &\quad + (-38592\beta + 44352 + 24624\beta^2 + 2592\beta^3)\lambda_i^2\lambda_j^2 \\ &\quad + (96384 + 6192\beta^2 + 48864\beta + 1440\beta^3)\lambda_i^3\lambda_j \\ &\quad + (39552 + 1584\beta^2 + 17376\beta)\lambda_i^4, \\ \xi_{03}^{(1)}(\lambda_j; \lambda_i; \beta) &= (194688\beta + 1172736 + 6912\beta^2)\lambda_j^3 \\ &\quad + (311904\beta - 870912 + 58320\beta^2 + 1728\beta^3)\lambda_i\lambda_j^2 \\ &\quad + (-77184\beta + 88704 + 49248\beta^2 + 5184\beta^3)\lambda_i^2\lambda_j \\ &\quad + (96384 + 6192\beta^2 + 48864\beta + 1440\beta^3)\lambda_i^3, \\ \xi_{03}^{(2)}(\lambda_j; \lambda_i; \beta) &= (584064\beta + 3518208 + 20736\beta^2)\lambda_j^2 \\ &\quad + (623808\beta - 1741824 + 116640\beta^2 + 3456\beta^3)\lambda_i\lambda_j \\ &\quad + (-77184\beta + 88704 + 49248\beta^2 + 5184\beta^3)\lambda_i^2, \\ \xi_{03}^{(3)}(\lambda_j; \lambda_i; \beta) &= (1168128\beta + 7036416 + 41472\beta^2)\lambda_j \\ &\quad + (623808\beta - 1741824 + 116640\beta^2 + 3456\beta^3)\lambda_i, \\ \xi_{03}^{(4)}(\lambda_j; \lambda_i; \beta) &= 1168128\beta + 7036416 + 41472\beta^2 \geq 0 \text{ for all } \beta \geq -8.7286.\end{aligned}$$

Now

$$\begin{aligned}\xi_{03}(\lambda_i; \lambda_i; \beta) &= 576\lambda_i^4(\beta + 3)(\beta + 2)(8\beta + 53) \geq 0 \text{ for all } \beta \geq -2, \\ \xi_{03}^{(1)}(\lambda_i; \lambda_i; \beta) &= 96\lambda_i^3(4982\beta + 5072 + 1257\beta^2 + 87\beta^3) \geq 0 \text{ for all } \beta \geq -1.5772, \\ \xi_{03}^{(2)}(\lambda_i; \lambda_i; \beta) &= 576\lambda_i^2(1963\beta + 3238 + 324\beta^2 + 15\beta^3) \geq 0 \text{ for all } \beta \geq -2.7086, \\ \xi_{03}^{(3)}(\lambda_i; \lambda_i; \beta) &= 864\lambda_i(2074\beta + 6128 + 183\beta^2 + 4\beta^3) \geq 0 \text{ for all } \beta \geq -4.7120.\end{aligned}$$

Hence, $\xi_0^{(3)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{02}(\lambda_j; \lambda_i; \beta) = \xi_0^{(2)}(\lambda_j; \lambda_j; \lambda_i; \beta)/\lambda_j^2$, where all derivatives are with respect to λ_i . We wish to show that ξ_{02} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{02} with

respect to λ_j gives

$$\begin{aligned}
\xi_{02}(\lambda_j; \lambda_i; \beta) &= (102832 + 16696\beta + 624\beta^2)\lambda_j^4 \\
&\quad + (40896\beta - 116928 + 8208\beta^2 + 288\beta^3)\lambda_i\lambda_j^3 \\
&\quad + (28464 - 30936\beta + 11016\beta^2 + 1728\beta^3)\lambda_i^2\lambda_j^2 \\
&\quad + (21632\beta + 2400\beta^2 + 20480 + 1392\beta^3)\lambda_i^3\lambda_j \\
&\quad + (11520 + 10800\beta + 2520\beta^2)\lambda_i^4, \\
\xi_{02}^{(1)}(\lambda_j; \lambda_i; \beta) &= (411328 + 66784\beta + 2496\beta^2)\lambda_j^3 \\
&\quad + (122688\beta - 350784 + 24624\beta^2 + 864\beta^3)\lambda_i\lambda_j^2 \\
&\quad + (56928 - 61872\beta + 22032\beta^2 + 3456\beta^3)\lambda_i^2\lambda_j \\
&\quad + (21632\beta + 2400\beta^2 + 20480 + 1392\beta^3)\lambda_i^3, \\
\xi_{02}^{(2)}(\lambda_j; \lambda_i; \beta) &= (1233984 + 200352\beta + 7488\beta^2)\lambda_j^2 \\
&\quad + (245376\beta - 701568 + 49248\beta^2 + 1728\beta^3)\lambda_i\lambda_j \\
&\quad + (56928 - 61872\beta + 22032\beta^2 + 3456\beta^3)\lambda_i^2, \\
\xi_{02}^{(3)}(\lambda_j; \lambda_i; \beta) &= (2467968 + 400704\beta + 14976\beta^2)\lambda_j \\
&\quad + (245376\beta - 701568 + 49248\beta^2 + 1728\beta^3)\lambda_i, \\
\xi_{02}^{(4)}(\lambda_j; \lambda_i; \beta) &= 2467968 + 400704\beta + 14976\beta^2 \geq 0 \text{ for all } \beta \geq -9.6124.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{02}(\lambda_i; \lambda_i; \beta) &= 48\lambda_i^4(\beta + 3)(\beta + 2)(71\beta + 161) \geq 0 \text{ for all } \beta \geq -2, \\
\xi_{02}^{(1)}(\lambda_i; \lambda_i; \beta) &= 48\lambda_i^3(\beta + 3)(\beta + 2)(119\beta + 479) \geq 0 \text{ for all } \beta \geq -2, \\
\xi_{02}^{(2)}(\lambda_i; \lambda_i; \beta) &= 48\lambda_i^2(12278 + 7997\beta + 1641\beta^2 + 108\beta^3) \geq 0 \text{ for all } \beta \geq -3.1439, \\
\xi_{02}^{(3)}(\lambda_i; \lambda_i; \beta) &= 96\lambda_i(\beta + 10)(18\beta^2 + 489\beta + 1840) \geq 0 \text{ for all } \beta \geq -4.5122.
\end{aligned}$$

Hence, $\xi_0^{(2)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{01}(\lambda_j; \lambda_i; \beta) = \xi_0^{(1)}(\lambda_j; \lambda_j; \lambda_i; \beta)/\lambda_j^3$, where all derivatives are with respect to λ_i . We wish to show that ξ_{01} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{01} with

respect to λ_j gives

$$\begin{aligned}
\xi_{01}(\lambda_j; \lambda_i; \beta) &= (27440 + 4424\beta + 168\beta^2)\lambda_j^4 \\
&\quad + (12672\beta - 36480 + 2592\beta^2 + 96\beta^3)\lambda_i\lambda_j^3 \\
&\quad + (11760 - 13080\beta + 3960\beta^2 + 720\beta^3)\lambda_i^2\lambda_j^2 \\
&\quad + (768\beta^2 + 4096 + 8704\beta + 768\beta^3)\lambda_i^3\lambda_j \\
&\quad + (2016\beta^2 + 2688 + 4704\beta)\lambda_i^4, \\
\xi_{01}^{(1)}(\lambda_j; \lambda_i; \beta) &= (109760 + 17696\beta + 672\beta^2)\lambda_j^3 \\
&\quad + (38016\beta - 109440 + 7776\beta^2 + 288\beta^3)\lambda_i\lambda_j^2 \\
&\quad + (23520 - 26160\beta + 7920\beta^2 + 1440\beta^3)\lambda_i^2\lambda_j \\
&\quad + (768\beta^2 + 4096 + 8704\beta + 768\beta^3)\lambda_i^3, \\
\xi_{01}^{(2)}(\lambda_j; \lambda_i; \beta) &= (329280 + 53088\beta + 2016\beta^2)\lambda_j^2 \\
&\quad + (76032\beta - 218880 + 15552\beta^2 + 576\beta^3)\lambda_i\lambda_j \\
&\quad + (23520 - 26160\beta + 7920\beta^2 + 1440\beta^3)\lambda_i^2, \\
\xi_{01}^{(3)}(\lambda_j; \lambda_i; \beta) &= (658560 + 106176\beta + 4032\beta^2)\lambda_j \\
&\quad + (76032\beta - 218880 + 15552\beta^2 + 576\beta^3)\lambda_i, \\
\xi_{01}^{(4)}(\lambda_j; \lambda_i; \beta) &= 1344(3\beta + 49)(\beta + 10) \geq 0 \text{ for all } \beta \geq -10.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{01}(\lambda_i; \lambda_i; \beta) &= 1584\lambda_i^4(\beta + 3)(\beta + 2)(\beta + 1) \geq 0 \text{ for all } \beta \geq -1, \\
\xi_{01}^{(1)}(\lambda_i; \lambda_i; \beta) &= 48\lambda_i^3(\beta + 3)(\beta + 2)(52\beta + 97) \geq 0 \text{ for all } \beta \geq -2, \\
\xi_{01}^{(2)}(\lambda_i; \lambda_i; \beta) &= 144\lambda_i^2(\beta + 3)(14\beta^2 + 135\beta + 310) \geq 0 \text{ for all } \beta \geq -3, \\
\xi_{01}^{(3)}(\lambda_i; \lambda_i; \beta) &= 192\lambda_i(\beta + 10)(3\beta^2 + 72\beta + 229) \geq 0 \text{ for all } \beta \geq -3.7740.
\end{aligned}$$

Hence, $\xi_0^{(1)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{00}(\lambda_j; \lambda_i; \beta) = \xi_0(\lambda_j; \lambda_j; \lambda_i; \beta)/\lambda_j^4$, where all derivatives are with respect to λ_i . We wish to show that ξ_{00} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{00} with

respect to λ_j gives

$$\begin{aligned} \xi_{00}(\lambda_j; \lambda_i; \beta) &= (948\beta + 5880 + 36\beta^2)\lambda_j^4 + (3168\beta - 9120 + 648\beta^2 + 24\beta^3)\lambda_i\lambda_j^3 \\ &\quad + (-3924\beta + 3528 + 1188\beta^2 + 216\beta^3)\lambda_i^2\lambda_j^2 \\ &\quad + (1536 + 3264\beta + 288\beta^2 + 288\beta^3)\lambda_i^3\lambda_j \\ &\quad + (1344 + 2352\beta + 1008\beta^2)\lambda_i^4, \\ \xi_{00}^{(1)}(\lambda_j; \lambda_i; \beta) &= (3792\beta + 23520 + 144\beta^2)\lambda_j^3 + (9504\beta - 27360 + 1944\beta^2 + 72\beta^3)\lambda_i\lambda_j^2 \\ &\quad + (-7848\beta + 7056 + 2376\beta^2 + 432\beta^3)\lambda_i^2\lambda_j \\ &\quad + (1536 + 3264\beta + 288\beta^2 + 288\beta^3)\lambda_i^3, \\ \xi_{00}^{(2)}(\lambda_j; \lambda_i; \beta) &= (11376\beta + 70560 + 432\beta^2)\lambda_j^2 + (19008\beta - 54720 + 3888\beta^2 + 144\beta^3)\lambda_i\lambda_j \\ &\quad + (-7848\beta + 7056 + 2376\beta^2 + 432\beta^3)\lambda_i^2, \\ \xi_{00}^{(3)}(\lambda_j; \lambda_i; \beta) &= (22752\beta + 141120 + 864\beta^2)\lambda_j + (19008\beta - 54720 + 3888\beta^2 + 144\beta^3)\lambda_i, \\ \xi_{00}^{(4)}(\lambda_j; \lambda_i; \beta) &= 288(3\beta + 49)(\beta + 10) \geq 0 \text{ for all } \beta \geq -10. \end{aligned}$$

Now

$$\begin{aligned} \xi_{00}(\lambda_i; \lambda_i; \beta) &= 528\lambda_i^4(\beta + 3)(\beta + 2)(\beta + 1) > 0 \text{ for all } \beta > -1, \\ \xi_{00}^{(1)}(\lambda_i; \lambda_i; \beta) &= 792\lambda_i^3(\beta + 3)(\beta + 2)(\beta + 1) > 0 \text{ for all } \beta > -1, \\ \xi_{00}^{(2)}(\lambda_i; \lambda_i; \beta) &= 72\lambda_i^2(\beta + 3)(\beta + 2)(8\beta + 53) > 0 \text{ for all } \beta > -2, \\ \xi_{00}^{(3)}(\lambda_i; \lambda_i; \beta) &= 144\lambda_i(\beta + 10)(\beta + 20)(\beta + 3) > 0 \text{ for all } \beta > -3. \end{aligned}$$

Hence, ξ_0 is positive for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$. This implies that $\xi(\lambda_l; \lambda_l; \lambda_j; \lambda_i; \beta)$ is positive for all $\lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Now

$$\begin{aligned} \xi_1(\lambda_l; \lambda_j; \lambda_i; \beta) &= (528\beta + 1824)(\lambda_j^2 + \lambda_i^2)\lambda_l^6 + (-2688 + 1344\beta)\lambda_i\lambda_j\lambda_l^6 \\ &\quad + (1512\beta - 6048 + 756\beta^2)(\lambda_i\lambda_j^2 + \lambda_i^2\lambda_j)\lambda_l^5 \\ &\quad + (10528 + 84\beta^2 + 1288\beta)(\lambda_j^3 + \lambda_i^3)\lambda_l^5 \\ &\quad + (-10944 + 648\beta^2 + 3888\beta + 72\beta^3)(\lambda_i\lambda_j^3 + \lambda_i^3\lambda_j)\lambda_l^4 \\ &\quad + (7248 + 36\beta^2 + 1344\beta)(\lambda_j^4 + \lambda_i^4)\lambda_l^4 \\ &\quad + (11232 + 216\beta^3 - 9504\beta + 1512\beta^2)\lambda_i^2\lambda_j^2\lambda_l^4 \\ &\quad + (-1800\beta + 1440 + 360\beta^3 - 180\beta^2)(\lambda_i^2\lambda_j^3 + \lambda_i^3\lambda_j^2)\lambda_l^3 \\ &\quad + (2760\beta - 7680 + 540\beta^2)(\lambda_i\lambda_j^4 + \lambda_i^4\lambda_j)\lambda_l^3 \\ &\quad + (1080\beta^2 + 960 - 1200\beta)(\lambda_i^2\lambda_j^4 + \lambda_i^4\lambda_j^2)\lambda_l^2 \\ &\quad + (3040\beta + 240\beta^3 + 640 - 240\beta^2)\lambda_i^3\lambda_j^3\lambda_l^2 \\ &\quad + (1200\beta + 360\beta^2 + 1920)(\lambda_i^3\lambda_j^4 + \lambda_i^4\lambda_j^3)\lambda_l \\ &\quad + (240 + 600\beta)\lambda_i^4\lambda_j^4, \end{aligned}$$

Define $\xi_{16}(\lambda_j; \lambda_i; \beta) = \xi_1^{(6)}(\lambda_l; \lambda_j; \lambda_i; \beta)$, where all derivatives are with respect to λ_l . We wish to show that ξ_{16} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{16} with respect

to λ_j gives

$$\begin{aligned}\xi_{16}(\lambda_j; \lambda_i; \beta) &= (380160\beta + 1313280)\lambda_j^2 + (967680\beta - 1935360)\lambda_i\lambda_j \\ &\quad + (380160\beta + 1313280)\lambda_i^2, \\ \xi_{16}^{(1)}(\lambda_j; \lambda_i; \beta) &= (760320\beta + 2626560)\lambda_j + (967680\beta - 1935360)\lambda_i, \\ \xi_{16}^{(2)}(\lambda_j; \lambda_i; \beta) &= 760320\beta + 2626560 \geq 0 \text{ for all } \beta \geq -3.4545.\end{aligned}$$

Now

$$\begin{aligned}\xi_{16}(\lambda_i; \lambda_i; \beta) &= 345600\lambda_i^2(2 + 5\beta) \geq 0 \text{ for all } \beta \geq -\frac{2}{5}, \\ \xi_{16}^{(1)}(\lambda_i; \lambda_i; \beta) &= 345600\lambda_i(2 + 5\beta) \geq 0 \text{ for all } \beta \geq -\frac{2}{5}.\end{aligned}$$

Hence, $\xi_1^{(6)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{15}(\lambda_j; \lambda_i; \beta) = \xi_1^{(5)}(\lambda_j; \lambda_j; \lambda_i; \beta)$, where all derivatives are with respect to λ_l . We wish to show that ξ_{15} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{15} with respect to λ_j gives

$$\begin{aligned}\xi_{15}(\lambda_j; \lambda_i; \beta) &= (534720\beta + 2576640 + 10080\beta^2)\lambda_j^3 \\ &\quad + (1149120\beta - 2661120 + 90720\beta^2)\lambda_i\lambda_j^2 \\ &\quad + (561600\beta + 587520 + 90720\beta^2)\lambda_i^2\lambda_j \\ &\quad + (1263360 + 10080\beta^2 + 154560\beta)\lambda_i^3, \\ \xi_{15}^{(1)}(\lambda_j; \lambda_i; \beta) &= (1604160\beta + 7729920 + 30240\beta^2)\lambda_j^2 \\ &\quad + (2298240\beta - 5322240 + 181440\beta^2)\lambda_i\lambda_j \\ &\quad + (561600\beta + 587520 + 90720\beta^2)\lambda_i^2, \\ \xi_{15}^{(2)}(\lambda_j; \lambda_i; \beta) &= (3208320\beta + 15459840 + 60480\beta^2)\lambda_j \\ &\quad + (2298240\beta - 5322240 + 181440\beta^2)\lambda_i, \\ \xi_{15}^{(3)}(\lambda_j; \lambda_i; \beta) &= 3208320\beta + 15459840 + 60480\beta^2 \geq 0 \text{ for all } \beta \geq -5.3603.\end{aligned}$$

Now

$$\begin{aligned}\xi_{15}(\lambda_i; \lambda_i; \beta) &= 9600\lambda_i^3(250\beta + 184 + 21\beta^2) \geq 0 \text{ for all } \beta \geq -0.7882, \\ \xi_{15}^{(1)}(\lambda_i; \lambda_i; \beta) &= 14400\lambda_i^2(310\beta + 208 + 21\beta^2) \geq 0 \text{ for all } \beta \geq -0.7046, \\ \xi_{15}^{(2)}(\lambda_i; \lambda_i; \beta) &= 11520\lambda_i(478\beta + 880 + 21\beta^2) \geq 0 \text{ for all } \beta \geq -2.0203.\end{aligned}$$

Hence, $\xi_1^{(5)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{14}(\lambda_j; \lambda_i; \beta) = \xi_1^{(4)}(\lambda_j; \lambda_j; \lambda_i; \beta)$, where all derivatives are with respect to λ_l . We wish to show that ξ_{14} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{14} with

respect to λ_j gives

$$\begin{aligned} \xi_{14}(\lambda_j; \lambda_i; \beta) &= (376896\beta + 2093952 + 10944\beta^2)\lambda_j^4 \\ &\quad + (106272\beta^2 + 758592\beta - 1956096 + 1728\beta^3)\lambda_i\lambda_j^3 \\ &\quad + (143424\beta + 200448 + 127008\beta^2 + 5184\beta^3)\lambda_i^2\lambda_j^2 \\ &\quad + (1000704 + 25632\beta^2 + 247872\beta + 1728\beta^3)\lambda_i^3\lambda_j \\ &\quad + (173952 + 32256\beta + 864\beta^2)\lambda_i^4, \\ \xi_{14}^{(1)}(\lambda_j; \lambda_i; \beta) &= (1507584\beta + 8375808 + 43776\beta^2)\lambda_j^3 \\ &\quad + (318816\beta^2 + 2275776\beta - 5868288 + 5184\beta^3)\lambda_i\lambda_j^2 \\ &\quad + (286848\beta + 400896 + 254016\beta^2 + 10368\beta^3)\lambda_i^2\lambda_j \\ &\quad + (1000704 + 25632\beta^2 + 247872\beta + 1728\beta^3)\lambda_i^3, \\ \xi_{14}^{(2)}(\lambda_j; \lambda_i; \beta) &= (4522752\beta + 25127424 + 131328\beta^2)\lambda_j^2 \\ &\quad + (637632\beta^2 + 4551552\beta - 11736576 + 10368\beta^3)\lambda_i\lambda_j \\ &\quad + (286848\beta + 400896 + 254016\beta^2 + 10368\beta^3)\lambda_i^2, \\ \xi_{14}^{(3)}(\lambda_j; \lambda_i; \beta) &= (9045504\beta + 50254848 + 262656\beta^2)\lambda_j \\ &\quad + (637632\beta^2 + 4551552\beta - 11736576 + 10368\beta^3)\lambda_i, \\ \xi_{14}^{(4)}(\lambda_j; \lambda_i; \beta) &= 9045504\beta + 50254848 + 262656\beta^2 \geq 0 \text{ for all } \beta \geq -6.9640. \end{aligned}$$

Now

$$\begin{aligned} \xi_{14}(\lambda_i; \lambda_i; \beta) &= 960\lambda_i^4(1624\beta + 1576 + 282\beta^2 + 9\beta^3) \geq 0 \text{ for all } \beta \geq -1.2181, \\ \xi_{14}^{(1)}(\lambda_i; \lambda_i; \beta) &= 960\lambda_i^3(4498\beta + 4072 + 669\beta^2 + 18\beta^3) \geq 0 \text{ for all } \beta \geq -1.0710, \\ \xi_{14}^{(2)}(\lambda_i; \lambda_i; \beta) &= 2304\lambda_i^2(4063\beta + 5986 + 444\beta^2 + 9\beta^3) \geq 0 \text{ for all } \beta \geq -1.8231, \\ \xi_{14}^{(3)}(\lambda_i; \lambda_i; \beta) &= 576\lambda_i(23606\beta + 66872 + 1563\beta^2 + 18\beta^3) \geq 0 \text{ for all } \beta \geq -3.7012. \end{aligned}$$

Hence, $\xi_1^{(4)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{13}(\lambda_j; \lambda_i; \beta) = \xi_1^{(3)}(\lambda_j; \lambda_j; \lambda_i; \beta)/\lambda_j$, where all derivatives are with respect to λ_i . We wish to show that ξ_{13} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{13} with

respect to λ_j gives

$$\begin{aligned}
\xi_{13}(\lambda_j; \lambda_i; \beta) &= (1024512 + 172896\beta + 5904\beta^2)\lambda_j^4 \\
&\quad + (-994176 + 361872\beta + 64152\beta^2 + 1728\beta^3)\lambda_i\lambda_j^3 \\
&\quad + (134208 - 84816\beta + 80568\beta^2 + 7344\beta^3)\lambda_i^2\lambda_j^2 \\
&\quad + (19512\beta^2 + 159792\beta + 377664 + 3888\beta^3)\lambda_i^3\lambda_j \\
&\quad + (127872 + 48816\beta + 4104\beta^2)\lambda_i^4, \\
\xi_{13}^{(1)}(\lambda_j; \lambda_i; \beta) &= (4098048 + 691584\beta + 23616\beta^2)\lambda_j^3 \\
&\quad + (-2982528 + 1085616\beta + 192456\beta^2 + 5184\beta^3)\lambda_i\lambda_j^2 \\
&\quad + (268416 - 169632\beta + 161136\beta^2 + 14688\beta^3)\lambda_i^2\lambda_j \\
&\quad + (19512\beta^2 + 159792\beta + 377664 + 3888\beta^3)\lambda_i^3, \\
\xi_{13}^{(2)}(\lambda_j; \lambda_i; \beta) &= (12294144 + 2074752\beta + 70848\beta^2)\lambda_j^2 \\
&\quad + (-5965056 + 2171232\beta + 384912\beta^2 + 10368\beta^3)\lambda_i\lambda_j \\
&\quad + (268416 - 169632\beta + 161136\beta^2 + 14688\beta^3)\lambda_i^2, \\
\xi_{13}^{(3)}(\lambda_j; \lambda_i; \beta) &= (24588288 + 4149504\beta + 141696\beta^2)\lambda_j \\
&\quad + (-5965056 + 2171232\beta + 384912\beta^2 + 10368\beta^3)\lambda_i, \\
\xi_{13}^{(4)}(\lambda_j; \lambda_i; \beta) &= 24588288 + 4149504\beta + 141696\beta^2 \geq 0 \text{ for all } \beta \geq -8.2495.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{13}(\lambda_i; \lambda_i; \beta) &= 480\lambda_i^4(1396 + 1372\beta + 363\beta^2 + 27\beta^3) \geq 0 \text{ for all } \beta \geq -1.6479, \\
\xi_{13}^{(1)}(\lambda_i; \lambda_i; \beta) &= 240\lambda_i^3(\beta + 10)(99\beta^2 + 663\beta + 734) \geq 0 \text{ for all } \beta \geq -1.3996, \\
\xi_{13}^{(2)}(\lambda_i; \lambda_i; \beta) &= 864\lambda_i^2(7636 + 4718\beta + 714\beta^2 + 29\beta^3) \geq 0 \text{ for all } \beta \geq -2.4138, \\
\xi_{13}^{(3)}(\lambda_i; \lambda_i; \beta) &= 144\lambda_i(129328 + 43894\beta + 3657\beta^2 + 72\beta^3) \geq 0 \text{ for all } \beta \geq -4.4547.
\end{aligned}$$

Hence, $\xi_1^{(3)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{12}(\lambda_j; \lambda_i; \beta) = \xi_1^{(2)}(\lambda_j; \lambda_j; \lambda_i; \beta)/\lambda_j^2$, where all derivatives are with respect to λ_i . We wish to show that ξ_{12} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{12} with

respect to λ_j gives

$$\begin{aligned} \xi_{12}(\lambda_j; \lambda_i; \beta) &= (352256 + 2112\beta^2 + 57728\beta)\lambda_j^4 \\ &\quad + (864\beta^3 - 379008 + 133776\beta + 26136\beta^2)\lambda_i\lambda_j^3 \\ &\quad + (34344\beta^2 + 4752\beta^3 + 79104 - 81168\beta)\lambda_i^2\lambda_j^2 \\ &\quad + (89152 + 67696\beta + 3504\beta^3 + 7896\beta^2)\lambda_i^3\lambda_j \\ &\quad + (42816 + 30288\beta + 5832\beta^2)\lambda_i^4, \\ \xi_{12}^{(1)}(\lambda_j; \lambda_i; \beta) &= (1409024 + 8448\beta^2 + 230912\beta)\lambda_j^3 \\ &\quad + (2592\beta^3 - 1137024 + 401328\beta + 78408\beta^2)\lambda_i\lambda_j^2 \\ &\quad + (68688\beta^2 + 9504\beta^3 + 158208 - 162336\beta)\lambda_i^2\lambda_j \\ &\quad + (89152 + 67696\beta + 3504\beta^3 + 7896\beta^2)\lambda_i^3, \\ \xi_{12}^{(2)}(\lambda_j; \lambda_i; \beta) &= (4227072 + 25344\beta^2 + 692736\beta)\lambda_j^2 \\ &\quad + (5184\beta^3 - 2274048 + 802656\beta + 156816\beta^2)\lambda_i\lambda_j \\ &\quad + (68688\beta^2 + 9504\beta^3 + 158208 - 162336\beta)\lambda_i^2, \\ \xi_{12}^{(3)}(\lambda_j; \lambda_i; \beta) &= (8454144 + 50688\beta^2 + 1385472\beta)\lambda_j \\ &\quad + (5184\beta^3 - 2274048 + 802656\beta + 156816\beta^2)\lambda_i, \\ \xi_{12}^{(4)}(\lambda_j; \lambda_i; \beta) &= 8454144 + 50688\beta^2 + 1385472\beta \geq 0 \text{ for all } \beta \geq -9.1957. \end{aligned}$$

Now

$$\begin{aligned} \xi_{12}(\lambda_i; \lambda_i; \beta) &= 480\lambda_i^4(\beta + 3)(\beta + 2)(19\beta + 64) \geq 0 \text{ for all } \beta \geq -2, \\ \xi_{12}^{(1)}(\lambda_i; \lambda_i; \beta) &= 240\lambda_i^3(2164 + 681\beta^2 + 2240\beta + 65\beta^3) \geq 0 \text{ for all } \beta \geq -1.7098, \\ \xi_{12}^{(2)}(\lambda_i; \lambda_i; \beta) &= 96\lambda_i^2(21992 + 2613\beta^2 + 13886\beta + 153\beta^3) \geq 0 \text{ for all } \beta \geq -2.8871, \\ \xi_{12}^{(3)}(\lambda_i; \lambda_i; \beta) &= 48\lambda_i(128752 + 4323\beta^2 + 45586\beta + 108\beta^3) \geq 0 \text{ for all } \beta \geq -4.6015. \end{aligned}$$

Hence, $\xi_1^{(2)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{11}(\lambda_j; \lambda_i; \beta) = \xi_1^{(1)}(\lambda_j; \lambda_j; \lambda_i; \beta)/\lambda_j^3$, where all derivatives are with respect to λ_i . We wish to show that ξ_{11} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{11} with

respect to λ_j gives

$$\begin{aligned}
\xi_{11}(\lambda_j; \lambda_i; \beta) &= (92576 + 14984\beta + 564\beta^2)\lambda_j^4 \\
&\quad + (288\beta^3 + 7992\beta^2 + 39456\beta - 113184)\lambda_i\lambda_j^3 \\
&\quad + (-35088\beta + 11448\beta^2 + 31872 + 1944\beta^3)\lambda_i^2\lambda_j^2 \\
&\quad + (16384 + 23872\beta + 1848\beta^3 + 2352\beta^2)\lambda_i^3\lambda_j \\
&\quad + (9792 + 12456\beta + 4284\beta^2)\lambda_i^4, \\
\xi_{11}^{(1)}(\lambda_j; \lambda_i; \beta) &= (370304 + 59936\beta + 2256\beta^2)\lambda_j^3 \\
&\quad + (864\beta^3 + 23976\beta^2 + 118368\beta - 339552)\lambda_i\lambda_j^2 \\
&\quad + (-70176\beta + 22896\beta^2 + 63744 + 3888\beta^3)\lambda_i^2\lambda_j \\
&\quad + (16384 + 23872\beta + 1848\beta^3 + 2352\beta^2)\lambda_i^3, \\
\xi_{11}^{(2)}(\lambda_j; \lambda_i; \beta) &= (1110912 + 179808\beta + 6768\beta^2)\lambda_j^2 \\
&\quad + (1728\beta^3 + 47952\beta^2 + 236736\beta - 679104)\lambda_i\lambda_j \\
&\quad + (-70176\beta + 22896\beta^2 + 63744 + 3888\beta^3)\lambda_i^2, \\
\xi_{11}^{(3)}(\lambda_j; \lambda_i; \beta) &= (2221824 + 359616\beta + 13536\beta^2)\lambda_j \\
&\quad + (1728\beta^3 + 47952\beta^2 + 236736\beta - 679104)\lambda_i, \\
\xi_{11}^{(4)}(\lambda_j; \lambda_i; \beta) &= 2221824 + 359616\beta + 13536\beta^2 \geq 0 \text{ for all } \beta \geq -9.7745.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{11}(\lambda_i; \lambda_i; \beta) &= 240\lambda_i^4(17\beta + 26)(\beta + 3)(\beta + 2) \geq 0 \text{ for all } \beta \geq -\frac{26}{17}, \\
\xi_{11}^{(1)}(\lambda_i; \lambda_i; \beta) &= 1320\lambda_i^3(5\beta + 14)(\beta + 3)(\beta + 2) \geq 0 \text{ for all } \beta \geq -2, \\
\xi_{11}^{(2)}(\lambda_i; \lambda_i; \beta) &= 48\lambda_i^2(10324 + 7216\beta + 1617\beta^2 + 117\beta^3) \geq 0 \text{ for all } \beta \geq -3.1148, \\
\xi_{11}^{(3)}(\lambda_i; \lambda_i; \beta) &= 48\lambda_i(\beta + 10)(36\beta^2 + 921\beta + 3214) \geq 0 \text{ for all } \beta \geq -4.1691.
\end{aligned}$$

Hence, $\xi_1^{(1)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{10}(\lambda_j; \lambda_i; \beta) = \xi_1(\lambda_j; \lambda_j; \lambda_i; \beta)/\lambda_j^4$, where all derivatives are with respect to λ_i . We wish to show that ξ_{10} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{10} with

respect to λ_j gives

$$\begin{aligned}
 \xi_{10}(\lambda_j; \lambda_i; \beta) &= (19600 + 3160\beta + 120\beta^2)\lambda_j^4 \\
 &\quad + (72\beta^3 + 1944\beta^2 + 9504\beta - 27360)\lambda_i\lambda_j^3 \\
 &\quad + (-10464\beta + 3168\beta^2 + 9408 + 576\beta^3)\lambda_i^2\lambda_j^2 \\
 &\quad + (3584 + 7616\beta + 672\beta^3 + 672\beta^2)\lambda_i^3\lambda_j \\
 &\quad + (2688 + 4704\beta + 2016\beta^2)\lambda_i^4, \\
 \xi_{10}^{(1)}(\lambda_j; \lambda_i; \beta) &= (78400 + 12640\beta + 480\beta^2)\lambda_j^3 \\
 &\quad + (216\beta^3 + 5832\beta^2 + 28512\beta - 82080)\lambda_i\lambda_j^2 \\
 &\quad + (-20928\beta + 6336\beta^2 + 18816 + 1152\beta^3)\lambda_i^2\lambda_j \\
 &\quad + (3584 + 7616\beta + 672\beta^3 + 672\beta^2)\lambda_i^3, \\
 \xi_{10}^{(2)}(\lambda_j; \lambda_i; \beta) &= (235200 + 37920\beta + 1440\beta^2)\lambda_j^2 \\
 &\quad + (432\beta^3 + 11664\beta^2 + 57024\beta - 164160)\lambda_i\lambda_j \\
 &\quad + (-20928\beta + 6336\beta^2 + 18816 + 1152\beta^3)\lambda_i^2, \\
 \xi_{10}^{(3)}(\lambda_j; \lambda_i; \beta) &= (470400 + 75840\beta + 2880\beta^2)\lambda_j \\
 &\quad + (432\beta^3 + 11664\beta^2 + 57024\beta - 164160)\lambda_i, \\
 \xi_{10}^{(4)}(\lambda_j; \lambda_i; \beta) &= 960(3\beta + 49)(\beta + 10) \geq 0 \text{ for all } \beta \geq -10.
 \end{aligned}$$

Now

$$\begin{aligned}
 \xi_{10}(\lambda_i; \lambda_i; \beta) &= 1320\lambda_i^4(\beta + 3)(\beta + 2)(\beta + 1) > 0 \text{ for all } \beta > -1, \\
 \xi_{10}^{(1)}(\lambda_i; \lambda_i; \beta) &= 120\lambda_i^3(17\beta + 26)(\beta + 3)(\beta + 2) > 0 \text{ for all } \beta > -\frac{26}{17}, \\
 \xi_{10}^{(2)}(\lambda_i; \lambda_i; \beta) &= 144\lambda_i^2(\beta + 3)(11\beta^2 + 102\beta + 208) > 0 \text{ for all } \beta > -3, \\
 \xi_{10}^{(3)}(\lambda_i; \lambda_i; \beta) &= 48\lambda_i(\beta + 10)(9\beta^2 + 213\beta + 638) > 0 \text{ for all } \beta > -3.5184.
 \end{aligned}$$

Hence, ξ_1 is positive for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$. This implies that $\xi(\lambda_l; \lambda_l; \lambda_j; \lambda_i; \beta)$ is positive for all $\lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Now

$$\begin{aligned}
\xi_2(\lambda_l; \lambda_j; \lambda_i; \beta) = & (5472 + 1584\beta)(\lambda_j^2 + \lambda_i^2)\lambda_l^6 + (-8064 + 4032\beta)\lambda_i\lambda_j\lambda_l^6 \\
& + (3888\beta + 1944\beta^2 - 15552)(\lambda_i\lambda_j^2 + \lambda_i^2\lambda_j)\lambda_l^5 \\
& + (3312\beta + 216\beta^2 + 27072)(\lambda_j^3 + \lambda_i^3)\lambda_l^5 \\
& + (-23616 + 144\beta^3 + 8208\beta + 1512\beta^2)(\lambda_i\lambda_j^3 + \lambda_i^3\lambda_j)\lambda_l^4 \\
& + (2952\beta + 15408 + 72\beta^2)(\lambda_j^4 + \lambda_i^4)\lambda_l^4 \\
& + (25056 + 3672\beta^2 + 432\beta^3 - 21600\beta)\lambda_i^2\lambda_j^2\lambda_l^4 \\
& + (576\beta^3 + 144\beta^2 - 1152 - 2016\beta)(\lambda_i^2\lambda_j^3 + \lambda_i^3\lambda_j^2)\lambda_l^3 \\
& + (-14976 + 5760\beta + 864\beta^2)(\lambda_i\lambda_j^4 + \lambda_i^4\lambda_j)\lambda_l^3 \\
& + (288\beta + 2880 + 1296\beta^2)(\lambda_i^2\lambda_j^4 + \lambda_i^4\lambda_j^2)\lambda_l^2 \\
& + (-144\beta^2 + 8832 + 6240\beta + 288\beta^3)\lambda_i^3\lambda_j^3\lambda_l^2 \\
& + (7296 + 1536\beta + 288\beta^2)(\lambda_i^3\lambda_j^4 + \lambda_i^4\lambda_j^3)\lambda_l \\
& + (264\beta + 912)\lambda_i^4\lambda_j^4,
\end{aligned}$$

Define $\xi_{26}(\lambda_j; \lambda_i; \beta) = \xi_2^{(6)}(\lambda_l; \lambda_j; \lambda_i; \beta)$, where all derivatives are with respect to λ_l . We wish to show that ξ_{26} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{26} with respect to λ_j gives

$$\begin{aligned}
\xi_{26}(\lambda_j; \lambda_i; \beta) &= (1140480\beta + 3939840)\lambda_j^2 + (-5806080 + 2903040\beta)\lambda_i\lambda_j \\
&\quad + (1140480\beta + 3939840)\lambda_i^2, \\
\xi_{26}^{(1)}(\lambda_j; \lambda_i; \beta) &= (2280960\beta + 7879680)\lambda_j + (-5806080 + 2903040\beta)\lambda_i, \\
\xi_{26}^{(2)}(\lambda_j; \lambda_i; \beta) &= 2280960\beta + 7879680 \geq 0 \text{ for all } \beta \geq 3.4545.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{26}(\lambda_i; \lambda_i; \beta) &= 1036800\lambda_i^2(2 + 5\beta) \geq 0 \text{ for all } \beta \geq -\frac{2}{5}, \\
\xi_{26}^{(1)}(\lambda_i; \lambda_i; \beta) &= 1036800\lambda_i(2 + 5\beta) \geq 0 \text{ for all } \beta \geq -\frac{2}{5}.
\end{aligned}$$

Hence, $\xi_2^{(6)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{25}(\lambda_j; \lambda_i; \beta) = \xi_2^{(5)}(\lambda_j; \lambda_j; \lambda_i; \beta)$, where all derivatives are with respect to λ_l . We wish to show that ξ_{25} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{25} with

respect to λ_j gives

$$\begin{aligned}\xi_{25}(\lambda_j; \lambda_i; \beta) &= (1537920\beta + 7188480 + 25920\beta^2)\lambda_j^3 \\ &\quad + (-7672320 + 3369600\beta + 233280\beta^2)\lambda_i\lambda_j^2 \\ &\quad + (1607040\beta + 2073600 + 233280\beta^2)\lambda_i^2\lambda_j \\ &\quad + (397440\beta + 25920\beta^2 + 3248640)\lambda_i^3, \\ \xi_{25}^{(1)}(\lambda_j; \lambda_i; \beta) &= (4613760\beta + 21565440 + 77760\beta^2)\lambda_j^2 \\ &\quad + (-15344640 + 6739200\beta + 466560\beta^2)\lambda_i\lambda_j \\ &\quad + (1607040\beta + 2073600 + 233280\beta^2)\lambda_i^2, \\ \xi_{25}^{(2)}(\lambda_j; \lambda_i; \beta) &= (9227520\beta + 43130880 + 155520\beta^2)\lambda_j \\ &\quad + (-15344640 + 6739200\beta + 466560\beta^2)\lambda_i, \\ \xi_{25}^{(3)}(\lambda_j; \lambda_i; \beta) &= 9227520\beta + 43130880 + 155520\beta^2 \geq 0 \text{ for all } \beta \geq -5.1151.\end{aligned}$$

Now

$$\begin{aligned}\xi_{25}(\lambda_i; \lambda_i; \beta) &= 172800\lambda_i^3(28 + 40\beta + 3\beta^2) \geq 0 \text{ for all } \beta \geq -0.7412, \\ \xi_{25}^{(1)}(\lambda_i; \lambda_i; \beta) &= 259200\lambda_i^2(\beta + 16)(3\beta + 2) \geq 0 \text{ for all } \beta \geq -\frac{2}{3}, \\ \xi_{25}^{(2)}(\lambda_i; \lambda_i; \beta) &= 207360\lambda_i(77\beta + 134 + 3\beta^2) \geq 0 \text{ for all } \beta \geq -1.8776.\end{aligned}$$

Hence, $\xi_2^{(5)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{24}(\lambda_j; \lambda_i; \beta) = \xi_2^{(4)}(\lambda_j; \lambda_j; \lambda_i; \beta)$, where all derivatives are with respect to λ_l . We wish to show that ξ_{24} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{24} with respect to λ_j gives

$$\begin{aligned}\xi_{24}(\lambda_j; \lambda_i; \beta) &= (1038528\beta + 5588352 + 27648\beta^2)\lambda_j^4 \\ &\quad + (2115072\beta - 5336064 + 269568\beta^2 + 3456\beta^3)\lambda_i\lambda_j^3 \\ &\quad + (518400\beta + 705024 + 10368\beta^3 + 321408\beta^2)\lambda_i^2\lambda_j^2 \\ &\quad + (594432\beta + 62208\beta^2 + 2681856 + 3456\beta^3)\lambda_i^3\lambda_j \\ &\quad + (369792 + 70848\beta + 1728\beta^2)\lambda_i^4, \\ \xi_{24}^{(1)}(\lambda_j; \lambda_i; \beta) &= (4154112\beta + 22353408 + 110592\beta^2)\lambda_j^3 \\ &\quad + (6345216\beta - 16008192 + 808704\beta^2 + 10368\beta^3)\lambda_i\lambda_j^2 \\ &\quad + (1036800\beta + 1410048 + 20736\beta^3 + 642816\beta^2)\lambda_i^2\lambda_j \\ &\quad + (594432\beta + 62208\beta^2 + 2681856 + 3456\beta^3)\lambda_i^3, \\ \xi_{24}^{(2)}(\lambda_j; \lambda_i; \beta) &= (12462336\beta + 67060224 + 331776\beta^2)\lambda_j^2 \\ &\quad + (12690432\beta - 32016384 + 1617408\beta^2 + 20736\beta^3)\lambda_i\lambda_j \\ &\quad + (1036800\beta + 1410048 + 20736\beta^3 + 642816\beta^2)\lambda_i^2, \\ \xi_{24}^{(3)}(\lambda_j; \lambda_i; \beta) &= (24924672\beta + 134120448 + 663552\beta^2)\lambda_j \\ &\quad + (12690432\beta - 32016384 + 1617408\beta^2 + 20736\beta^3)\lambda_i, \\ \xi_{24}^{(4)}(\lambda_j; \lambda_i; \beta) &= 24924672\beta + 134120448 + 663552\beta^2 \geq 0 \text{ for all } \beta \geq -6.5089.\end{aligned}$$

Now

$$\begin{aligned}\xi_{24}(\lambda_i; \lambda_i; \beta) &= 8640\lambda_i^4(502\beta + 464 + 79\beta^2 + 2\beta^3) \geq 0 \text{ for all } \beta \geq -1.1141, \\ \xi_{24}^{(1)}(\lambda_i; \lambda_i; \beta) &= 34560\lambda_i^3(351\beta + 302 + 47\beta^2 + \beta^3) \geq 0 \text{ for all } \beta \geq -0.9885, \\ \xi_{24}^{(2)}(\lambda_i; \lambda_i; \beta) &= 20736\lambda_i^2(1263\beta + 1758 + 125\beta^2 + 2\beta^3) \geq 0 \text{ for all } \beta \geq -1.6562, \\ \xi_{24}^{(3)}(\lambda_i; \lambda_i; \beta) &= 20736\lambda_i(1814\beta + 4924 + 110\beta^2 + \beta^3) \geq 0 \text{ for all } \beta \geq -3.3897.\end{aligned}$$

Hence, $\xi_2^{(4)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{23}(\lambda_j; \lambda_i; \beta) = \xi_2^{(3)}(\lambda_j; \lambda_j; \lambda_i; \beta)/\lambda_j$, where all derivatives are with respect to λ_i . We wish to show that ξ_{23} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{23} with respect to λ_j gives

$$\begin{aligned}\xi_{23}(\lambda_j; \lambda_i; \beta) &= (2650752 + 459648\beta + 14688\beta^2)\lambda_j^4 \\ &\quad + (948672\beta + 3456\beta^3 + 158112\beta^2 - 2557440)\lambda_i\lambda_j^3 \\ &\quad + (205632\beta^2 + 13824\beta^3 + 317952 - 107136\beta)\lambda_i^2\lambda_j^2 \\ &\quad + (1050624 + 383616\beta + 6912\beta^3 + 50112\beta^2)\lambda_i^3\lambda_j \\ &\quad + (279936 + 105408\beta + 6912\beta^2)\lambda_i^4, \\ \xi_{23}^{(1)}(\lambda_j; \lambda_i; \beta) &= (10603008 + 1838592\beta + 58752\beta^2)\lambda_j^3 \\ &\quad + (2846016\beta + 10368\beta^3 + 474336\beta^2 - 7672320)\lambda_i\lambda_j^2 \\ &\quad + (411264\beta^2 + 27648\beta^3 + 635904 - 214272\beta)\lambda_i^2\lambda_j \\ &\quad + (1050624 + 383616\beta + 6912\beta^3 + 50112\beta^2)\lambda_i^3, \\ \xi_{23}^{(2)}(\lambda_j; \lambda_i; \beta) &= (31809024 + 5515776\beta + 176256\beta^2)\lambda_j^2 \\ &\quad + (5692032\beta + 20736\beta^3 + 948672\beta^2 - 15344640)\lambda_i\lambda_j \\ &\quad + (411264\beta^2 + 27648\beta^3 + 635904 - 214272\beta)\lambda_i^2, \\ \xi_{23}^{(3)}(\lambda_j; \lambda_i; \beta) &= (63618048 + 11031552\beta + 352512\beta^2)\lambda_j \\ &\quad + (5692032\beta + 20736\beta^3 + 948672\beta^2 - 15344640)\lambda_i, \\ \xi_{23}^{(4)}(\lambda_j; \lambda_i; \beta) &= 63618048 + 11031552\beta + 352512\beta^2 \geq 0 \text{ for all } \beta \geq -7.6246.\end{aligned}$$

Now

$$\begin{aligned}\xi_{23}(\lambda_i; \lambda_i; \beta) &= 24192\lambda_i^4(\beta + 4)(\beta^2 + 14\beta + 18) \geq 0 \text{ for all } \beta \geq -1.4322, \\ \xi_{23}^{(1)}(\lambda_i; \lambda_i; \beta) &= 864\lambda_i^3(5344 + 5618\beta + 1151\beta^2 + 52\beta^3) \geq 0 \text{ for all } \beta \geq -1.2562, \\ \xi_{23}^{(2)}(\lambda_i; \lambda_i; \beta) &= 1728\lambda_i^2(9896 + 6362\beta + 889\beta^2 + 28\beta^3) \geq 0 \text{ for all } \beta \geq -2.1667, \\ \xi_{23}^{(3)}(\lambda_i; \lambda_i; \beta) &= 5184\lambda_i(9312 + 3226\beta + 251\beta^2 + 4\beta^3) \geq 0 \text{ for all } \beta \geq -4.1213.\end{aligned}$$

Hence, $\xi_2^{(3)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{22}(\lambda_j; \lambda_i; \beta) = \xi_2^{(2)}(\lambda_j; \lambda_j; \lambda_i; \beta)/\lambda_j^2$, where all derivatives are with respect to λ_i . We wish to show that ξ_{22} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{22} with

respect to λ_j gives

$$\begin{aligned}
 \xi_{22}(\lambda_j; \lambda_i; \beta) &= (890496 + 149184\beta + 5184\beta^2)\lambda_j^4 \\
 &\quad + (1728\beta^3 + 62208\beta^2 - 926208 + 331776\beta)\lambda_i\lambda_j^3 \\
 &\quad + (-145440\beta + 86400\beta^2 + 152640 + 8640\beta^3)\lambda_i^2\lambda_j^2 \\
 &\quad + (268800 + 165120\beta + 5760\beta^3 + 23040\beta^2)\lambda_i^3\lambda_j \\
 &\quad + (100800 + 70560\beta + 8640\beta^2)\lambda_i^4, \\
 \xi_{22}^{(1)}(\lambda_j; \lambda_i; \beta) &= (3561984 + 596736\beta + 20736\beta^2)\lambda_j^3 \\
 &\quad + (5184\beta^3 + 186624\beta^2 - 2778624 + 995328\beta)\lambda_i\lambda_j^2 \\
 &\quad + (-290880\beta + 172800\beta^2 + 305280 + 17280\beta^3)\lambda_i^2\lambda_j \\
 &\quad + (268800 + 165120\beta + 5760\beta^3 + 23040\beta^2)\lambda_i^3, \\
 \xi_{22}^{(2)}(\lambda_j; \lambda_i; \beta) &= (10685952 + 1790208\beta + 62208\beta^2)\lambda_j^2 \\
 &\quad + (10368\beta^3 + 373248\beta^2 - 5557248 + 1990656\beta)\lambda_i\lambda_j \\
 &\quad + (-290880\beta + 172800\beta^2 + 305280 + 17280\beta^3)\lambda_i^2, \\
 \xi_{22}^{(3)}(\lambda_j; \lambda_i; \beta) &= (21371904 + 3580416\beta + 124416\beta^2)\lambda_j \\
 &\quad + (10368\beta^3 + 373248\beta^2 - 5557248 + 1990656\beta)\lambda_i, \\
 \xi_{22}^{(4)}(\lambda_j; \lambda_i; \beta) &= 21371904 + 3580416\beta + 124416\beta^2 \geq 0 \text{ for all } \beta \geq -8.4507.
 \end{aligned}$$

Now

$$\begin{aligned}
 \xi_{22}(\lambda_i; \lambda_i; \beta) &= 1344\lambda_i^4(362 + 425\beta + 138\beta^2 + 12\beta^3) \geq 0 \text{ for all } \beta \geq -1.4432, \\
 \xi_{22}^{(1)}(\lambda_i; \lambda_i; \beta) &= 1344\lambda_i^3(1010 + 1091\beta + 300\beta^2 + 21\beta^3) \geq 0 \text{ for all } \beta \geq -1.4354, \\
 \xi_{22}^{(2)}(\lambda_i; \lambda_i; \beta) &= 576\lambda_i^2(9434 + 6059\beta + 1056\beta^2 + 48\beta^3) \geq 0 \text{ for all } \beta \geq -2.5842, \\
 \xi_{22}^{(3)}(\lambda_i; \lambda_i; \beta) &= 3456\lambda_i(4576 + 1612\beta + 144\beta^2 + 3\beta^3) \geq 0 \text{ for all } \beta \geq -4.4300.
 \end{aligned}$$

Hence, $\xi_2^{(2)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{21}(\lambda_j; \lambda_i; \beta) = \xi_2^{(1)}(\lambda_j; \lambda_j; \lambda_i; \beta)/\lambda_j^3$, where all derivatives are with respect to λ_l . We wish to show that ξ_{21} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{21} with

respect to λ_j gives

$$\begin{aligned}
\xi_{21}(\lambda_j; \lambda_i; \beta) &= (229824 + 1368\beta^2 + 37872\beta)\lambda_j^4 \\
&\quad + (-265536 + 18360\beta^2 + 576\beta^3 + 93744\beta)\lambda_i\lambda_j^3 \\
&\quad + (-62928\beta + 27432\beta^2 + 3456\beta^3 + 57600)\lambda_i^2\lambda_j^2 \\
&\quad + (62400 + 7560\beta^2 + 2880\beta^3 + 57360\beta)\lambda_i^3\lambda_j \\
&\quad + (29760 + 31200\beta + 5760\beta^2)\lambda_i^4, \\
\xi_{21}^{(1)}(\lambda_j; \lambda_i; \beta) &= (919296 + 5472\beta^2 + 151488\beta)\lambda_j^3 \\
&\quad + (-796608 + 55080\beta^2 + 1728\beta^3 + 281232\beta)\lambda_i\lambda_j^2 \\
&\quad + (-125856\beta + 54864\beta^2 + 6912\beta^3 + 115200)\lambda_i^2\lambda_j \\
&\quad + (62400 + 7560\beta^2 + 2880\beta^3 + 57360\beta)\lambda_i^3, \\
\xi_{21}^{(2)}(\lambda_j; \lambda_i; \beta) &= (2757888 + 16416\beta^2 + 454464\beta)\lambda_j^2 \\
&\quad + (-1593216 + 110160\beta^2 + 3456\beta^3 + 562464\beta)\lambda_i\lambda_j \\
&\quad + (-125856\beta + 54864\beta^2 + 6912\beta^3 + 115200)\lambda_i^2, \\
\xi_{21}^{(3)}(\lambda_j; \lambda_i; \beta) &= (5515776 + 32832\beta^2 + 908928\beta)\lambda_j \\
&\quad + (-1593216 + 110160\beta^2 + 3456\beta^3 + 562464\beta)\lambda_i, \\
\xi_{21}^{(4)}(\lambda_j; \lambda_i; \beta) &= 5515776 + 32832\beta^2 + 908928\beta \geq 0 \text{ for all } \beta \geq -13.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{21}(\lambda_i; \lambda_i; \beta) &= 1728\lambda_i^4(\beta + 3)(4\beta^2 + 23\beta + 22) \geq 0 \text{ for all } \beta \geq -1.2120, \\
\xi_{21}^{(1)}(\lambda_i; \lambda_i; \beta) &= 96\lambda_i^3(3128 + 1281\beta^2 + 3794\beta + 120\beta^3) \geq 0 \text{ for all } \beta \geq -1.3977, \\
\xi_{21}^{(2)}(\lambda_i; \lambda_i; \beta) &= 576\lambda_i^2(2222 + 315\beta^2 + 1547\beta + 18\beta^3) \geq 0 \text{ for all } \beta \geq -2.6483, \\
\xi_{21}^{(3)}(\lambda_i; \lambda_i; \beta) &= 432\lambda_i(9080 + 331\beta^2 + 3406\beta + 8\beta^3) \geq 0 \text{ for all } \beta \geq -4.2200.
\end{aligned}$$

Hence, $\xi_2^{(1)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{20}(\lambda_j; \lambda_i; \beta) = \xi_2(\lambda_j; \lambda_j; \lambda_i; \beta)/\lambda_j^4$, where all derivatives are with respect to λ_i . We wish to show that ξ_{20} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{20} with

respect to λ_j gives

$$\begin{aligned}
 \xi_{20}(\lambda_j; \lambda_i; \beta) &= (47952 + 288\beta^2 + 7848\beta)\lambda_j^4 \\
 &\quad + (4320\beta^2 + 21888\beta + 144\beta^3 - 62208)\lambda_i\lambda_j^3 \\
 &\quad + (-17856\beta + 7056\beta^2 + 1008\beta^3 + 16704)\lambda_i^2\lambda_j^2 \\
 &\quad + (18432 + 17280\beta + 2016\beta^2 + 1008\beta^3)\lambda_i^3\lambda_j \\
 &\quad + (11520 + 10800\beta + 2520\beta^2)\lambda_i^4, \\
 \xi_{20}^{(1)}(\lambda_j; \lambda_i; \beta) &= (191808 + 1152\beta^2 + 31392\beta)\lambda_j^3 \\
 &\quad + (12960\beta^2 + 65664\beta + 432\beta^3 - 186624)\lambda_i\lambda_j^2 \\
 &\quad + (-35712\beta + 14112\beta^2 + 2016\beta^3 + 33408)\lambda_i^2\lambda_j \\
 &\quad + (18432 + 17280\beta + 2016\beta^2 + 1008\beta^3)\lambda_i^3, \\
 \xi_{20}^{(2)}(\lambda_j; \lambda_i; \beta) &= (575424 + 3456\beta^2 + 94176\beta)\lambda_j^2 \\
 &\quad + (25920\beta^2 + 131328\beta + 864\beta^3 - 373248)\lambda_i\lambda_j \\
 &\quad + (-35712\beta + 14112\beta^2 + 2016\beta^3 + 33408)\lambda_i^2, \\
 \xi_{20}^{(3)}(\lambda_j; \lambda_i; \beta) &= (1150848 + 6912\beta^2 + 188352\beta)\lambda_j \\
 &\quad + (25920\beta^2 + 131328\beta + 864\beta^3 - 373248)\lambda_i, \\
 \xi_{20}^{(4)}(\lambda_j; \lambda_i; \beta) &= 1728(\beta + 18)(4\beta + 37) \geq 0 \text{ for all } \beta \geq -\frac{37}{4}.
 \end{aligned}$$

Now

$$\begin{aligned}
 \xi_{20}(\lambda_i; \lambda_i; \beta) &= 1080\lambda_i^4(2\beta + 5)(\beta + 3)(\beta + 2) > 0 \text{ for all } \beta > -2, \\
 \xi_{20}^{(1)}(\lambda_i; \lambda_i; \beta) &= 864\lambda_i^3(\beta + 3)(4\beta^2 + 23\beta + 22) > 0 \text{ for all } \beta > -1.2120, \\
 \xi_{20}^{(2)}(\lambda_i; \lambda_i; \beta) &= 288\lambda_i^2(818 + 151\beta^2 + 659\beta + 10\beta^3) > 0 \text{ for all } \beta > -2.1484, \\
 \xi_{20}^{(3)}(\lambda_i; \lambda_i; \beta) &= 864\lambda_i(\beta + 10)(\beta^2 + 28\beta + 90) > 0 \text{ for all } \beta > -3.7044.
 \end{aligned}$$

Hence, ξ_2 is positive for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$. This implies that $\xi(\lambda_l; \lambda_l; \lambda_j; \lambda_i; \beta)$ is positive for all $\lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Now

$$\begin{aligned}
\xi_3(\lambda_l; \lambda_j; \lambda_i; \beta) = & (10944 + 3168\beta)(\lambda_i^2 + \lambda_j^2)\lambda_l^5 + (8064\beta - 16128)\lambda_i\lambda_j\lambda_l^5 \\
& + (6480\beta + 3240\beta^2 - 25920)(\lambda_i\lambda_j^2 + \lambda_i^2\lambda_j)\lambda_l^4 \\
& + (360\beta^2 + 45120 + 5520\beta)(\lambda_j^3 + \lambda_i^3)\lambda_l^4 \\
& + (2592\beta^2 - 32256 + 10368\beta + 144\beta^3)(\lambda_i\lambda_j^3 + \lambda_i^3\lambda_j) \\
& + (72\beta^2 + 19968 + 4272\beta)(\lambda_j^4 + \lambda_i^4) \\
& + (432\beta^3 + 38016 - 34560\beta + 6912\beta^2)\lambda_i^2\lambda_j^2\lambda_l^3 \\
& + (432\beta^3 - 13824 + 1728\beta + 1728\beta^2)(\lambda_i^2\lambda_j^3 + \lambda_i^3\lambda_j^2)\lambda_l^2 \\
& + (-21312 + 9360\beta + 648\beta^2)(\lambda_i\lambda_j^4 + \lambda_i^4\lambda_j)\lambda_l^2 \\
& + (5760 + 4464\beta + 648\beta^2)(\lambda_i^2\lambda_j^4 + \lambda_i^4\lambda_j^2)\lambda_l \\
& + (144\beta^3 + 288\beta^2 + 9600\beta + 24576)\lambda_i^3\lambda_j^3\lambda_l \\
& + (9024 + 1104\beta + 72\beta^2)\lambda_i^3\lambda_j^4 + (9024 + 1104\beta + 72\beta^2)\lambda_i^4\lambda_j^3.
\end{aligned}$$

Define $\xi_{35}(\lambda_j; \lambda_i; \beta) = \xi_3^{(4)}(\lambda_l; \lambda_j; \lambda_i; \beta)$, where all derivatives are with respect to λ_l . We wish to show that ξ_{35} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{35} with respect to λ_j gives

$$\begin{aligned}
\xi_{35}(\lambda_j; \lambda_i; \beta) &= (380160\beta + 1313280)\lambda_j^2 + (967680\beta - 1935360)\lambda_i\lambda_j \\
&\quad + (380160\beta + 1313280)\lambda_i^2, \\
\xi_{35}^{(1)}(\lambda_j; \lambda_i; \beta) &= (760320\beta + 2626560)\lambda_j + (967680\beta - 1935360)\lambda_i, \\
\xi_{35}^{(2)}(\lambda_j; \lambda_i; \beta) &= 760320\beta + 2626560 \geq 0 \text{ for all } \beta \geq -3.4545.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{35}(\lambda_i; \lambda_i; \beta) &= 345600\lambda_i^2(2 + 5\beta) \geq 0 \text{ for all } \beta \geq -\frac{2}{5}, \\
\xi_{35}^{(1)}(\lambda_i; \lambda_i; \beta) &= 345600\lambda_i(2 + 5\beta) \geq 0 \text{ for all } \beta \geq -\frac{2}{5}.
\end{aligned}$$

Hence, $\xi_3^{(4)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{34}(\lambda_j; \lambda_i; \beta) = \xi_3^{(4)}(\lambda_l; \lambda_j; \lambda_i; \beta)$, where all derivatives are with respect to λ_l . We wish to show that ξ_{34} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{34} with respect

to λ_j gives

$$\begin{aligned}\xi_{34}(\lambda_j; \lambda_i; \beta) &= (512640\beta + 2396160 + 8640\beta^2)\lambda_j^3 \\ &\quad + (1123200\beta - 2557440 + 77760\beta^2)\lambda_i\lambda_j^2 \\ &\quad + (535680\beta + 691200 + 77760\beta^2)\lambda_i^2\lambda_j \\ &\quad + (1082880 + 132480\beta + 8640\beta^2)\lambda_i^3, \\ \xi_{34}^{(1)}(\lambda_j; \lambda_i; \beta) &= (1537920\beta + 7188480 + 25920\beta^2)\lambda_j^2 \\ &\quad + (2246400\beta - 5114880 + 155520\beta^2)\lambda_i\lambda_j \\ &\quad + (535680\beta + 691200 + 77760\beta^2)\lambda_i^2, \\ \xi_{34}^{(2)}(\lambda_j; \lambda_i; \beta) &= (3075840\beta + 14376960 + 51840\beta^2)\lambda_j \\ &\quad + (2246400\beta - 5114880 + 155520\beta^2)\lambda_i, \\ \xi_{34}^{(3)}(\lambda_j; \lambda_i; \beta) &= 3075840\beta + 14376960 + 51840\beta^2 \geq 0 \text{ for all } \beta \geq -5.1151.\end{aligned}$$

Now

$$\begin{aligned}\xi_{34}(\lambda_i; \lambda_i; \beta) &= 57600\lambda_i^3(28 + 40\beta + 3\beta^2) \geq 0 \text{ for all } \beta \geq -0.7412, \\ \xi_{34}^{(1)}(\lambda_i; \lambda_i; \beta) &= 86400\lambda_i^2(\beta + 16)(3\beta + 2) \geq 0 \text{ for all } \beta \geq -\frac{2}{3}, \\ \xi_{34}^{(2)}(\lambda_i; \lambda_i; \beta) &= 69120\lambda_i(77\beta + 134 + 3\beta^2) \geq 0 \text{ for all } \beta \geq -1.8776.\end{aligned}$$

Hence, $\xi_3^{(4)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{33}(\lambda_j; \lambda_i; \beta) = \xi_3^{(3)}(\lambda_j; \lambda_j; \lambda_i; \beta)$, where all derivatives are with respect to λ_l . We wish to show that ξ_{33} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{33} with respect to λ_j gives

$$\begin{aligned}\xi_{33}(\lambda_j; \lambda_i; \beta) &= (1859328 + 9072\beta^2 + 348192\beta)\lambda_j^4 \\ &\quad + (701568\beta + 93312\beta^2 + 864\beta^3 - 1783296)\lambda_i\lambda_j^3 \\ &\quad + (138240\beta + 119232\beta^2 + 2592\beta^3 + 262656)\lambda_i^2\lambda_j^2 \\ &\quad + (889344 + 864\beta^3 + 194688\beta + 24192\beta^2)\lambda_i^3\lambda_j \\ &\quad + (432\beta^2 + 25632\beta + 119808)\lambda_i^4, \\ \xi_{33}^{(1)}(\lambda_j; \lambda_i; \beta) &= (7437312 + 36288\beta^2 + 1392768\beta)\lambda_j^3 \\ &\quad + (2104704\beta + 279936\beta^2 + 2592\beta^3 - 5349888)\lambda_i\lambda_j^2 \\ &\quad + (276480\beta + 238464\beta^2 + 5184\beta^3 + 525312)\lambda_i^2\lambda_j \\ &\quad + (889344 + 864\beta^3 + 194688\beta + 24192\beta^2)\lambda_i^3, \\ \xi_{33}^{(2)}(\lambda_j; \lambda_i; \beta) &= (22311936 + 108864\beta^2 + 4178304\beta)\lambda_j^2 \\ &\quad + (4209408\beta + 559872\beta^2 + 5184\beta^3 - 10699776)\lambda_i\lambda_j \\ &\quad + (276480\beta + 238464\beta^2 + 5184\beta^3 + 525312)\lambda_i^2, \\ \xi_{33}^{(3)}(\lambda_j; \lambda_i; \beta) &= (44623872 + 217728\beta^2 + 8356608\beta)\lambda_j \\ &\quad + (4209408\beta + 559872\beta^2 + 5184\beta^3 - 10699776)\lambda_i, \\ \xi_{33}^{(4)}(\lambda_j; \lambda_i; \beta) &= 44623872 + 217728\beta^2 + 8356608\beta \geq 0 \text{ for all } \beta \geq -6.4107.\end{aligned}$$

Now

$$\begin{aligned}
\xi_{33}(\lambda_i; \lambda_i; \beta) &= 4320\lambda_i^4(312 + 57\beta^2 + 326\beta + \beta^3) \geq 0 \text{ for all } \beta \geq -1.2060, \\
\xi_{33}^{(1)}(\lambda_i; \lambda_i; \beta) &= 2880\lambda_i^3(1216 + 201\beta^2 + 1378\beta + 3\beta^3) \geq 0 \text{ for all } \beta \geq -1.0368, \\
\xi_{33}^{(2)}(\lambda_i; \lambda_i; \beta) &= 1728\lambda_i^2(7024 + 525\beta^2 + 5014\beta + 6\beta^3) \geq 0 \text{ for all } \beta \geq -1.6963, \\
\xi_{33}^{(3)}(\lambda_i; \lambda_i; \beta) &= 5184\lambda_i(6544 + 150\beta^2 + 2424\beta + \beta^3) \geq 0 \text{ for all } \beta \geq -3.3980.
\end{aligned}$$

Hence, $\xi_3^{(3)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{32}(\lambda_j; \lambda_i; \beta) = \xi_3^{(2)}(\lambda_j; \lambda_j; \lambda_i; \beta)/\lambda_j$. We wish to show that ξ_{32} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{32} with respect to λ_j gives

$$\begin{aligned}
\xi_{32}(\lambda_j; \lambda_i; \beta) &= (155232\beta + 880128 + 4752\beta^2)\lambda_j^4 \\
&\quad + (-869760 + 319968\beta + 864\beta^3 + 55728\beta^2)\lambda_i\lambda_j^3 \\
&\quad + (83808\beta^2 + 3456\beta^3 + 108288 - 62784\beta)\lambda_i^2\lambda_j^2 \\
&\quad + (23328\beta^2 + 1728\beta^3 + 320256 + 131904\beta)\lambda_i^3\lambda_j \\
&\quad + (77184 + 1728\beta^2 + 44352\beta)\lambda_i^4, \\
\xi_{32}^{(1)}(\lambda_j; \lambda_i; \beta) &= (620928\beta + 3520512 + 19008\beta^2)\lambda_j^3 \\
&\quad + (-2609280 + 959904\beta + 2592\beta^3 + 167184\beta^2)\lambda_i\lambda_j^2 \\
&\quad + (167616\beta^2 + 6912\beta^3 + 216576 - 125568\beta)\lambda_i^2\lambda_j \\
&\quad + (23328\beta^2 + 1728\beta^3 + 320256 + 131904\beta)\lambda_i^3, \\
\xi_{32}^{(2)}(\lambda_j; \lambda_i; \beta) &= (1862784\beta + 10561536 + 57024\beta^2)\lambda_j^2 \\
&\quad + (-5218560 + 1919808\beta + 5184\beta^3 + 334368\beta^2)\lambda_i\lambda_j \\
&\quad + (167616\beta^2 + 6912\beta^3 + 216576 - 125568\beta)\lambda_i^2, \\
\xi_{32}^{(3)}(\lambda_j; \lambda_i; \beta) &= (3725568\beta + 21123072 + 114048\beta^2)\lambda_j \\
&\quad + (-5218560 + 1919808\beta + 5184\beta^3 + 334368\beta^2)\lambda_i, \\
\xi_{32}^{(4)}(\lambda_j; \lambda_i; \beta) &= 3725568\beta + 21123072 + 114048\beta^2 \geq 0 \text{ for all } \beta \geq -7.3020.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{32}(\lambda_i; \lambda_i; \beta) &= 2016\lambda_i^4(292\beta + 256 + 84\beta^2 + 3\beta^3) \geq 0 \text{ for all } \beta \geq -1.4510, \\
\xi_{32}^{(1)}(\lambda_i; \lambda_i; \beta) &= 432\lambda_i^3(3674\beta + 3352 + 873\beta^2 + 26\beta^3) \geq 0 \text{ for all } \beta \geq -1.2961, \\
\xi_{32}^{(2)}(\lambda_i; \lambda_i; \beta) &= 288\lambda_i^2(12698\beta + 19304 + 1941\beta^2 + 42\beta^3) \geq 0 \text{ for all } \beta \geq -2.2678, \\
\xi_{32}^{(3)}(\lambda_i; \lambda_i; \beta) &= 2592\lambda_i(2178\beta + 6136 + 173\beta^2 + 2\beta^3) \geq 0 \text{ for all } \beta \geq -4.0728.
\end{aligned}$$

Hence, $\xi_3^{(2)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{31}(\lambda_j; \lambda_i; \beta) = \xi_3^{(1)}(\lambda_j; \lambda_j; \lambda_i; \beta)/\lambda_j^2$. We wish to show that ξ_{31} is non-negative

for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{31} with respect to λ_j gives

$$\begin{aligned} \xi_{31}(\lambda_j; \lambda_i; \beta) &= (295104 + 1656\beta^2 + 50736\beta)\lambda_j^4 \\ &\quad + (432\beta^3 + 116064\beta - 323712 + 22032\beta^2)\lambda_i\lambda_j^3 \\ &\quad + (43200 + 37800\beta^2 + 2160\beta^3 - 54000\beta)\lambda_i^2\lambda_j^2 \\ &\quad + (80640 + 66240\beta + 1440\beta^3 + 12960\beta^2)\lambda_i^3\lambda_j \\ &\quad + (23040 + 36000\beta + 2160\beta^2)\lambda_i^4, \\ \xi_{31}^{(1)}(\lambda_j; \lambda_i; \beta) &= (1180416 + 6624\beta^2 + 202944\beta)\lambda_j^3 \\ &\quad + (1296\beta^3 + 348192\beta - 971136 + 66096\beta^2)\lambda_i\lambda_j^2 \\ &\quad + (86400 + 75600\beta^2 + 4320\beta^3 - 108000\beta)\lambda_i^2\lambda_j \\ &\quad + (80640 + 66240\beta + 1440\beta^3 + 12960\beta^2)\lambda_i^3, \\ \xi_{31}^{(2)}(\lambda_j; \lambda_i; \beta) &= (3541248 + 19872\beta^2 + 608832\beta)\lambda_j^2 \\ &\quad + (2592\beta^3 + 696384\beta - 1942272 + 132192\beta^2)\lambda_i\lambda_j \\ &\quad + (86400 + 75600\beta^2 + 4320\beta^3 - 108000\beta)\lambda_i^2, \\ \xi_{31}^{(3)}(\lambda_j; \lambda_i; \beta) &= (7082496 + 39744\beta^2 + 1217664\beta)\lambda_j \\ &\quad + (2592\beta^3 + 696384\beta - 1942272 + 132192\beta^2)\lambda_i, \\ \xi_{31}^{(4)}(\lambda_j; \lambda_i; \beta) &= 7082496 + 39744\beta^2 + 1217664\beta \geq 0 \text{ for all } \beta \geq -7.8046. \end{aligned}$$

Now

$$\begin{aligned} \xi_{31}(\lambda_i; \lambda_i; \beta) &= 1344\lambda_i^4(88 + 57\beta^2 + 160\beta + 3\beta^3) \geq 0 \text{ for all } \beta \geq -0.7350, \\ \xi_{31}^{(1)}(\lambda_i; \lambda_i; \beta) &= 336\lambda_i^3(1120 + 480\beta^2 + 1516\beta + 21\beta^3) \geq 0 \text{ for all } \beta \geq -1.1099, \\ \xi_{31}^{(2)}(\lambda_i; \lambda_i; \beta) &= 144\lambda_i^2(11704 + 1581\beta^2 + 8314\beta + 48\beta^3) \geq 0 \text{ for all } \beta \geq -2.5399, \\ \xi_{31}^{(3)}(\lambda_i; \lambda_i; \beta) &= 288\lambda_i(17848 + 597\beta^2 + 6646\beta + 9\beta^3) \geq 0 \text{ for all } \beta \geq -4.1067. \end{aligned}$$

Hence, $\xi_3^{(1)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{30}(\lambda_j; \lambda_i; \beta) = \xi_3(\lambda_j; \lambda_j; \lambda_i; \beta)/\lambda_j^3$. We wish to show that ξ_{30} is non-negative for

all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{30} with respect to λ_j gives

$$\begin{aligned}\xi_{30}(\lambda_j; \lambda_i; \beta) &= (76032 + 432\beta^2 + 12960\beta)\lambda_j^4 \\ &\quad + (144\beta^3 + 6480\beta^2 - 95616 + 34272\beta)\lambda_i\lambda_j^3 \\ &\quad + (-18720\beta + 12528\beta^2 + 14976 + 864\beta^3)\lambda_i^2\lambda_j^2 \\ &\quad + (32640 + 28320\beta + 720\beta^3 + 5040\beta^2)\lambda_i^3\lambda_j \\ &\quad + (13440 + 19200\beta + 1440\beta^2)\lambda_i^4, \\ \xi_{30}^{(1)}(\lambda_j; \lambda_i; \beta) &= (304128 + 1728\beta^2 + 51840\beta)\lambda_j^3 \\ &\quad + (432\beta^3 + 19440\beta^2 - 286848 + 102816\beta)\lambda_i\lambda_j^2 \\ &\quad + (-37440\beta + 25056\beta^2 + 29952 + 1728\beta^3)\lambda_i^2\lambda_j \\ &\quad + (32640 + 28320\beta + 720\beta^3 + 5040\beta^2)\lambda_i^3, \\ \xi_{30}^{(2)}(\lambda_j; \lambda_i; \beta) &= (912384 + 5184\beta^2 + 155520\beta)\lambda_j^2 \\ &\quad + (864\beta^3 + 38880\beta^2 - 573696 + 205632\beta)\lambda_i\lambda_j \\ &\quad + (-37440\beta + 25056\beta^2 + 29952 + 1728\beta^3)\lambda_i^2, \\ \xi_{30}^{(3)}(\lambda_j; \lambda_i; \beta) &= (1824768 + 10368\beta^2 + 311040\beta)\lambda_j \\ &\quad + (864\beta^3 + 38880\beta^2 - 573696 + 205632\beta)\lambda_i, \\ \xi_{30}^{(4)}(\lambda_j; \lambda_i; \beta) &= 10368(\beta + 22)(8 + \beta) \geq 0 \text{ for all } \beta \geq -8.\end{aligned}$$

Now

$$\begin{aligned}\xi_{30}(\lambda_i; \lambda_i; \beta) &= 1728\lambda_i^4(\beta + 3)(\beta^2 + 12\beta + 8) > 0 \text{ for all } \beta > -0.7085, \\ \xi_{30}^{(1)}(\lambda_i; \lambda_i; \beta) &= 192\lambda_i^3(416 + 267\beta^2 + 758\beta + 15\beta^3) > 0 \text{ for all } \beta > -0.7277, \\ \xi_{30}^{(2)}(\lambda_i; \lambda_i; \beta) &= 288\lambda_i^2(1280 + 240\beta^2 + 1124\beta + 9\beta^3) > 0 \text{ for all } \beta > -1.7494, \\ \xi_{30}^{(3)}(\lambda_i; \lambda_i; \beta) &= 864\lambda_i(1448 + 57\beta^2 + 598\beta + \beta^3) > 0 \text{ for all } \beta > -3.5444.\end{aligned}$$

Hence, ξ_3 is positive for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$. This implies that $\xi(\lambda_l; \lambda_l; \lambda_j; \lambda_i; \beta)$ is positive for all $\lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Differentiating $\xi_4(\lambda_l; \lambda_j; \lambda_i; \beta)$ with respect to λ_l we obtain

$$\begin{aligned}\xi_4(\lambda_l; \lambda_j; \lambda_i; \beta) &= (8064\beta - 16128)\lambda_i\lambda_j\lambda_l^4 + (10944 + 3168\beta)(\lambda_i^2 + \lambda_j^2)\lambda_l^4 \\ &\quad + (10944 + 3168\beta)(\lambda_j^4 + \lambda_i^4)\lambda_l^2 \\ &\quad + (5184\beta + 2592\beta^2 - 20736)(\lambda_i^3\lambda_j + \lambda_i\lambda_j^3)\lambda_l^2 \\ &\quad + (7776\beta^2 + 31104 - 31104\beta)\lambda_i^2\lambda_j^2\lambda_l^2 \\ &\quad + (4416\beta + 288\beta^2 + 36096)(\lambda_j^3 + \lambda_i^3)\lambda_l^3 \\ &\quad + (5184\beta + 2592\beta^2 - 20736)(\lambda_i\lambda_j^2 + \lambda_i^2\lambda_j)\lambda_l^3 \\ &\quad + (10944 + 3168\beta)(\lambda_i^2\lambda_j^4 + \lambda_i^4\lambda_j^2) \\ &\quad + (8064\beta - 16128)(\lambda_i\lambda_j^4 + \lambda_i^4\lambda_j)\lambda_l \\ &\quad + (5184\beta + 2592\beta^2 - 20736)(\lambda_i^3\lambda_j^2 + \lambda_i^2\lambda_j^3)\lambda_l \\ &\quad + (4416\beta + 288\beta^2 + 36096)\lambda_i^3\lambda_j^3.\end{aligned}$$

Define $\xi_{44}(\lambda_j; \lambda_i; \beta) = \xi_4^{(4)}(\lambda_i; \lambda_j; \lambda_i; \beta)$. We wish to show that ξ_{44} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{44} with respect to λ_j gives

$$\begin{aligned}\xi_{44}(\lambda_j; \lambda_i; \beta) &= (76032\beta + 262656)\lambda_j^2 + (193536\beta - 387072)\lambda_i\lambda_j \\ &\quad + (76032\beta + 262656)\lambda_i^2, \\ \xi_{44}^{(1)}(\lambda_j; \lambda_i; \beta) &= (152064\beta + 525312)\lambda_j + (193536\beta - 387072)\lambda_i, \\ \xi_{44}^{(2)}(\lambda_j; \lambda_i; \beta) &= 152064\beta + 525312 \geq 0 \text{ for all } \beta \geq -3.4545.\end{aligned}$$

Now

$$\begin{aligned}\xi_{44}(\lambda_i; \lambda_i; \beta) &= 69120\lambda_i^2(2 + 5\beta) \geq 0 \text{ for all } \beta \geq -\frac{2}{5}, \\ \xi_{44}^{(1)}(\lambda_i; \lambda_i; \beta) &= 69120\lambda_i(2 + 5\beta) \geq 0 \text{ for all } \beta \geq -\frac{2}{5}.\end{aligned}$$

Hence, $\xi_4^{(4)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{43}(\lambda_j; \lambda_i; \beta) = \xi_4^{(3)}(\lambda_j; \lambda_j; \lambda_i; \beta)$. We wish to show that ξ_{43} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{43} with respect to λ_j gives

$$\begin{aligned}\xi_{43}(\lambda_j; \lambda_i; \beta) &= (102528\beta + 479232 + 1728\beta^2)\lambda_j^3 + (224640\beta + 15552\beta^2 - 511488)\lambda_i\lambda_j^2 \\ &\quad + (107136\beta + 138240 + 15552\beta^2)\lambda_i^2\lambda_j + (26496\beta + 216576 + 1728\beta^2)\lambda_i^3, \\ \xi_{43}^{(1)}(\lambda_j; \lambda_i; \beta) &= (307584\beta + 1437696 + 5184\beta^2)\lambda_j^2 + (449280\beta + 31104\beta^2 - 1022976)\lambda_i\lambda_j \\ &\quad + (107136\beta + 138240 + 15552\beta^2)\lambda_i^2, \\ \xi_{43}^{(2)}(\lambda_j; \lambda_i; \beta) &= (615168\beta + 2875392 + 10368\beta^2)\lambda_j + (449280\beta + 31104\beta^2 - 1022976)\lambda_i, \\ \xi_{43}^{(3)}(\lambda_j; \lambda_i; \beta) &= 615168\beta + 2875392 + 10368\beta^2 \geq 0 \text{ for all } \beta \geq -5.1151.\end{aligned}$$

Now

$$\begin{aligned}\xi_{43}(\lambda_i; \lambda_i; \beta) &= 11520\lambda_i^3(28 + 40\beta + 3\beta^2) \geq 0 \text{ for all } \beta \geq -0.7412, \\ \xi_{43}^{(1)}(\lambda_i; \lambda_i; \beta) &= 17280\lambda_i^2(\beta + 16)(3\beta + 2) \geq 0 \text{ for all } \beta \geq -\frac{2}{3}, \\ \xi_{43}^{(2)}(\lambda_i; \lambda_i; \beta) &= 13824\lambda_i(77\beta + 134 + 3\beta^2) \geq 0 \text{ for all } \beta \geq -1.8776, \\ \xi_{43}^{(3)}(\lambda_i; \lambda_i; \beta) &= 615168\beta + 2875392 + 10368\beta^2 \geq 0 \text{ for all } \beta \geq -5.1151.\end{aligned}$$

Hence, $\xi_4^{(3)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{42}(\lambda_j; \lambda_i; \beta) = \xi_4^{(2)}(\lambda_j; \lambda_j; \lambda_i; \beta)$. We wish to show that ξ_{42} is non-negative for

all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{42} with respect to λ_j gives

$$\begin{aligned}
\xi_{42}(\lambda_j; \lambda_i; \beta) &= (369792 + 70848\beta + 1728\beta^2)\lambda_j^4 \\
&\quad + (138240\beta + 20736\beta^2 - 359424)\lambda_i\lambda_j^3 \\
&\quad + (69120 + 6912\beta + 31104\beta^2)\lambda_i^2\lambda_j^2 \\
&\quad + (175104 + 36864\beta + 6912\beta^2)\lambda_i^3\lambda_j \\
&\quad + (21888 + 6336\beta)\lambda_i^4, \\
\xi_{42}^{(1)}(\lambda_j; \lambda_i; \beta) &= (1479168 + 283392\beta + 6912\beta^2)\lambda_j^3 \\
&\quad + (414720\beta + 62208\beta^2 - 1078272)\lambda_i\lambda_j^2 \\
&\quad + (138240 + 13824\beta + 62208\beta^2)\lambda_i^2\lambda_j \\
&\quad + (175104 + 36864\beta + 6912\beta^2)\lambda_i^3, \\
\xi_{42}^{(2)}(\lambda_j; \lambda_i; \beta) &= (4437504 + 850176\beta + 20736\beta^2)\lambda_j^2 \\
&\quad + (829440\beta + 124416\beta^2 - 2156544)\lambda_i\lambda_j \\
&\quad + (138240 + 13824\beta + 62208\beta^2)\lambda_i^2, \\
\xi_{42}^{(3)}(\lambda_j; \lambda_i; \beta) &= (8875008 + 1700352\beta + 41472\beta^2)\lambda_j \\
&\quad + (829440\beta + 124416\beta^2 - 2156544)\lambda_i, \\
\xi_{42}^{(4)}(\lambda_j; \lambda_i; \beta) &= 8875008 + 1700352\beta + 41472\beta^2 \geq 0 \text{ for all } \beta \geq -6.1386.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{42}(\lambda_i; \lambda_i; \beta) &= 8640\lambda_i^4(7\beta + 16)(\beta + 2) \geq 0 \text{ for all } \beta \geq -2, \\
\xi_{42}^{(1)}(\lambda_i; \lambda_i; \beta) &= 11520\lambda_i^3(62 + 65\beta + 12\beta^2) \geq 0 \text{ for all } \beta \geq -1.2358, \\
\xi_{42}^{(2)}(\lambda_i; \lambda_i; \beta) &= 34560\lambda_i^2(70 + 49\beta + 6\beta^2) \geq 0 \text{ for all } \beta \geq -1.8457, \\
\xi_{42}^{(3)}(\lambda_i; \lambda_i; \beta) &= 41472\lambda_i(162 + 61\beta + 4\beta^2) \geq 0 \text{ for all } \beta \geq -3.4249.
\end{aligned}$$

Hence, $\xi_4^{(2)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{41}(\lambda_j; \lambda_i; \beta) = \xi_4^{(1)}(\lambda_j; \lambda_j; \lambda_i; \beta)/\lambda_j$. We wish to show that ξ_{41} is non-negative

for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{41} with respect to λ_j gives

$$\begin{aligned} \xi_{41}(\lambda_j; \lambda_i; \beta) &= (32256\beta + 173952 + 864\beta^2)\lambda_j^4 \\ &\quad + (-184320 + 66240\beta + 12960\beta^2)\lambda_i\lambda_j^3 \\ &\quad + (25920\beta^2 + 23040 - 28800\beta)\lambda_i^2\lambda_j^2 \\ &\quad + (28800\beta + 8640\beta^2 + 46080)\lambda_i^3\lambda_j \\ &\quad + (14400\beta + 5760)\lambda_i^4, \\ \xi_{41}^{(1)}(\lambda_j; \lambda_i; \beta) &= (129024\beta + 695808 + 3456\beta^2)\lambda_j^3 \\ &\quad + (-552960 + 198720\beta + 38880\beta^2)\lambda_i\lambda_j^2 \\ &\quad + (51840\beta^2 + 46080 - 57600\beta)\lambda_i^2\lambda_j + (28800\beta + 8640\beta^2 + 46080)\lambda_i^3, \\ \xi_{41}^{(2)}(\lambda_j; \lambda_i; \beta) &= (387072\beta + 2087424 + 10368\beta^2)\lambda_j^2 \\ &\quad + (-1105920 + 397440\beta + 77760\beta^2)\lambda_i\lambda_j \\ &\quad + (51840\beta^2 + 46080 - 57600\beta)\lambda_i^2, \\ \xi_{41}^{(3)}(\lambda_j; \lambda_i; \beta) &= (774144\beta + 4174848 + 20736\beta^2)\lambda_j \\ &\quad + (-1105920 + 397440\beta + 77760\beta^2)\lambda_i, \\ \xi_{41}^{(4)}(\lambda_j; \lambda_i; \beta) &= 774144\beta + 4174848 + 20736\beta^2 \geq 0 \text{ for all } \beta \geq -6.5377. \end{aligned}$$

Now

$$\begin{aligned} \xi_{41}(\lambda_i; \lambda_i; \beta) &= 16128\lambda_i^4(3\beta + 4)(\beta + 1) \geq 0 \text{ for all } \beta \geq -1, \\ \xi_{41}^{(1)}(\lambda_i; \lambda_i; \beta) &= 864\lambda_i^3(119\beta^2 + 272 + 346\beta) \geq 0 \text{ for all } \beta, \\ \xi_{41}^{(2)}(\lambda_i; \lambda_i; \beta) &= 576\lambda_i^2(1262\beta + 1784 + 243\beta^2) \geq 0 \text{ for all } \beta, \\ \xi_{41}^{(3)}(\lambda_i; \lambda_i; \beta) &= 5184\lambda_i(8 + \beta)(19\beta + 74) \geq 0 \text{ for all } \beta \geq -\frac{74}{19}. \end{aligned}$$

Hence, $\xi_4^{(1)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{40}(\lambda_j; \lambda_i; \beta) = \xi_4(\lambda_j; \lambda_j; \lambda_i; \beta)/\lambda_j^2$. We wish to show that ξ_{40} is non-negative for

all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{40} with respect to λ_j gives

$$\begin{aligned} \xi_{40}(\lambda_j; \lambda_i; \beta) &= (57984 + 10752\beta + 288\beta^2)\lambda_j^4 + (5184\beta^2 - 73728 + 26496\beta)\lambda_i\lambda_j^3 \\ &\quad + (11520 - 14400\beta + 12960\beta^2)\lambda_i^2\lambda_j^2 \\ &\quad + (5760\beta^2 + 19200\beta + 30720)\lambda_i^3\lambda_j \\ &\quad + (14400\beta + 5760)\lambda_i^4, \\ \xi_{40}^{(1)}(\lambda_j; \lambda_i; \beta) &= (231936 + 43008\beta + 1152\beta^2)\lambda_j^3 \\ &\quad + (15552\beta^2 - 221184 + 79488\beta)\lambda_i\lambda_j^2 \\ &\quad + (23040 - 28800\beta + 25920\beta^2)\lambda_i^2\lambda_j \\ &\quad + (5760\beta^2 + 19200\beta + 30720)\lambda_i^3, \\ \xi_{40}^{(2)}(\lambda_j; \lambda_i; \beta) &= (695808 + 129024\beta + 3456\beta^2)\lambda_j^2 \\ &\quad + (31104\beta^2 - 442368 + 158976\beta)\lambda_i\lambda_j \\ &\quad + (23040 - 28800\beta + 25920\beta^2)\lambda_i^2, \\ \xi_{40}^{(3)}(\lambda_j; \lambda_i; \beta) &= (1391616 + 258048\beta + 6912\beta^2)\lambda_j \\ &\quad + (31104\beta^2 - 442368 + 158976\beta)\lambda_i, \\ \xi_{40}^{(4)}(\lambda_j; \lambda_i; \beta) &= 1391616 + 258048\beta + 6912\beta^2 \geq 0 \text{ for all } \beta \geq -6.5377. \end{aligned}$$

Now

$$\begin{aligned} \xi_{40}(\lambda_i; \lambda_i; \beta) &= 8064\lambda_i^4(3\beta + 4)(\beta + 1) > 0 \text{ for all } \beta > -1, \\ \xi_{40}^{(1)}(\lambda_i; \lambda_i; \beta) &= 16128\lambda_i^3(3\beta + 4)(\beta + 1) > 0 \text{ for all } \beta > -1, \\ \xi_{40}^{(2)}(\lambda_i; \lambda_i; \beta) &= 8640\lambda_i^2(7\beta + 16)(\beta + 2) > 0 \text{ for all } \beta > -2, \\ \xi_{40}^{(3)}(\lambda_i; \lambda_i; \beta) &= 1152\lambda_i(824 + 362\beta + 33\beta^2) > 0 \text{ for all } \beta > -3.2235. \end{aligned}$$

Hence, ξ_4 is positive for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$. This implies that $\xi(\lambda_l; \lambda_l; \lambda_j; \lambda_i; \beta)$ is positive for all $\lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

□

Lemma B.14 Let

$$\begin{aligned} \xi(\lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta) = & (648\beta^3 + 15552 - 10368\beta)t(2, 2, 2, 3) \\ & + (2208\beta + 432\beta^2 - 6144)t(1, 1, 3, 4) \\ & + (1296\beta^2 - 1296\beta - 2592)t(1, 2, 2, 4) \\ & + (576\beta^2 + 3168\beta + 216\beta^3 - 10368)t(1, 2, 3, 3) \\ & + (144\beta^2 + 1296\beta + 5472)t(0, 2, 3, 4) \\ & + (240 + 600\beta)t(0, 1, 4, 4) \\ & + (144\beta^2 + 3072\beta + 16128 + 24\beta^3)t(0, 3, 3, 3), \end{aligned}$$

where

$$t(a, b, c, d) = \sum_{\substack{(n_1, n_2, n_3, n_4) \\ \in \text{perm}(a, b, c, d)}} \lambda_i^{n_1} \lambda_j^{n_2} \lambda_l^{n_3} \lambda_k^{n_4}.$$

The expression $\xi(\lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta)$ is positive for all $\lambda_k \geq \lambda_l \geq \lambda_j \geq \lambda_i$, and $\beta \geq -\frac{2}{5}$.

Proof.

Let

$$\hat{s}(a, b, c) = \sum_{\substack{(n_1, n_2, n_3) \\ \in \text{perm}(a, b, c)}} \lambda_i^{n_1} \lambda_j^{n_2} \lambda_l^{n_3}.$$

Differentiating $\xi(\lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta)$ with respect to λ_k we obtain

$$\begin{aligned}
\xi^{(1)}(\lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta) &= (1728\beta^2 - 24576 + 8832\beta)\lambda_k^3\hat{s}(1, 1, 3) \\
&+ (2400\beta + 960)\lambda_k^3\hat{s}(0, 1, 4) \\
&+ (-5184\beta + 5184\beta^2 - 10368)\lambda_k^3\hat{s}(1, 2, 2) \\
&+ (21888 + 5184\beta + 576\beta^2)\lambda_k^3\hat{s}(0, 2, 3) \\
&+ (-31104\beta + 46656 + 1944\beta^3)\lambda_k^2\hat{s}(2, 2, 2) \\
&+ (1296\beta^2 + 6624\beta - 18432)\lambda_k^2\hat{s}(1, 1, 4) \\
&+ (-31104 + 648\beta^3 + 1728\beta^2 + 9504\beta)\lambda_k^2\hat{s}(1, 2, 3) \\
&+ (3888\beta + 16416 + 432\beta^2)\lambda_k^2\hat{s}(0, 2, 4) \\
&+ (432\beta^2 + 72\beta^3 + 48384 + 9216\beta)\lambda_k^2\hat{s}(0, 3, 3) \\
&+ (-20736\beta + 1296\beta^3 + 31104)\lambda_k\hat{s}(2, 2, 3) \\
&+ (6336\beta + 432\beta^3 - 20736 + 1152\beta^2)\lambda_k\hat{s}(1, 3, 3) \\
&+ (-5184 + 2592\beta^2 - 2592\beta)\lambda_k\hat{s}(1, 2, 4) \\
&+ (10944 + 288\beta^2 + 2592\beta)\lambda_k\hat{s}(0, 3, 4) \\
&+ (240 + 600\beta)\hat{s}(0, 4, 4) \\
&+ (-2592 + 1296\beta^2 - 1296\beta)\hat{s}(2, 2, 4) \\
&+ (576\beta^2 + 3168\beta + 216\beta^3 - 10368)\hat{s}(2, 3, 3) \\
&+ (2208\beta + 432\beta^2 - 6144)\hat{s}(1, 3, 4), \\
\xi^{(2)}(\lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta) &= 3(1728\beta^2 - 24576 + 8832\beta)\lambda_k^2\hat{s}(1, 1, 3) \\
&+ 3(2400\beta + 960)\lambda_k^2\hat{s}(0, 1, 4) \\
&+ 3(-5184\beta + 5184\beta^2 - 10368)\lambda_k^2\hat{s}(1, 2, 2) \\
&+ 3(21888 + 5184\beta + 576\beta^2)\lambda_k^2\hat{s}(0, 2, 3) \\
&+ 2(-31104\beta + 46656 + 1944\beta^3)\lambda_k\hat{s}(2, 2, 2) \\
&+ 2(1296\beta^2 + 6624\beta - 18432)\lambda_k\hat{s}(1, 1, 4) \\
&+ 2(-31104 + 648\beta^3 + 1728\beta^2 + 9504\beta)\lambda_k\hat{s}(1, 2, 3) \\
&+ 2(3888\beta + 16416 + 432\beta^2)\lambda_k\hat{s}(0, 2, 4) \\
&+ 2(432\beta^2 + 72\beta^3 + 48384 + 9216\beta)\lambda_k\hat{s}(0, 3, 3) \\
&+ (-20736\beta + 1296\beta^3 + 31104)\hat{s}(2, 2, 3) \\
&+ (6336\beta + 432\beta^3 - 20736 + 1152\beta^2)\hat{s}(1, 3, 3) \\
&+ (-5184 + 2592\beta^2 - 2592\beta)\hat{s}(1, 2, 4) \\
&+ (10944 + 288\beta^2 + 2592\beta)\hat{s}(0, 3, 4),
\end{aligned}$$

$$\begin{aligned}
 \xi^{(3)}(\lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta) &= 6(1728\beta^2 - 24576 + 8832\beta)\lambda_k\hat{s}(1, 1, 3) \\
 &\quad + 6(2400\beta + 960)\lambda_k\hat{s}(0, 1, 4) \\
 &\quad + 6(-5184\beta + 5184\beta^2 - 10368)\lambda_k\hat{s}(1, 2, 2) \\
 &\quad + 6(21888 + 5184\beta + 576\beta^2)\lambda_k\hat{s}(0, 2, 3) \\
 &\quad + 2(-31104\beta + 46656 + 1944\beta^3)\hat{s}(2, 2, 2) \\
 &\quad + 2(1296\beta^2 + 6624\beta - 18432)\hat{s}(1, 1, 4) \\
 &\quad + 2(-31104 + 648\beta^3 + 1728\beta^2 + 9504\beta)\hat{s}(1, 2, 3) \\
 &\quad + 2(3888\beta + 16416 + 432\beta^2)\hat{s}(0, 2, 4) \\
 &\quad + 2(432\beta^2 + 72\beta^3 + 48384 + 9216\beta)\hat{s}(0, 3, 3), \\
 \xi^{(4)}(\lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta) &= 6(1728\beta^2 - 24576 + 8832\beta)\hat{s}(1, 1, 3) + 6(2400\beta + 960)\hat{s}(0, 1, 4) \\
 &\quad + 6(-5184\beta + 5184\beta^2 - 10368)\hat{s}(1, 2, 2) \\
 &\quad + 6(21888 + 5184\beta + 576\beta^2)\hat{s}(0, 2, 3).
 \end{aligned}$$

Define

$$\begin{aligned}
 \xi_0(\lambda_l; \lambda_j; \lambda_i; \beta) &\stackrel{\text{def}}{=} \xi(\lambda_l; \lambda_l; \lambda_j; \lambda_i; \beta)/\lambda_l, \\
 \xi_1(\lambda_l; \lambda_j; \lambda_i; \beta) &\stackrel{\text{def}}{=} \xi^{(1)}(\lambda_l; \lambda_l; \lambda_j; \lambda_i; \beta), \\
 \xi_2(\lambda_l; \lambda_j; \lambda_i; \beta) &\stackrel{\text{def}}{=} \xi^{(2)}(\lambda_l; \lambda_l; \lambda_j; \lambda_i; \beta), \\
 \xi_3(\lambda_l; \lambda_j; \lambda_i; \beta) &\stackrel{\text{def}}{=} \xi^{(3)}(\lambda_l; \lambda_l; \lambda_j; \lambda_i; \beta), \\
 \xi_4(\lambda_l; \lambda_j; \lambda_i; \beta) &\stackrel{\text{def}}{=} \xi^{(4)}(\lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta).
 \end{aligned}$$

We will prove that the above equations are non-negative for all $\lambda_k \geq \lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$ and, hence, deduce that $\xi(\lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta)$ is positive for all $\lambda_k \geq \lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Now

$$\begin{aligned}
\xi_0(\lambda_i; \lambda_j; \lambda_i; \beta) = & ((240 + 600\beta)\lambda_i + (240 + 600\beta)\lambda_j)\lambda_l^7 \\
& + (288\beta^2 + 2592\beta + 10944)(\lambda_j^2 + \lambda_i^2)\lambda_l^6 \\
& + (4416\beta - 12288 + 864\beta^2)\lambda_i\lambda_j\lambda_l^6 \\
& + (-15552 + 576\beta + 216\beta^3 + 3168\beta^2)(\lambda_i\lambda_j^2 + \lambda_i^2\lambda_j)\lambda_l^5 \\
& + (432\beta^2 + 24\beta^3 + 27072 + 5664\beta)(\lambda_j^3 + \lambda_i^3)\lambda_l^5 \\
& + (10752\beta - 33024 + 2016\beta^2 + 432\beta^3)(\lambda_i\lambda_j^3 + \lambda_i^3\lambda_j)\lambda_l^4 \\
& + (11424 + 3792\beta + 288\beta^2)(\lambda_j^4 + \lambda_i^4)\lambda_l^4 \\
& + (2592\beta^2 + 25920 + 1296\beta^3 - 23328\beta)\lambda_i^2\lambda_j^2\lambda_l^4 \\
& + (1440\beta^2 + 5760 - 1440\beta + 1080\beta^3)(\lambda_i^2\lambda_j^3 + \lambda_i^3\lambda_j^2)\lambda_l^3 \\
& + (2160\beta^2 - 14400 + 4320\beta)(\lambda_i\lambda_j^4 + \lambda_i^4\lambda_j)\lambda_l^3 \\
& + (2880\beta^2 + 5760)(\lambda_i^2\lambda_j^4 + \lambda_i^4\lambda_j^2)\lambda_l^2 \\
& + (11520 + 12480\beta + 1440\beta^2 + 480\beta^3)\lambda_i^3\lambda_j^3\lambda_l^2 \\
& + (4800 + 4800\beta + 720\beta^2)(\lambda_i^3\lambda_j^4 + \lambda_i^4\lambda_j^3)\lambda_l \\
& + (480 + 1200\beta)\lambda_i^4\lambda_j^4.
\end{aligned}$$

Define $\xi_{07}(\lambda_j; \lambda_i; \beta) = \xi_0^{(7)}(\lambda_l; \lambda_j; \lambda_i; \beta)$. We wish to show that ξ_{07} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{07} with respect to λ_j gives

$$\begin{aligned}
\xi_{07}(\lambda_j; \lambda_i; \beta) &= (3024000\beta + 1209600)\lambda_j + (3024000\beta + 1209600)\lambda_i, \\
\xi_{07}^{(1)}(\lambda_j; \lambda_i; \beta) &= 3024000\beta + 1209600\beta \geq -\frac{2}{5}.
\end{aligned}$$

Now

$$\xi_{07}(\lambda_i; \lambda_i; \beta) = 1209600(2 + 5\beta)\lambda_i \geq 0 \text{ for all } \beta \geq -\frac{2}{5}.$$

Hence, $\xi_0^{(7)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{06}(\lambda_j; \lambda_i; \beta) = \xi_0^{(6)}(\lambda_j; \lambda_j; \lambda_i; \beta)$. We wish to show that ξ_{06} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{06} with respect to λ_j gives

$$\begin{aligned}
\xi_{06}(\lambda_j; \lambda_i; \beta) &= (4890240\beta + 9089280 + 207360\beta^2)\lambda_j^2 \\
&+ (6203520\beta - 7637760 + 622080\beta^2)\lambda_i\lambda_j \\
&+ (7879680 + 1866240\beta + 207360\beta^2)\lambda_i^2, \\
\xi_{06}^{(1)}(\lambda_j; \lambda_i; \beta) &= (9780480\beta + 18178560 + 414720\beta^2)\lambda_j \\
&+ (6203520\beta - 7637760 + 622080\beta^2)\lambda_i, \\
\xi_{06}^{(2)}(\lambda_j; \lambda_i; \beta) &= 9780480\beta + 18178560 + 414720\beta^2. \geq 0 \text{ for all } \beta \geq -2.0341.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{06}(\lambda_i; \lambda_i; \beta) &= 518400\lambda_i^2(25\beta + 18 + 2\beta^2) \geq 0 \text{ for all } \beta \geq -0.7671, \\
\xi_{06}^{(1)}(\lambda_i; \lambda_i; \beta) &= 86400\lambda_i(185\beta + 122 + 12\beta^2) \geq 0 \text{ for all } \beta \geq -0.6904.
\end{aligned}$$

Hence, $\xi_0^{(6)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{05}(\lambda_j; \lambda_i; \beta) = \xi_0^{(5)}(\lambda_j; \lambda_j; \lambda_i; \beta)$. We wish to show that ξ_{05} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{05} with respect to λ_j gives

$$\begin{aligned} \xi_{05}(\lambda_j; \lambda_i; \beta) &= (4057920\beta + 11733120 + 259200\beta^2 + 2880\beta^3)\lambda_j^3 \\ &\quad + (4760640\beta - 10108800 + 1002240\beta^2 + 25920\beta^3)\lambda_i\lambda_j^2 \\ &\quad + (6013440 + 1935360\beta + 587520\beta^2 + 25920\beta^3)\lambda_i^2\lambda_j \\ &\quad + (679680\beta + 3248640 + 51840\beta^2 + 2880\beta^3)\lambda_i^3, \\ \xi_{05}^{(1)}(\lambda_j; \lambda_i; \beta) &= (12173760\beta + 35199360 + 777600\beta^2 + 8640\beta^3)\lambda_j^2 \\ &\quad + (9521280\beta - 20217600 + 2004480\beta^2 + 51840\beta^3)\lambda_i\lambda_j \\ &\quad + (6013440 + 1935360\beta + 587520\beta^2 + 25920\beta^3)\lambda_i^2, \\ \xi_{05}^{(2)}(\lambda_j; \lambda_i; \beta) &= (24347520\beta + 70398720 + 1555200\beta^2 + 17280\beta^3)\lambda_j \\ &\quad + (9521280\beta - 20217600 + 2004480\beta^2 + 51840\beta^3)\lambda_i, \\ \xi_{05}^{(3)}(\lambda_j; \lambda_i; \beta) &= 24347520\beta + 70398720 + 1555200\beta^2 + 17280\beta^3 \geq 0 \text{ for all } \beta \geq -3.7540. \end{aligned}$$

Now

$$\begin{aligned} \xi_{05}(\lambda_i; \lambda_i; \beta) &= 28800\lambda_i^3(397\beta + 378 + 66\beta^2 + 2\beta^3) \geq 0 \text{ for all } \beta \geq -1.1726, \\ \xi_{05}^{(1)}(\lambda_i; \lambda_i; \beta) &= 43200\lambda_i^2(547\beta + 486 + 78\beta^2 + 2\beta^3) \geq 0 \text{ for all } \beta \geq -1.0380, \\ \xi_{05}^{(2)}(\lambda_i; \lambda_i; \beta) &= 34560\lambda_i(980\beta + 1452 + 103\beta^2 + 2\beta^3) \geq 0 \text{ for all } \beta \geq -1.8160. \end{aligned}$$

Hence, $\xi_0^{(5)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{04}(\lambda_j; \lambda_i; \beta) = \xi_0^{(4)}(\lambda_j; \lambda_j; \lambda_i; \beta)$. We wish to show that ξ_{04} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{04} with respect to λ_j gives

$$\begin{aligned} \xi_{04}(\lambda_j; \lambda_i; \beta) &= (7664256 + 2207808\beta + 162432\beta^2 + 2880\beta^3)\lambda_j^4 \\ &\quad + (-6880896 + 2420928\beta + 739584\beta^2 + 36288\beta^3)\lambda_i\lambda_j^3 \\ &\quad + (442368\beta + 2695680 + 546048\beta^2 + 57024\beta^3)\lambda_i^2\lambda_j^2 \\ &\quad + (937728\beta + 2456064 + 100224\beta^2 + 13248\beta^3)\lambda_i^3\lambda_j \\ &\quad + (91008\beta + 6912\beta^2 + 274176)\lambda_i^4, \\ \xi_{04}^{(1)}(\lambda_j; \lambda_i; \beta) &= (30657024 + 8831232\beta + 649728\beta^2 + 11520\beta^3)\lambda_j^3 \\ &\quad + (-20642688 + 7262784\beta + 2218752\beta^2 + 108864\beta^3)\lambda_i\lambda_j^2 \\ &\quad + (884736\beta + 5391360 + 1092096\beta^2 + 114048\beta^3)\lambda_i^2\lambda_j \\ &\quad + (937728\beta + 2456064 + 100224\beta^2 + 13248\beta^3)\lambda_i^3, \\ \xi_{04}^{(2)}(\lambda_j; \lambda_i; \beta) &= (91971072 + 26493696\beta + 1949184\beta^2 + 34560\beta^3)\lambda_j^2 \\ &\quad + (-41285376 + 14525568\beta + 4437504\beta^2 + 217728\beta^3)\lambda_i\lambda_j \\ &\quad + (884736\beta + 5391360 + 1092096\beta^2 + 114048\beta^3)\lambda_i^2, \\ \xi_{04}^{(3)}(\lambda_j; \lambda_i; \beta) &= (183942144 + 52987392\beta + 3898368\beta^2 + 69120\beta^3)\lambda_j \\ &\quad + (-41285376 + 14525568\beta + 4437504\beta^2 + 217728\beta^3)\lambda_i, \\ \xi_{04}^{(4)}(\lambda_j; \lambda_i; \beta) &= 183942144 + 52987392\beta + 3898368\beta^2 + 69120\beta^3 \geq 0 \text{ for all } \beta \geq -5.4358. \end{aligned}$$

Now

$$\begin{aligned}\xi_{04}(\lambda_i; \lambda_i; \beta) &= 5760\lambda_i^4(1078 + 1059\beta + 270\beta^2 + 19\beta^3) \geq 0 \text{ for all } \beta \geq -1.5912, \\ \xi_{04}^{(1)}(\lambda_i; \lambda_i; \beta) &= 2880\lambda_i^3(6202 + 6221\beta + 1410\beta^2 + 86\beta^3) \geq 0 \text{ for all } \beta \geq -1.4073, \\ \xi_{04}^{(2)}(\lambda_i; \lambda_i; \beta) &= 3456\lambda_i^2(16226 + 12125\beta + 2164\beta^2 + 106\beta^3) \geq 0 \text{ for all } \beta \geq -1.9551, \\ \xi_{04}^{(3)}(\lambda_i; \lambda_i; \beta) &= 3456\lambda_i(41278 + 19535\beta + 2412\beta^2 + 83\beta^3) \geq 0 \text{ for all } \beta \geq -3.3153.\end{aligned}$$

Hence, $\xi_0^{(4)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{03}(\lambda_j; \lambda_i; \beta) = \xi_0^{(3)}(\lambda_j; \lambda_j; \lambda_i; \beta)/\lambda_j$. We wish to show that ξ_{03} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{03} with respect to λ_j gives

$$\begin{aligned}\xi_{03}(\lambda_j; \lambda_i; \beta) &= (3262176 + 867888\beta + 67392\beta^2 + 1440\beta^3)\lambda_j^4 \\ &\quad + (974448\beta + 355104\beta^2 - 3236256 + 23328\beta^3)\lambda_i\lambda_j^3 \\ &\quad + (295488\beta^2 + 1036800 - 222912\beta + 50544\beta^3)\lambda_i^2\lambda_j^2 \\ &\quad + (82944\beta^2 + 589248\beta + 18288\beta^3 + 866304)\lambda_i^3\lambda_j \\ &\quad + (116928\beta + 19872\beta^2 + 187776)\lambda_i^4, \\ \xi_{03}^{(1)}(\lambda_j; \lambda_i; \beta) &= (13048704 + 3471552\beta + 269568\beta^2 + 5760\beta^3)\lambda_j^3 \\ &\quad + (2923344\beta + 1065312\beta^2 - 9708768 + 69984\beta^3)\lambda_i\lambda_j^2 \\ &\quad + (590976\beta^2 + 2073600 - 445824\beta + 101088\beta^3)\lambda_i^2\lambda_j \\ &\quad + (82944\beta^2 + 589248\beta + 18288\beta^3 + 866304)\lambda_i^3, \\ \xi_{03}^{(2)}(\lambda_j; \lambda_i; \beta) &= (39146112 + 10414656\beta + 808704\beta^2 + 17280\beta^3)\lambda_j^2 \\ &\quad + (5846688\beta + 2130624\beta^2 - 19417536 + 139968\beta^3)\lambda_i\lambda_j \\ &\quad + (590976\beta^2 + 2073600 - 445824\beta + 101088\beta^3)\lambda_i^2, \\ \xi_{03}^{(3)}(\lambda_j; \lambda_i; \beta) &= (78292224 + 20829312\beta + 1617408\beta^2 + 34560\beta^3)\lambda_j \\ &\quad + (5846688\beta + 2130624\beta^2 - 19417536 + 139968\beta^3)\lambda_i, \\ \xi_{03}^{(4)}(\lambda_j; \lambda_i; \beta) &= 78292224 + 20829312\beta + 1617408\beta^2 + 34560\beta^3 \geq 0 \text{ for all } \beta \geq -6.9654.\end{aligned}$$

Now

$$\begin{aligned}\xi_{03}(\lambda_i; \lambda_i; \beta) &= 7200\lambda_i^4(13\beta + 49)(\beta + 3)(\beta + 2) \geq 0 \text{ for all } \beta \geq -2, \\ \xi_{03}^{(1)}(\lambda_i; \lambda_i; \beta) &= 720\lambda_i^3(8722 + 9081\beta + 2790\beta^2 + 271\beta^3) \geq 0 \text{ for all } \beta \geq -1.7083, \\ \xi_{03}^{(2)}(\lambda_i; \lambda_i; \beta) &= 864\lambda_i^2(25234 + 18305\beta + 4086\beta^2 + 299\beta^3) \geq 0 \text{ for all } \beta \geq -2.6023, \\ \xi_{03}^{(3)}(\lambda_i; \lambda_i; \beta) &= 864\lambda_i(68142 + 30875\beta + 4338\beta^2 + 202\beta^3) \geq 0 \text{ for all } \beta \geq -4.2046.\end{aligned}$$

Hence, $\xi_0^{(3)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{02}(\lambda_j; \lambda_i; \beta) = \xi_0^{(2)}(\lambda_j; \lambda_j; \lambda_i; \beta)/\lambda_j^2$. We wish to show that ξ_{02} is non-negative

for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{02} with respect to λ_j gives

$$\begin{aligned} \xi_{02}(\lambda_j; \lambda_i; \beta) &= (261744\beta + 1016928 + 480\beta^3 + 20736\beta^2)\lambda_j^4 \\ &\quad + (126432\beta^2 - 1152288 + 324144\beta + 9504\beta^3)\lambda_i\lambda_j^3 \\ &\quad + (117504\beta^2 - 199296\beta + 374400 + 26352\beta^3)\lambda_i^2\lambda_j^2 \\ &\quad + (13104\beta^3 + 44352\beta^2 + 258624\beta + 202752)\lambda_i^3\lambda_j \\ &\quad + (22176\beta^2 + 62208 + 71424\beta)\lambda_i^4, \\ \xi_{02}^{(1)}(\lambda_j; \lambda_i; \beta) &= (1046976\beta + 4067712 + 1920\beta^3 + 82944\beta^2)\lambda_j^3 \\ &\quad + (379296\beta^2 - 3456864 + 972432\beta + 28512\beta^3)\lambda_i\lambda_j^2 \\ &\quad + (235008\beta^2 - 398592\beta + 748800 + 52704\beta^3)\lambda_i^2\lambda_j \\ &\quad + (13104\beta^3 + 44352\beta^2 + 258624\beta + 202752)\lambda_i^3, \\ \xi_{02}^{(2)}(\lambda_j; \lambda_i; \beta) &= (3140928\beta + 12203136 + 5760\beta^3 + 248832\beta^2)\lambda_j^2 \\ &\quad + (758592\beta^2 - 6913728 + 1944864\beta + 57024\beta^3)\lambda_i\lambda_j \\ &\quad + (235008\beta^2 - 398592\beta + 748800 + 52704\beta^3)\lambda_i^2, \\ \xi_{02}^{(3)}(\lambda_j; \lambda_i; \beta) &= (6281856\beta + 24406272 + 11520\beta^3 + 497664\beta^2)\lambda_j \\ &\quad + (758592\beta^2 - 6913728 + 1944864\beta + 57024\beta^3)\lambda_i, \\ \xi_{02}^{(4)}(\lambda_j; \lambda_i; \beta) &= 6281856\beta + 24406272 + 11520\beta^3 + 497664\beta^2 \geq 0 \text{ for all } \beta \geq -8.2141. \end{aligned}$$

Now

$$\begin{aligned} \xi_{02}(\lambda_i; \lambda_i; \beta) &= 480\lambda_i^4(103\beta + 175)(\beta + 3)(\beta + 2) \geq 0 \text{ for all } \beta \geq -\frac{175}{103}, \\ \xi_{02}^{(1)}(\lambda_i; \lambda_i; \beta) &= 240\lambda_i^3(401\beta + 1085)(\beta + 3)(\beta + 2) \geq 0 \text{ for all } \beta \geq -\frac{1085}{401}, \\ \xi_{02}^{(2)}(\lambda_i; \lambda_i; \beta) &= 288\lambda_i^2(16275\beta + 20966 + 401\beta^3 + 4314\beta^2) \geq 0 \text{ for all } \beta \geq -3.1231, \\ \xi_{02}^{(3)}(\lambda_i; \lambda_i; \beta) &= \geq 0 \text{ for all } \beta \geq -4.2173. \end{aligned}$$

Hence, $\xi_0^{(2)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{01}(\lambda_j; \lambda_i; \beta) = \xi_0^{(1)}(\lambda_j; \lambda_j; \lambda_i; \beta)/\lambda_j^3$. We wish to show that ξ_{01} is non-negative

for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{01} with respect to λ_j gives

$$\begin{aligned}
\xi_{01}(\lambda_j; \lambda_i; \beta) &= (248400 + 63240\beta + 120\beta^3 + 5040\beta^2)\lambda_j^4 \\
&\quad + (35568\beta^2 - 325104 + 89544\beta + 2808\beta^3)\lambda_i\lambda_j^3 \\
&\quad + (38016\beta^2 - 79200\beta + 120384 + 9504\beta^3)\lambda_i^2\lambda_j^2 \\
&\quad + (96768\beta + 6048\beta^3 + 18144\beta^2 + 48384)\lambda_i^3\lambda_j \\
&\quad + (14112\beta^2 + 32928\beta + 18816)\lambda_i^4, \\
\xi_{01}^{(1)}(\lambda_j; \lambda_i; \beta) &= (993600 + 252960\beta + 480\beta^3 + 20160\beta^2)\lambda_j^3 \\
&\quad + (106704\beta^2 - 975312 + 268632\beta + 8424\beta^3)\lambda_i\lambda_j^2 \\
&\quad + (76032\beta^2 - 158400\beta + 240768 + 19008\beta^3)\lambda_i^2\lambda_j \\
&\quad + (96768\beta + 6048\beta^3 + 18144\beta^2 + 48384)\lambda_i^3, \\
\xi_{01}^{(2)}(\lambda_j; \lambda_i; \beta) &= (2980800 + 758880\beta + 1440\beta^3 + 60480\beta^2)\lambda_j^2 \\
&\quad + (213408\beta^2 - 1950624 + 537264\beta + 16848\beta^3)\lambda_i\lambda_j \\
&\quad + (76032\beta^2 - 158400\beta + 240768 + 19008\beta^3)\lambda_i^2, \\
\xi_{01}^{(3)}(\lambda_j; \lambda_i; \beta) &= (5961600 + 1517760\beta + 2880\beta^3 + 120960\beta^2)\lambda_j \\
&\quad + (213408\beta^2 - 1950624 + 537264\beta + 16848\beta^3)\lambda_i, \\
\xi_{01}^{(4)}(\lambda_j; \lambda_i; \beta) &= 2880(\beta + 10)(\beta + 9)(\beta + 23) \geq 0 \text{ for all } \beta \geq -9.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{01}(\lambda_i; \lambda_i; \beta) &= 18480\lambda_i^4(\beta + 3)(\beta + 2)(\beta + 1) \geq 0 \text{ for all } \beta \geq -1, \\
\xi_{01}^{(1)}(\lambda_i; \lambda_i; \beta) &= 120\lambda_i^3(283\beta + 427)(\beta + 3)(\beta + 2) \geq 0 \text{ for all } \beta \geq -\frac{427}{283}, \\
\xi_{01}^{(2)}(\lambda_i; \lambda_i; \beta) &= 144\lambda_i^2(\beta + 3)(259\beta^2 + 1653\beta + 2942) \geq 0 \text{ for all } \beta \geq -3, \\
\xi_{01}^{(3)}(\lambda_i; \lambda_i; \beta) &= 144\lambda_i(27854 + 14271\beta + 137\beta^3 + 2322\beta^2) \geq 0 \text{ for all } \beta \geq -3.6636.
\end{aligned}$$

Hence, $\xi_0^{(1)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{00}(\lambda_j; \lambda_i; \beta) = \xi_0(\lambda_j; \lambda_j; \lambda_i; \beta)/\lambda_j^4$. We wish to show that ξ_{00} is non-negative for

all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{00} with respect to λ_j gives

$$\begin{aligned} \xi_{00}(\lambda_j; \lambda_i; \beta) &= (49680 + 12648\beta + 1008\beta^2 + 24\beta^3)\lambda_j^4 \\ &\quad + (20664\beta - 75024 + 8208\beta^2 + 648\beta^3)\lambda_i\lambda_j^3 \\ &\quad + (10368\beta^2 - 21600\beta + 32832 + 2592\beta^3)\lambda_i^2\lambda_j^2 \\ &\quad + (6048\beta^2 + 2016\beta^3 + 16128 + 32256\beta)\lambda_i^3\lambda_j \\ &\quad + (8064 + 14112\beta + 6048\beta^2)\lambda_i^4, \\ \xi_{00}^{(1)}(\lambda_j; \lambda_i; \beta) &= (198720 + 50592\beta + 4032\beta^2 + 96\beta^3)\lambda_j^3 \\ &\quad + (61992\beta - 225072 + 24624\beta^2 + 1944\beta^3)\lambda_i\lambda_j^2 \\ &\quad + (20736\beta^2 - 43200\beta + 65664 + 5184\beta^3)\lambda_i^2\lambda_j \\ &\quad + (6048\beta^2 + 2016\beta^3 + 16128 + 32256\beta)\lambda_i^3, \\ \xi_{00}^{(2)}(\lambda_j; \lambda_i; \beta) &= (596160 + 151776\beta + 12096\beta^2 + 288\beta^3)\lambda_j^2 \\ &\quad + (123984\beta - 450144 + 49248\beta^2 + 3888\beta^3)\lambda_i\lambda_j \\ &\quad + (20736\beta^2 - 43200\beta + 65664 + 5184\beta^3)\lambda_i^2, \\ \xi_{00}^{(3)}(\lambda_j; \lambda_i; \beta) &= (1192320 + 303552\beta + 24192\beta^2 + 576\beta^3)\lambda_j \\ &\quad + (123984\beta - 450144 + 49248\beta^2 + 3888\beta^3)\lambda_i, \\ \xi_{00}^{(4)}(\lambda_j; \lambda_i; \beta) &= 1192320 + 303552\beta + 24192\beta^2 + 576\beta^3 \geq 0 \text{ for all } \beta \geq -9. \end{aligned}$$

Now

$$\begin{aligned} \xi_{00}(\lambda_i; \lambda_i; \beta) &= 5280\lambda_i^4(\beta + 3)(\beta + 2)(\beta + 1) > 0 \text{ for all } \beta > -1, \\ \xi_{00}^{(1)}(\lambda_i; \lambda_i; \beta) &= 9240\lambda_i^3(\beta + 3)(\beta + 2)(\beta + 1) > 0 \text{ for all } \beta > -1, \\ \xi_{00}^{(2)}(\lambda_i; \lambda_i; \beta) &= 720\lambda_i^2(13\beta + 49)(\beta + 3)(\beta + 2) > 0 \text{ for all } \beta > -2, \\ \xi_{00}^{(3)}(\lambda_i; \lambda_i; \beta) &= 144\lambda_i(\beta + 3)(31\beta^2 + 417\beta + 1718) > 0 \text{ for all } \beta > -3. \end{aligned}$$

Hence, ξ_0 is positive for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$. This implies that $\xi(\lambda_l; \lambda_l; \lambda_j; \lambda_i; \beta)$ is positive for all $\lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Now

$$\begin{aligned}
\xi_1(\lambda_l; \lambda_j; \lambda_i; \beta) &= (2400\beta + 960)(\lambda_j + \lambda_i)\lambda_l^7 \\
&+ (38304 + 9072\beta + 1008\beta^2)(\lambda_j^2 + \lambda_i^2)\lambda_l^6 \\
&+ (-43008 + 3024\beta^2 + 15456\beta)\lambda_i\lambda_j\lambda_l^6 \\
&+ (1728\beta + 9504\beta^2 - 46656 + 648\beta^3)(\lambda_i\lambda_j^2 + \lambda_i^2\lambda_j)\lambda_l^5 \\
&+ (1296\beta^2 + 81216 + 72\beta^3 + 16992\beta)(\lambda_j^3 + \lambda_i^3)\lambda_l^5 \\
&+ (1080\beta^3 + 5040\beta^2 - 82560 + 26880\beta)(\lambda_i\lambda_j^3 + \lambda_i^3\lambda_j)\lambda_l^4 \\
&+ (720\beta^2 + 28560 + 9480\beta)(\lambda_j^4 + \lambda_i^4)\lambda_l^4 \\
&+ (6480\beta^2 - 58320\beta + 64800 + 3240\beta^3)\lambda_i^2\lambda_j^2\lambda_l^4 \\
&+ (2160\beta^3 + 11520 - 2880\beta + 2880\beta^2)(\lambda_i^2\lambda_j^3 + \lambda_i^3\lambda_j^2)\lambda_l^3 \\
&+ (-28800 + 4320\beta^2 + 8640\beta)(\lambda_i\lambda_j^4 + \lambda_i^4\lambda_j)\lambda_l^3 \\
&+ (4320\beta^2 + 8640)(\lambda_i^2\lambda_j^4 + \lambda_i^4\lambda_j^2)\lambda_l^2 \\
&+ (18720\beta + 17280 + 2160\beta^2 + 720\beta^3)\lambda_i^3\lambda_j^3\lambda_l^2 \\
&+ (4800 + 4800\beta + 720\beta^2)(\lambda_i^3\lambda_j^4 + \lambda_i^4\lambda_j^3)\lambda_l \\
&+ (240 + 600\beta)\lambda_i^4\lambda_j^4,
\end{aligned}$$

and, hence,

$$\xi_1^{(7)}(\lambda_l; \lambda_j; \lambda_i; \beta) = (12096000\beta + 4838400)(\lambda_j + \lambda_i) \geq 0 \text{ for all } \beta \geq -\frac{2}{5}.$$

Define $\xi_{16}(\lambda_j; \lambda_i; \beta) = \xi_1^{(6)}(\lambda_l; \lambda_j; \lambda_i; \beta)$. We wish to show that ξ_{16} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{16} with respect to λ_j gives

$$\begin{aligned}
\xi_{16}(\lambda_j; \lambda_i; \beta) &= (18627840\beta + 32417280 + 725760\beta^2)\lambda_j^2 \\
&+ (23224320\beta - 26127360 + 2177280\beta^2)\lambda_i\lambda_j \\
&+ (27578880 + 725760\beta^2 + 6531840\beta)\lambda_i^2, \\
\xi_{16}^{(1)}(\lambda_j; \lambda_i; \beta) &= (37255680\beta + 64834560 + 1451520\beta^2)\lambda_j \\
&+ (23224320\beta - 26127360 + 2177280\beta^2)\lambda_i, \\
\xi_{16}^{(2)}(\lambda_j; \lambda_i; \beta) &= 37255680\beta + 64834560 + 1451520\beta^2 \geq 0 \text{ for all } \beta \geq -1.8776.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{16}(\lambda_i; \lambda_i; \beta) &= 1209600\lambda_i^2(28 + 40\beta + 3\beta^2) \geq 0 \text{ for all } \beta \geq -0.7412, \\
\xi_{16}^{(1)}(\lambda_i; \lambda_i; \beta) &= 1209600\lambda_i(\beta + 16)(3\beta + 2) \geq 0 \text{ for all } \beta \geq -\frac{2}{3}.
\end{aligned}$$

Hence, $\xi_1^{(6)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{15}(\lambda_j; \lambda_i; \beta) = \xi_1^{(5)}(\lambda_i; \lambda_j; \lambda_i; \beta)$. We wish to show that ξ_{15} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{15} with respect to λ_j gives

$$\begin{aligned} \xi_{15}(\lambda_j; \lambda_i; \beta) &= (14618880\beta + 39744000 + 881280\beta^2 + 8640\beta^3)\lambda_j^3 \\ &\quad + (17383680\beta - 34145280 + 77760\beta^3 + 3317760\beta^2)\lambda_i\lambda_j^2 \\ &\quad + (21980160 + 1866240\beta^2 + 6739200\beta + 77760\beta^3)\lambda_i^2\lambda_j \\ &\quad + (9745920 + 8640\beta^3 + 155520\beta^2 + 2039040\beta)\lambda_i^3, \\ \xi_{15}^{(1)}(\lambda_j; \lambda_i; \beta) &= (43856640\beta + 119232000 + 2643840\beta^2 + 25920\beta^3)\lambda_j^2 \\ &\quad + (34767360\beta - 68290560 + 155520\beta^3 + 6635520\beta^2)\lambda_i\lambda_j \\ &\quad + (21980160 + 1866240\beta^2 + 6739200\beta + 77760\beta^3)\lambda_i^2, \\ \xi_{15}^{(2)}(\lambda_j; \lambda_i; \beta) &= (87713280\beta + 238464000 + 5287680\beta^2 + 51840\beta^3)\lambda_j \\ &\quad + (34767360\beta - 68290560 + 155520\beta^3 + 6635520\beta^2)\lambda_i, \\ \xi_{15}^{(3)}(\lambda_j; \lambda_i; \beta) &= 87713280\beta + 238464000 + 5287680\beta^2 + 51840\beta^3 \geq 0 \text{ for all } \beta \geq -3.3874. \end{aligned}$$

Now

$$\begin{aligned} \xi_{15}(\lambda_i; \lambda_i; \beta) &= 172800\lambda_i^3(236\beta + 216 + 36\beta^2 + \beta^3) \geq 0 \text{ for all } \beta \geq -1.0915, \\ \xi_{15}^{(1)}(\lambda_i; \lambda_i; \beta) &= 86400\lambda_i^2(988\beta + 844 + 129\beta^2 + 3\beta^3) \geq 0 \text{ for all } \beta \geq -0.9757, \\ \xi_{15}^{(2)}(\lambda_i; \lambda_i; \beta) &= 34560\lambda_i(3544\beta + 4924 + 345\beta^2 + 6\beta^3) \geq 0 \text{ for all } \beta \geq -1.6454. \end{aligned}$$

Hence, $\xi_1^{(5)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{14}(\lambda_j; \lambda_i; \beta) = \xi_1^{(4)}(\lambda_i; \lambda_j; \lambda_i; \beta)$. We wish to show that ξ_{14} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{14} with respect to λ_j gives

$$\begin{aligned} \xi_{14}(\lambda_j; \lambda_i; \beta) &= (25027200 + 7548480\beta + 8640\beta^3 + 535680\beta^2)\lambda_j^4 \\ &\quad + (-22256640 + 8432640\beta + 103680\beta^3 + 2350080\beta^2)\lambda_i\lambda_j^3 \\ &\quad + (2073600\beta + 9745920 + 1658880\beta^2 + 155520\beta^3)\lambda_i^2\lambda_j^2 \\ &\quad + (7764480 + 34560\beta^3 + 276480\beta^2 + 2684160\beta)\lambda_i^3\lambda_j \\ &\quad + (227520\beta + 17280\beta^2 + 685440)\lambda_i^4, \\ \xi_{14}^{(1)}(\lambda_j; \lambda_i; \beta) &= (100108800 + 30193920\beta + 34560\beta^3 + 2142720\beta^2)\lambda_j^3 \\ &\quad + (-66769920 + 25297920\beta + 311040\beta^3 + 7050240\beta^2)\lambda_i\lambda_j^2 \\ &\quad + (4147200\beta + 19491840 + 3317760\beta^2 + 311040\beta^3)\lambda_i^2\lambda_j \\ &\quad + (7764480 + 34560\beta^3 + 276480\beta^2 + 2684160\beta)\lambda_i^3, \\ \xi_{14}^{(2)}(\lambda_j; \lambda_i; \beta) &= (300326400 + 90581760\beta + 103680\beta^3 + 6428160\beta^2)\lambda_j^2 \\ &\quad + (-133539840 + 50595840\beta + 622080\beta^3 + 14100480\beta^2)\lambda_i\lambda_j \\ &\quad + (4147200\beta + 19491840 + 3317760\beta^2 + 311040\beta^3)\lambda_i^2, \\ \xi_{14}^{(3)}(\lambda_j; \lambda_i; \beta) &= (600652800 + 181163520\beta + 207360\beta^3 + 12856320\beta^2)\lambda_j \\ &\quad + (-133539840 + 50595840\beta + 622080\beta^3 + 14100480\beta^2)\lambda_i, \\ \xi_{14}^{(4)}(\lambda_j; \lambda_i; \beta) &= 600652800 + 181163520\beta + 207360\beta^3 + 12856320\beta^2 \geq 0 \text{ for all } \beta \geq -4.8608. \end{aligned}$$

Now

$$\begin{aligned}\xi_{14}(\lambda_i; \lambda_i; \beta) &= 100800\lambda_i^4(208 + 208\beta + 3\beta^3 + 48\beta^2) \geq 0 \text{ for all } \beta \geq -1.4294, \\ \xi_{14}^{(1)}(\lambda_i; \lambda_i; \beta) &= 115200\lambda_i^3(526 + 541\beta + 6\beta^3 + 111\beta^2) \geq 0 \text{ for all } \beta \geq -1.2898, \\ \xi_{14}^{(2)}(\lambda_i; \lambda_i; \beta) &= 172800\lambda_i^2(1078 + 841\beta + 6\beta^3 + 138\beta^2) \geq 0 \text{ for all } \beta \geq -1.7421, \\ \xi_{14}^{(3)}(\lambda_i; \lambda_i; \beta) &= 69120\lambda_i(6758 + 3353\beta + 12\beta^3 + 390\beta^2) \geq 0 \text{ for all } \beta \geq -2.9155.\end{aligned}$$

Hence, $\xi_1^{(4)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{13}(\lambda_j; \lambda_i; \beta) = \xi_1^{(3)}(\lambda_j; \lambda_j; \lambda_i; \beta)/\lambda_j$. We wish to show that ξ_{13} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{13} with respect to λ_j gives

$$\begin{aligned}\xi_{13}(\lambda_j; \lambda_i; \beta) &= (10356480 + 2839680\beta + 216000\beta^2 + 4320\beta^3)\lambda_j^4 \\ &\quad + (1080000\beta^2 - 9912960 + 3159360\beta + 64800\beta^3)\lambda_i\lambda_j^3 \\ &\quad + (3421440 + 864000\beta^2 - 224640\beta + 129600\beta^3)\lambda_i^2\lambda_j^2 \\ &\quad + (216000\beta^2 + 2960640 + 43200\beta^3 + 1647360\beta)\lambda_i^3\lambda_j \\ &\quad + (279360\beta + 43200\beta^2 + 512640)\lambda_i^4, \\ \xi_{13}^{(1)}(\lambda_j; \lambda_i; \beta) &= (41425920 + 11358720\beta + 864000\beta^2 + 17280\beta^3)\lambda_j^3 \\ &\quad + (3240000\beta^2 - 29738880 + 9478080\beta + 194400\beta^3)\lambda_i\lambda_j^2 \\ &\quad + (6842880 + 1728000\beta^2 - 449280\beta + 259200\beta^3)\lambda_i^2\lambda_j \\ &\quad + (216000\beta^2 + 2960640 + 43200\beta^3 + 1647360\beta)\lambda_i^3, \\ \xi_{13}^{(2)}(\lambda_j; \lambda_i; \beta) &= (124277760 + 34076160\beta + 2592000\beta^2 + 51840\beta^3)\lambda_j^2 \\ &\quad + (6480000\beta^2 - 59477760 + 18956160\beta + 388800\beta^3)\lambda_i\lambda_j \\ &\quad + (6842880 + 1728000\beta^2 - 449280\beta + 259200\beta^3)\lambda_i^2, \\ \xi_{13}^{(3)}(\lambda_j; \lambda_i; \beta) &= (248555520 + 68152320\beta + 5184000\beta^2 + 103680\beta^3)\lambda_j \\ &\quad + (6480000\beta^2 - 59477760 + 18956160\beta + 388800\beta^3)\lambda_i, \\ \xi_{13}^{(4)}(\lambda_j; \lambda_i; \beta) &= 248555520 + 68152320\beta + 5184000\beta^2 + 103680\beta^3 \geq 0 \text{ for all } \beta \geq -6.2366.\end{aligned}$$

Now

$$\begin{aligned}\xi_{13}(\lambda_i; \lambda_i; \beta) &= 40320\lambda_i^4(182 + 191\beta + 60\beta^2 + 6\beta^3) \geq 0 \text{ for all } \beta \geq -1.7321, \\ \xi_{13}^{(1)}(\lambda_i; \lambda_i; \beta) &= 10080\lambda_i^3(2132 + 2186\beta + 600\beta^2 + 51\beta^3) \geq 0 \text{ for all } \beta \geq -1.5432, \\ \xi_{13}^{(2)}(\lambda_i; \lambda_i; \beta) &= 8640\lambda_i^2(8292 + 6086\beta + 1250\beta^2 + 81\beta^3) \geq 0 \text{ for all } \beta \geq -2.2524, \\ \xi_{13}^{(3)}(\lambda_i; \lambda_i; \beta) &= 8640\lambda_i(21884 + 10082\beta + 1350\beta^2 + 57\beta^3) \geq 0 \text{ for all } \beta \geq -3.7735.\end{aligned}$$

Hence, $\xi_1^{(3)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{12}(\lambda_j; \lambda_i; \beta) = \xi_1^{(2)}(\lambda_j; \lambda_j; \lambda_i; \beta)/\lambda_j^2$. We wish to show that ξ_{12} is non-negative

for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{12} with respect to λ_j gives

$$\begin{aligned} \xi_{12}(\lambda_j; \lambda_i; \beta) &= (826560\beta + 3156480 + 1440\beta^3 + 64800\beta^2)\lambda_j^4 \\ &\quad + (367200\beta^2 + 973440\beta - 3346560 + 25920\beta^3)\lambda_i\lambda_j^3 \\ &\quad + (1080000 - 410400\beta + 324000\beta^2 + 64800\beta^3)\lambda_i^2\lambda_j^2 \\ &\quad + (737280 + 682560\beta + 28800\beta^3 + 108000\beta^2)\lambda_i^3\lambda_j \\ &\quad + (187200 + 43200\beta^2 + 165600\beta)\lambda_i^4, \\ \xi_{12}^{(1)}(\lambda_j; \lambda_i; \beta) &= (3306240\beta + 12625920 + 5760\beta^3 + 259200\beta^2)\lambda_j^3 \\ &\quad + (1101600\beta^2 + 2920320\beta - 10039680 + 77760\beta^3)\lambda_i\lambda_j^2 \\ &\quad + (2160000 - 820800\beta + 648000\beta^2 + 129600\beta^3)\lambda_i^2\lambda_j \\ &\quad + (737280 + 682560\beta + 28800\beta^3 + 108000\beta^2)\lambda_i^3, \\ \xi_{12}^{(2)}(\lambda_j; \lambda_i; \beta) &= (9918720\beta + 37877760 + 17280\beta^3 + 777600\beta^2)\lambda_j^2 \\ &\quad + (2203200\beta^2 + 5840640\beta - 20079360 + 155520\beta^3)\lambda_i\lambda_j \\ &\quad + (2160000 - 820800\beta + 648000\beta^2 + 129600\beta^3)\lambda_i^2, \\ \xi_{12}^{(3)}(\lambda_j; \lambda_i; \beta) &= (19837440\beta + 75755520 + 34560\beta^3 + 1555200\beta^2)\lambda_j \\ &\quad + (2203200\beta^2 + 5840640\beta - 20079360 + 155520\beta^3)\lambda_i, \\ \xi_{12}^{(4)}(\lambda_j; \lambda_i; \beta) &= 19837440\beta + 75755520 + 34560\beta^3 + 1555200\beta^2 \geq 0 \text{ for all } \beta \geq -7.4474. \end{aligned}$$

Now

$$\begin{aligned} \xi_{12}(\lambda_i; \lambda_i; \beta) &= 60480\lambda_i^4(2\beta + 5)(\beta + 3)(\beta + 2) \geq 0 \text{ for all } \beta \geq -2, \\ \xi_{12}^{(1)}(\lambda_i; \lambda_i; \beta) &= 20160\lambda_i^3(302\beta + 272 + 12\beta^3 + 105\beta^2) \geq 0 \text{ for all } \beta \geq -1.7670, \\ \xi_{12}^{(2)}(\lambda_i; \lambda_i; \beta) &= 60480\lambda_i^2(247\beta + 330 + 5\beta^3 + 60\beta^2) \geq 0 \text{ for all } \beta \geq -2.7671, \\ \xi_{12}^{(3)}(\lambda_i; \lambda_i; \beta) &= 8640\lambda_i(2972\beta + 6444 + 22\beta^3 + 435\beta^2) \geq 0 \text{ for all } \beta \geq -4.2105. \end{aligned}$$

Hence, $\xi_1^{(2)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{11}(\lambda_j; \lambda_i; \beta) = \xi_1^{(1)}(\lambda_j; \lambda_j; \lambda_i; \beta)/\lambda_j^3$. We wish to show that ξ_{11} is non-negative

for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{11} with respect to λ_j gives

$$\begin{aligned}
\xi_{11}(\lambda_j; \lambda_i; \beta) &= (756864 + 194112\beta + 360\beta^3 + 15408\beta^2)\lambda_j^4 \\
&\quad + (251616\beta - 901248 + 98784\beta^2 + 7560\beta^3)\lambda_i\lambda_j^3 \\
&\quad + (307584 + 96768\beta^2 - 178848\beta + 22680\beta^3)\lambda_i^2\lambda_j^2 \\
&\quad + (226080\beta + 149760 + 12600\beta^3 + 40320\beta^2)\lambda_i^3\lambda_j \\
&\quad + (49920 + 68640\beta + 25200\beta^2)\lambda_i^4, \\
\xi_{11}^{(1)}(\lambda_j; \lambda_i; \beta) &= (3027456 + 776448\beta + 1440\beta^3 + 61632\beta^2)\lambda_j^3 \\
&\quad + (754848\beta - 2703744 + 296352\beta^2 + 22680\beta^3)\lambda_i\lambda_j^2 \\
&\quad + (615168 + 193536\beta^2 - 357696\beta + 45360\beta^3)\lambda_i^2\lambda_j \\
&\quad + (226080\beta + 149760 + 12600\beta^3 + 40320\beta^2)\lambda_i^3, \\
\xi_{11}^{(2)}(\lambda_j; \lambda_i; \beta) &= (9082368 + 2329344\beta + 4320\beta^3 + 184896\beta^2)\lambda_j^2 \\
&\quad + (1509696\beta - 5407488 + 592704\beta^2 + 45360\beta^3)\lambda_i\lambda_j \\
&\quad + (615168 + 193536\beta^2 - 357696\beta + 45360\beta^3)\lambda_i^2, \\
\xi_{11}^{(3)}(\lambda_j; \lambda_i; \beta) &= (18164736 + 4658688\beta + 8640\beta^3 + 369792\beta^2)\lambda_j \\
&\quad + (1509696\beta - 5407488 + 592704\beta^2 + 45360\beta^3)\lambda_i, \\
\xi_{11}^{(4)}(\lambda_j; \lambda_i; \beta) &= 18164736 + 4658688\beta + 8640\beta^3 + 369792\beta^2 \geq 0 \text{ for all } \beta \geq -8.4121.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{11}(\lambda_i; \lambda_i; \beta) &= 8640\lambda_i^4(5\beta + 7)(\beta + 3)(\beta + 2) \geq 0 \text{ for all } \beta \geq -\frac{7}{4}, \\
\xi_{11}^{(1)}(\lambda_i; \lambda_i; \beta) &= 4320\lambda_i^3(19\beta + 42)(\beta + 3)(\beta + 2) \geq 0 \text{ for all } \beta \geq -2, \\
\xi_{11}^{(2)}(\lambda_i; \lambda_i; \beta) &= 576\lambda_i^2(7448 + 6044\beta + 165\beta^3 + 1686\beta^2) \geq 0 \text{ for all } \beta \geq -3.0958, \\
\xi_{11}^{(3)}(\lambda_i; \lambda_i; \beta) &= 144\lambda_i(88592 + 42836\beta + 375\beta^3 + 6684\beta^2) \geq 0 \text{ for all } \beta \geq -4.0263.
\end{aligned}$$

Hence, $\xi_1^{(1)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{10}(\lambda_j; \lambda_i; \beta) = \xi_1(\lambda_j; \lambda_j; \lambda_i; \beta)/\lambda_j^4$. We wish to show that ξ_{10} is non-negative for

all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{10} with respect to λ_j gives

$$\begin{aligned} \xi_{10}(\lambda_j; \lambda_i; \beta) &= (37944\beta + 149040 + 3024\beta^2 + 72\beta^3)\lambda_j^4 \\ &\quad + (-200064 + 21888\beta^2 + 55104\beta + 1728\beta^3)\lambda_i\lambda_j^3 \\ &\quad + (76608 - 50400\beta + 24192\beta^2 + 6048\beta^3)\lambda_i^2\lambda_j^2 \\ &\quad + (12096\beta^2 + 32256 + 4032\beta^3 + 64512\beta)\lambda_i^3\lambda_j \\ &\quad + (10080\beta^2 + 13440 + 23520\beta)\lambda_i^4, \\ \xi_{10}^{(1)}(\lambda_j; \lambda_i; \beta) &= (151776\beta + 596160 + 12096\beta^2 + 288\beta^3)\lambda_j^3 \\ &\quad + (-600192 + 65664\beta^2 + 165312\beta + 5184\beta^3)\lambda_i\lambda_j^2 \\ &\quad + (153216 - 100800\beta + 48384\beta^2 + 12096\beta^3)\lambda_i^2\lambda_j \\ &\quad + (12096\beta^2 + 32256 + 4032\beta^3 + 64512\beta)\lambda_i^3, \\ \xi_{10}^{(2)}(\lambda_j; \lambda_i; \beta) &= (455328\beta + 1788480 + 36288\beta^2 + 864\beta^3)\lambda_j^2 \\ &\quad + (-1200384 + 131328\beta^2 + 330624\beta + 10368\beta^3)\lambda_i\lambda_j \\ &\quad + (153216 - 100800\beta + 48384\beta^2 + 12096\beta^3)\lambda_i^2, \\ \xi_{10}^{(3)}(\lambda_j; \lambda_i; \beta) &= (910656\beta + 3576960 + 72576\beta^2 + 1728\beta^3)\lambda_j \\ &\quad + (-1200384 + 131328\beta^2 + 330624\beta + 10368\beta^3)\lambda_i, \\ \xi_{10}^{(4)}(\lambda_j; \lambda_i; \beta) &= 1728(\beta + 10)(\beta + 9)(\beta + 23) \geq 0 \text{ for all } \beta \geq -9. \end{aligned}$$

Now

$$\begin{aligned} \xi_{10}(\lambda_i; \lambda_i; \beta) &= 11880\lambda_i^4(\beta + 3)(\beta + 2)(\beta + 1) > 0 \text{ for all } \beta > -1, \\ \xi_{10}^{(1)}(\lambda_i; \lambda_i; \beta) &= 4320\lambda_i^3(5\beta + 7)(\beta + 3)(\beta + 2) > 0 \text{ for all } \beta > -\frac{7}{5}, \\ \xi_{10}^{(2)}(\lambda_i; \lambda_i; \beta) &= 864\lambda_i^2(\beta + 3)(27\beta^2 + 169\beta + 286) > 0 \text{ for all } \beta > -3, \\ \xi_{10}^{(3)}(\lambda_i; \lambda_i; \beta) &= 576\lambda_i(2155\beta + 4126 + 354\beta^2 + 21\beta^3) > 0 \text{ for all } \beta > -3.5443. \end{aligned}$$

Hence, ξ_1 is positive for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$. This implies that $\xi(\lambda_l; \lambda_l; \lambda_j; \lambda_i; \beta)$ is positive for all $\lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Now

$$\begin{aligned}
\xi_2(\lambda_l; \lambda_j; \lambda_i; \beta) = & (7200\beta + 2880)(\lambda_j + \lambda_i)\lambda_l^6 \\
& +(98496 + 2592\beta^2 + 23328\beta)(\lambda_j^2 + \lambda_i^2)\lambda_l^5 \\
& +(39744\beta + 7776\beta^2 - 110592)\lambda_i\lambda_j\lambda_l^5 \\
& +(21600\beta^2 + 864\beta - 98496 + 1296\beta^3)(\lambda_i\lambda_j^2 + \lambda_i^2\lambda_j)\lambda_l^4 \\
& +(36576\beta + 144\beta^3 + 173376 + 2880\beta^2)(\lambda_j^3 + \lambda_i^3)\lambda_l^4 \\
& +(-156672 + 1728\beta^3 + 51840\beta + 9792\beta^2)(\lambda_i\lambda_j^3 + \lambda_i^3\lambda_j)\lambda_l^3 \\
& +(1152\beta^2 + 46656 + 17568\beta)(\lambda_j^4 + \lambda_i^4)\lambda_l^3 \\
& +(-98496\beta + 15552\beta^2 + 5184\beta^3 + 93312)\lambda_i^2\lambda_j^2\lambda_l^3 \\
& +(13824\beta + 2592\beta^3 + 34560 + 5184\beta^2)(\lambda_i^2\lambda_j^3 + \lambda_i^3\lambda_j^2)\lambda_l^2 \\
& +(5184\beta^2 + 17856\beta - 39168)(\lambda_i\lambda_j^4 + \lambda_i^4\lambda_j)\lambda_l^2 \\
& +(27648 + 5184\beta + 3456\beta^2)(\lambda_i^2\lambda_j^4 + \lambda_i^4\lambda_j^2)\lambda_l \\
& +(576\beta^3 + 24768\beta + 76032 + 2016\beta^2)\lambda_i^3\lambda_j^3\lambda_l \\
& +(288\beta^2 + 2592\beta + 10944)(\lambda_i^3\lambda_j^4 + \lambda_i^4\lambda_j^3),
\end{aligned}$$

and, hence,

$$\xi_2^{(6)}(\lambda_l; \lambda_j; \lambda_i; \beta) = (5184000\beta + 2073600)(\lambda_j + \lambda_i) \geq 0 \text{ for all } \beta \geq -\frac{2}{5}.$$

Define $\xi_{25}(\lambda_j; \lambda_i; \beta) = \xi_2^{(5)}(\lambda_j; \lambda_j; \lambda_i; \beta)$. We wish to show that ξ_{25} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{25} with respect to λ_j gives

$$\begin{aligned}
\xi_{25}(\lambda_j; \lambda_i; \beta) = & (7983360\beta + 13893120 + 311040\beta^2)\lambda_j^2 \\
& +(9953280\beta - 11197440 + 933120\beta^2)\lambda_i\lambda_j \\
& +(11819520 + 311040\beta^2 + 2799360\beta)\lambda_i^2, \\
\xi_{25}^{(1)}(\lambda_j; \lambda_i; \beta) = & (15966720\beta + 27786240 + 622080\beta^2)\lambda_j \\
& +(9953280\beta - 11197440 + 933120\beta^2)\lambda_i, \\
\xi_{25}^{(2)}(\lambda_j; \lambda_i; \beta) = & 15966720\beta + 27786240 + 622080\beta^2 \geq 0 \text{ for all } \beta \geq -1.8776.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{25}(\lambda_i; \lambda_i; \beta) = & 518400\lambda_i^2(28 + 40\beta + 3\beta^2) \geq 0 \text{ for all } \beta \geq -0.7412, \\
\xi_{25}^{(1)}(\lambda_i; \lambda_i; \beta) = & 518400\lambda_i(\beta + 16)(3\beta + 2) \geq 0 \text{ for all } \beta \geq -\frac{2}{3}.
\end{aligned}$$

Hence, $\xi_2^{(5)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{24}(\lambda_j; \lambda_i; \beta) = \xi_2^{(4)}(\lambda_j; \lambda_j; \lambda_i; \beta)$. We wish to show that ξ_{24} is non-negative for

all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{24} with respect to λ_j gives

$$\begin{aligned} \xi_{24}(\lambda_j; \lambda_i; \beta) &= (6269184\beta + 17017344 + 380160\beta^2 + 3456\beta^3)\lambda_j^3 \\ &\quad + (7382016\beta - 14598144 + 1451520\beta^2 + 31104\beta^3)\lambda_i\lambda_j^2 \\ &\quad + (9455616 + 829440\beta^2 + 2820096\beta + 31104\beta^3)\lambda_i^2\lambda_j \\ &\quad + (877824\beta + 3456\beta^3 + 4161024 + 69120\beta^2)\lambda_i^3, \\ \xi_{24}^{(1)}(\lambda_j; \lambda_i; \beta) &= (18807552\beta + 51052032 + 1140480\beta^2 + 10368\beta^3)\lambda_j^2 \\ &\quad + (14764032\beta - 29196288 + 2903040\beta^2 + 62208\beta^3)\lambda_i\lambda_j \\ &\quad + (9455616 + 829440\beta^2 + 2820096\beta + 31104\beta^3)\lambda_i^2, \\ \xi_{24}^{(2)}(\lambda_j; \lambda_i; \beta) &= (37615104\beta + 102104064 + 2280960\beta^2 + 20736\beta^3)\lambda_j \\ &\quad + (14764032\beta - 29196288 + 2903040\beta^2 + 62208\beta^3)\lambda_i, \\ \xi_{24}^{(3)}(\lambda_j; \lambda_i; \beta) &= 37615104\beta + 102104064 + 2280960\beta^2 + 20736\beta^3 \geq 0 \text{ for all } \beta \geq -3.3897. \end{aligned}$$

Now

$$\begin{aligned} \xi_{24}(\lambda_i; \lambda_i; \beta) &= 34560\lambda_i^3(502\beta + 464 + 79\beta^2 + 2\beta^3) \geq 0 \text{ for all } \beta \geq -1.1141, \\ \xi_{24}^{(1)}(\lambda_i; \lambda_i; \beta) &= 103680\lambda_i^2(351\beta + 302 + 47\beta^2 + \beta^3) \geq 0 \text{ for all } \beta \geq -0.9885, \\ \xi_{24}^{(2)}(\lambda_i; \lambda_i; \beta) &= 41472\lambda_i(1263\beta + 1758 + 125\beta^2 + 2\beta^3) \geq 0 \text{ for all } \beta \geq -1.6562. \end{aligned}$$

Hence, $\xi_2^{(4)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{23}(\lambda_j; \lambda_i; \beta) = \xi_2^{(3)}(\lambda_j; \lambda_j; \lambda_i; \beta)$. We wish to show that ξ_{23} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{23} with respect to λ_j gives

$$\begin{aligned} \xi_{23}(\lambda_j; \lambda_i; \beta) &= (10696320 + 3246912\beta + 3456\beta^3 + 231552\beta^2)\lambda_j^4 \\ &\quad + (-9593856 + 3580416\beta + 1043712\beta^2 + 41472\beta^3)\lambda_i\lambda_j^3 \\ &\quad + (4105728 + 767232\beta^2 + 829440\beta + 62208\beta^3)\lambda_i^2\lambda_j^2 \\ &\quad + (1188864\beta + 13824\beta^3 + 3220992 + 127872\beta^2)\lambda_i^3\lambda_j \\ &\quad + (6912\beta^2 + 105408\beta + 279936)\lambda_i^4, \\ \xi_{23}^{(1)}(\lambda_j; \lambda_i; \beta) &= (42785280 + 12987648\beta + 13824\beta^3 + 926208\beta^2)\lambda_j^3 \\ &\quad + (-28781568 + 10741248\beta + 3131136\beta^2 + 124416\beta^3)\lambda_i\lambda_j^2 \\ &\quad + (8211456 + 1534464\beta^2 + 1658880\beta + 124416\beta^3)\lambda_i^2\lambda_j \\ &\quad + (1188864\beta + 13824\beta^3 + 3220992 + 127872\beta^2)\lambda_i^3, \\ \xi_{23}^{(2)}(\lambda_j; \lambda_i; \beta) &= (128355840 + 38962944\beta + 41472\beta^3 + 2778624\beta^2)\lambda_j^2 \\ &\quad + (-57563136 + 21482496\beta + 6262272\beta^2 + 248832\beta^3)\lambda_i\lambda_j \\ &\quad + (8211456 + 1534464\beta^2 + 1658880\beta + 124416\beta^3)\lambda_i^2, \\ \xi_{23}^{(3)}(\lambda_j; \lambda_i; \beta) &= (256711680 + 77925888\beta + 82944\beta^3 + 5557248\beta^2)\lambda_j \\ &\quad + (-57563136 + 21482496\beta + 6262272\beta^2 + 248832\beta^3)\lambda_i, \\ \xi_{23}^{(4)}(\lambda_j; \lambda_i; \beta) &= 256711680 + 77925888\beta + 82944\beta^3 + 5557248\beta^2 \geq 0 \text{ for all } \beta \geq -4.8510. \end{aligned}$$

Now

$$\begin{aligned}\xi_{23}(\lambda_i; \lambda_i; \beta) &= 120960\lambda_i^4(\beta + 4)(\beta^2 + 14\beta + 18) \geq 0 \text{ for all } \beta \geq -1.4322, \\ \xi_{23}^{(1)}(\lambda_i; \lambda_i; \beta) &= 17280\lambda_i^3(1472 + 1538\beta + 16\beta^3 + 331\beta^2) \geq 0 \text{ for all } \beta \geq -1.2958, \\ \xi_{23}^{(2)}(\lambda_i; \lambda_i; \beta) &= 103680\lambda_i^2(762 + 599\beta + 4\beta^3 + 102\beta^2) \geq 0 \text{ for all } \beta \geq -1.7669, \\ \xi_{23}^{(3)}(\lambda_i; \lambda_i; \beta) &= 41472\lambda_i(4802 + 2397\beta + 8\beta^3 + 285\beta^2) \geq 0 \text{ for all } \beta \geq -2.9562.\end{aligned}$$

Hence, $\xi_2^{(3)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{22}(\lambda_j; \lambda_i; \beta) = \xi_2^{(2)}(\lambda_j; \lambda_j; \lambda_i; \beta)/\lambda_j$. We wish to show that ξ_{22} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{22} with respect to λ_j gives

$$\begin{aligned}\xi_{22}(\lambda_j; \lambda_i; \beta) &= (4416768 + 1226880\beta + 93312\beta^2 + 1728\beta^3)\lambda_j^4 \\ &\quad + (1368000\beta - 4325760 + 483840\beta^2 + 25920\beta^3)\lambda_i\lambda_j^3 \\ &\quad + (1416960 + 414720\beta^2 - 86400\beta + 51840\beta^3)\lambda_i^2\lambda_j^2 \\ &\quad + (1209600 + 103680\beta^2 + 17280\beta^3 + 777600\beta)\lambda_i^3\lambda_j \\ &\quad + (17280\beta^2 + 141120\beta + 201600)\lambda_i^4, \\ \xi_{22}^{(1)}(\lambda_j; \lambda_i; \beta) &= (17667072 + 4907520\beta + 373248\beta^2 + 6912\beta^3)\lambda_j^3 \\ &\quad + (4104000\beta - 12977280 + 1451520\beta^2 + 77760\beta^3)\lambda_i\lambda_j^2 \\ &\quad + (2833920 + 829440\beta^2 - 172800\beta + 103680\beta^3)\lambda_i^2\lambda_j \\ &\quad + (1209600 + 103680\beta^2 + 17280\beta^3 + 777600\beta)\lambda_i^3, \\ \xi_{22}^{(2)}(\lambda_j; \lambda_i; \beta) &= (53001216 + 14722560\beta + 1119744\beta^2 + 20736\beta^3)\lambda_j^2 \\ &\quad + (8208000\beta - 25954560 + 2903040\beta^2 + 155520\beta^3)\lambda_i\lambda_j \\ &\quad + (2833920 + 829440\beta^2 - 172800\beta + 103680\beta^3)\lambda_i^2, \\ \xi_{22}^{(3)}(\lambda_j; \lambda_i; \beta) &= (106002432 + 29445120\beta + 2239488\beta^2 + 41472\beta^3)\lambda_j \\ &\quad + (8208000\beta - 25954560 + 2903040\beta^2 + 155520\beta^3)\lambda_i, \\ \xi_{22}^{(4)}(\lambda_j; \lambda_i; \beta) &= 106002432 + 29445120\beta + 2239488\beta^2 + 41472\beta^3 \geq 0 \text{ for all } \beta \geq -6.1457.\end{aligned}$$

Now

$$\begin{aligned}\xi_{22}(\lambda_i; \lambda_i; \beta) &= 8064\lambda_i^4(362 + 425\beta + 138\beta^2 + 12\beta^3) \geq 0 \text{ for all } \beta \geq -1.4432, \\ \xi_{22}^{(1)}(\lambda_i; \lambda_i; \beta) &= 12096\lambda_i^3(722 + 795\beta + 228\beta^2 + 17\beta^3) \geq 0 \text{ for all } \beta \geq -1.4368, \\ \xi_{22}^{(2)}(\lambda_i; \lambda_i; \beta) &= 10368\lambda_i^2(2882 + 2195\beta + 468\beta^2 + 27\beta^3) \geq 0 \text{ for all } \beta \geq -2.2600, \\ \xi_{22}^{(3)}(\lambda_i; \lambda_i; \beta) &= 3456\lambda_i(23162 + 10895\beta + 1488\beta^2 + 57\beta^3) \geq 0 \text{ for all } \beta \geq -3.8598.\end{aligned}$$

Hence, $\xi_2^{(2)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{21}(\lambda_j; \lambda_i; \beta) = \xi_2^{(1)}(\lambda_j; \lambda_j; \lambda_i; \beta)/\lambda_j^2$. We wish to show that ξ_{21} is non-negative

for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{21} with respect to λ_j gives

$$\begin{aligned} \xi_{21}(\lambda_j; \lambda_i; \beta) &= (358848\beta + 1343232 + 27936\beta^2 + 576\beta^3)\lambda_j^4 \\ &\quad + (436608\beta - 1478016 + 165024\beta^2 + 10368\beta^3)\lambda_i\lambda_j^3 \\ &\quad + (-142560\beta + 159840\beta^2 + 475200 + 25920\beta^3)\lambda_i^2\lambda_j^2 \\ &\quad + (368640 + 53280\beta^2 + 354240\beta + 11520\beta^3)\lambda_i^3\lambda_j \\ &\quad + (89280 + 17280\beta^2 + 93600\beta)\lambda_i^4, \\ \xi_{21}^{(1)}(\lambda_j; \lambda_i; \beta) &= (1435392\beta + 5372928 + 111744\beta^2 + 2304\beta^3)\lambda_j^3 \\ &\quad + (1309824\beta - 4434048 + 495072\beta^2 + 31104\beta^3)\lambda_i\lambda_j^2 \\ &\quad + (-285120\beta + 319680\beta^2 + 950400 + 51840\beta^3)\lambda_i^2\lambda_j \\ &\quad + (368640 + 53280\beta^2 + 354240\beta + 11520\beta^3)\lambda_i^3, \\ \xi_{21}^{(2)}(\lambda_j; \lambda_i; \beta) &= (4306176\beta + 16118784 + 335232\beta^2 + 6912\beta^3)\lambda_j^2 \\ &\quad + (2619648\beta - 8868096 + 990144\beta^2 + 62208\beta^3)\lambda_i\lambda_j \\ &\quad + (-285120\beta + 319680\beta^2 + 950400 + 51840\beta^3)\lambda_i^2, \\ \xi_{21}^{(3)}(\lambda_j; \lambda_i; \beta) &= (8612352\beta + 32237568 + 670464\beta^2 + 13824\beta^3)\lambda_j \\ &\quad + (2619648\beta - 8868096 + 990144\beta^2 + 62208\beta^3)\lambda_i, \\ \xi_{21}^{(4)}(\lambda_j; \lambda_i; \beta) &= 8612352\beta + 32237568 + 670464\beta^2 + 13824\beta^3 \geq 0 \text{ for all } \beta \geq -7.0502. \end{aligned}$$

Now

$$\begin{aligned} \xi_{21}(\lambda_i; \lambda_i; \beta) &= 12096\lambda_i^4(\beta + 3)(4\beta^2 + 23\beta + 22) \geq 0 \text{ for all } \beta \geq -1.2120, \\ \xi_{21}^{(1)}(\lambda_i; \lambda_i; \beta) &= 4032\lambda_i^3(698\beta + 560 + 243\beta^2 + 24\beta^3) \geq 0 \text{ for all } \beta \geq -1.3588, \\ \xi_{21}^{(2)}(\lambda_i; \lambda_i; \beta) &= 12096\lambda_i^2(549\beta + 678 + 136\beta^2 + 10\beta^3) \geq 0 \text{ for all } \beta \geq -2.4869, \\ \xi_{21}^{(3)}(\lambda_i; \lambda_i; \beta) &= 1728\lambda_i(6500\beta + 13524 + 961\beta^2 + 44\beta^3) \geq 0 \text{ for all } \beta \geq -4.0951. \end{aligned}$$

Hence, $\xi_2^{(1)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{20}(\lambda_j; \lambda_i; \beta) = \xi_2(\lambda_j; \lambda_j; \lambda_i; \beta)/\lambda_j^3$. We wish to show that ξ_{20} is non-negative for

all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{20} with respect to λ_j gives

$$\begin{aligned} \xi_{20}(\lambda_j; \lambda_i; \beta) &= (84672\beta + 321408 + 144\beta^3 + 6624\beta^2)\lambda_j^4 \\ &\quad + (117504\beta + 44352\beta^2 - 402048 + 3024\beta^3)\lambda_i\lambda_j^3 \\ &\quad + (155520 + 48384\beta^2 - 55296\beta + 9072\beta^3)\lambda_i^2\lambda_j^2 \\ &\quad + (129600\beta + 5040\beta^3 + 138240 + 20160\beta^2)\lambda_i^3\lambda_j \\ &\quad + (10080\beta^2 + 46080 + 43200\beta)\lambda_i^4, \\ \xi_{20}^{(1)}(\lambda_j; \lambda_i; \beta) &= (338688\beta + 1285632 + 576\beta^3 + 26496\beta^2)\lambda_j^3 \\ &\quad + (352512\beta + 133056\beta^2 - 1206144 + 9072\beta^3)\lambda_i\lambda_j^2 \\ &\quad + (311040 + 96768\beta^2 - 110592\beta + 18144\beta^3)\lambda_i^2\lambda_j \\ &\quad + (129600\beta + 5040\beta^3 + 138240 + 20160\beta^2)\lambda_i^3, \\ \xi_{20}^{(2)}(\lambda_j; \lambda_i; \beta) &= (1016064\beta + 3856896 + 1728\beta^3 + 79488\beta^2)\lambda_j^2 \\ &\quad + (705024\beta + 266112\beta^2 - 2412288 + 18144\beta^3)\lambda_i\lambda_j \\ &\quad + (311040 + 96768\beta^2 - 110592\beta + 18144\beta^3)\lambda_i^2, \\ \xi_{20}^{(3)}(\lambda_j; \lambda_i; \beta) &= (2032128\beta + 7713792 + 3456\beta^3 + 158976\beta^2)\lambda_j \\ &\quad + (705024\beta + 266112\beta^2 - 2412288 + 18144\beta^3)\lambda_i, \\ \xi_{20}^{(4)}(\lambda_j; \lambda_i; \beta) &= 2032128\beta + 7713792 + 3456\beta^3 + 158976\beta^2 \geq 0 \text{ for all } \beta \geq -7.3254. \end{aligned}$$

Now

$$\begin{aligned} \xi_{20}(\lambda_i; \lambda_i; \beta) &= 8640\lambda_i^4(2\beta + 5)(\beta + 3)(\beta + 2) > 0 \text{ for all } \beta > -2, \\ \xi_{20}^{(1)}(\lambda_i; \lambda_i; \beta) &= 1728\lambda_i^3(\beta + 3)(19\beta^2 + 103\beta + 102) > 0 \text{ for all } \beta > -1.3039, \\ \xi_{20}^{(2)}(\lambda_i; \lambda_i; \beta) &= 3456\lambda_i^2(\beta + 2)(11\beta^2 + 106\beta + 254) > 0 \text{ for all } \beta > -2, \\ \xi_{20}^{(3)}(\lambda_i; \lambda_i; \beta) &= 864\lambda_i(3168\beta + 6136 + 25\beta^3 + 492\beta^2) > 0 \text{ for all } \beta > -3.5049. \end{aligned}$$

Hence, ξ_2 is positive for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$. This implies that $\xi(\lambda_l; \lambda_l; \lambda_j; \lambda_i; \beta)$ is positive for all $\lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Now

$$\begin{aligned}
 \xi_3(\lambda_l; \lambda_j; \lambda_i; \beta) = & (14400\beta + 5760)(\lambda_j + \lambda_i)\lambda_l^5 \\
 & + (4320\beta^2 + 38880\beta + 164160)(\lambda_j^2 + \lambda_i^2)\lambda_l^4 \\
 & + (12960\beta^2 - 184320 + 66240\beta)\lambda_i\lambda_j\lambda_l^4 \\
 & + (-124416 + 34560\beta^2 - 12096\beta + 1296\beta^3)(\lambda_i\lambda_j^2 + \lambda_i^2\lambda_j)\lambda_l^3 \\
 & + (4320\beta^2 + 228096 + 144\beta^3 + 49536\beta)(\lambda_j^3 + \lambda_i^3)\lambda_l^3 \\
 & + (1296\beta^3 + 13824\beta^2 - 209664 + 72000\beta)(\lambda_i\lambda_j^3 + \lambda_i^3\lambda_j)\lambda_l^2 \\
 & + (38592 + 864\beta^2 + 22176\beta)(\lambda_j^4 + \lambda_i^4)\lambda_l^2 \\
 & + (3888\beta^3 - 93312\beta + 31104 + 31104\beta^2)\lambda_i^2\lambda_j^2\lambda_l^2 \\
 & + (50112\beta + 6912\beta^2 + 69120 + 1296\beta^3)(\lambda_i^2\lambda_j^3 + \lambda_i^3\lambda_j^2)\lambda_l \\
 & + (2592\beta^2 - 31104 + 27648\beta)(\lambda_i\lambda_j^4 + \lambda_i^4\lambda_j)\lambda_l \\
 & + (144\beta^3 + 18432\beta + 96768 + 864\beta^2)\lambda_i^3\lambda_j^3 \\
 & + (864\beta^2 + 7776\beta + 32832)(\lambda_i^2\lambda_j^4 + \lambda_i^4\lambda_j^2),
 \end{aligned}$$

and, hence,

$$\xi_3^{(5)}(\lambda_l; \lambda_j; \lambda_i; \beta) = (691200 + 1728000\beta)(\lambda_j + \lambda_i) \geq 0 \text{ for all } \beta \geq -\frac{2}{5}.$$

Define $\xi_{34}(\lambda_j; \lambda_i; \beta) = \xi_3^{(4)}(\lambda_l; \lambda_j; \lambda_i; \beta)$. We wish to show that ξ_{34} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{34} with respect to λ_j gives

$$\begin{aligned}
 \xi_{34}(\lambda_j; \lambda_i; \beta) = & (4631040 + 2661120\beta + 103680\beta^2)\lambda_j^2 \\
 & + (-3732480 + 3317760\beta + 311040\beta^2)\lambda_i\lambda_j \\
 & + (3939840 + 103680\beta^2 + 933120\beta)\lambda_i^2, \\
 \xi_{34}^{(1)}(\lambda_j; \lambda_i; \beta) = & (9262080 + 5322240\beta + 207360\beta^2)\lambda_j \\
 & + (-3732480 + 3317760\beta + 311040\beta^2)\lambda_i, \\
 \xi_{34}^{(2)}(\lambda_j; \lambda_i; \beta) = & 9262080 + 5322240\beta + 207360\beta^2 \geq 0 \text{ for all } \beta \geq -1.8776.
 \end{aligned}$$

Now

$$\begin{aligned}
 \xi_{34}(\lambda_i; \lambda_i; \beta) = & 172800\lambda_i^2(28 + 40\beta + 3\beta^2) \geq 0 \text{ for all } \beta \geq -0.7412, \\
 \xi_{34}^{(1)}(\lambda_i; \lambda_i; \beta) = & 172800\lambda_i(\beta + 16)(3\beta + 2) \geq 0 \text{ for all } \beta \geq -\frac{2}{3}.
 \end{aligned}$$

Hence, $\xi_3^{(4)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{33}(\lambda_j; \lambda_i; \beta) = \xi_3^{(3)}(\lambda_l; \lambda_j; \lambda_i; \beta)$. We wish to show that ξ_{33} is non-negative for

all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{33} with respect to λ_j gives

$$\begin{aligned}\xi_{33}(\lambda_j; \lambda_i; \beta) &= (5654016 + 2094336\beta + 129600\beta^2 + 864\beta^3)\lambda_j^3 \\ &\quad + (-4824576 + 2381184\beta + 7776\beta^3 + 518400\beta^2)\lambda_i\lambda_j^2 \\ &\quad + (3193344 + 311040\beta^2 + 860544\beta + 7776\beta^3)\lambda_i^2\lambda_j \\ &\quad + (864\beta^3 + 1368576 + 25920\beta^2 + 297216\beta)\lambda_i^3, \\ \xi_{33}^{(1)}(\lambda_j; \lambda_i; \beta) &= (16962048 + 6283008\beta + 388800\beta^2 + 2592\beta^3)\lambda_j^2 \\ &\quad + (-9649152 + 4762368\beta + 15552\beta^3 + 1036800\beta^2)\lambda_i\lambda_j \\ &\quad + (3193344 + 311040\beta^2 + 860544\beta + 7776\beta^3)\lambda_i^2, \\ \xi_{33}^{(2)}(\lambda_j; \lambda_i; \beta) &= (33924096 + 12566016\beta + 777600\beta^2 + 5184\beta^3)\lambda_j \\ &\quad + (-9649152 + 4762368\beta + 15552\beta^3 + 1036800\beta^2)\lambda_i, \\ \xi_{33}^{(3)}(\lambda_j; \lambda_i; \beta) &= 33924096 + 12566016\beta + 777600\beta^2 + 5184\beta^3 \geq 0 \text{ for all } \beta \geq -3.40.\end{aligned}$$

Now

$$\begin{aligned}\xi_{33}(\lambda_i; \lambda_i; \beta) &= 17280\lambda_i^3(312 + 326\beta + 57\beta^2 + \beta^3) \geq 0 \text{ for all } \beta \geq -1.2060, \\ \xi_{33}^{(1)}(\lambda_i; \lambda_i; \beta) &= 8640\lambda_i^2(1216 + 1378\beta + 201\beta^2 + 3\beta^3) \geq 0 \text{ for all } \beta \geq -1.0368, \\ \xi_{33}^{(2)}(\lambda_i; \lambda_i; \beta) &= 3456\lambda_i(7024 + 5014\beta + 525\beta^2 + 6\beta^3) \geq 0 \text{ for all } \beta \geq -1.6963.\end{aligned}$$

Hence, $\xi_3^{(3)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{32}(\lambda_j; \lambda_i; \beta) = \xi_3^{(2)}(\lambda_j; \lambda_j; \lambda_i; \beta)$. We wish to show that ξ_{32} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{32} with respect to λ_j gives

$$\begin{aligned}\xi_{32}(\lambda_j; \lambda_i; \beta) &= (3530880 + 1096128\beta + 864\beta^3 + 79488\beta^2)\lambda_j^4 \\ &\quad + (-3262464 + 1154304\beta + 10368\beta^3 + 390528\beta^2)\lambda_i\lambda_j^3 \\ &\quad + (207360\beta + 1285632 + 321408\beta^2 + 15552\beta^3)\lambda_i^2\lambda_j^2 \\ &\quad + (3456\beta^3 + 949248 + 53568\beta^2 + 441216\beta)\lambda_i^3\lambda_j \\ &\quad + (44352\beta + 77184 + 1728\beta^2)\lambda_i^4, \\ \xi_{32}^{(1)}(\lambda_j; \lambda_i; \beta) &= (14123520 + 4384512\beta + 3456\beta^3 + 317952\beta^2)\lambda_j^3 \\ &\quad + (-9787392 + 3462912\beta + 31104\beta^3 + 1171584\beta^2)\lambda_i\lambda_j^2 \\ &\quad + (414720\beta + 2571264 + 642816\beta^2 + 31104\beta^3)\lambda_i^2\lambda_j \\ &\quad + (3456\beta^3 + 949248 + 53568\beta^2 + 441216\beta)\lambda_i^3, \\ \xi_{32}^{(2)}(\lambda_j; \lambda_i; \beta) &= (42370560 + 13153536\beta + 10368\beta^3 + 953856\beta^2)\lambda_j^2 \\ &\quad + (-19574784 + 6925824\beta + 62208\beta^3 + 2343168\beta^2)\lambda_i\lambda_j \\ &\quad + (414720\beta + 2571264 + 642816\beta^2 + 31104\beta^3)\lambda_i^2, \\ \xi_{32}^{(3)}(\lambda_j; \lambda_i; \beta) &= (84741120 + 26307072\beta + 20736\beta^3 + 1907712\beta^2)\lambda_j \\ &\quad + (-19574784 + 6925824\beta + 62208\beta^3 + 2343168\beta^2)\lambda_i, \\ \xi_{32}^{(4)}(\lambda_j; \lambda_i; \beta) &= 84741120 + 26307072\beta + 20736\beta^3 + 1907712\beta^2 \geq 0 \text{ for all } \beta \geq -4.8136.\end{aligned}$$

Now

$$\begin{aligned} \xi_{32}(\lambda_i; \lambda_i; \beta) &= 10080\lambda_i^4(256 + 292\beta + 3\beta^3 + 84\beta^2) \geq 0 \text{ for all } \beta \geq -1.4510, \\ \xi_{32}^{(1)}(\lambda_i; \lambda_i; \beta) &= 2880\lambda_i^3(2728 + 3022\beta + 24\beta^3 + 759\beta^2) \geq 0 \text{ for all } \beta \geq -1.3255, \\ \xi_{32}^{(2)}(\lambda_i; \lambda_i; \beta) &= 34560\lambda_i^2(734 + 593\beta + 3\beta^3 + 114\beta^2) \geq 0 \text{ for all } \beta \geq -1.8910, \\ \xi_{32}^{(3)}(\lambda_i; \lambda_i; \beta) &= 6912\lambda_i(9428 + 4808\beta + 12\beta^3 + 615\beta^2) \geq 0 \text{ for all } \beta \geq -3.1578. \end{aligned}$$

Hence, $\xi_3^{(2)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{31}(\lambda_j; \lambda_i; \beta) = \xi_3^{(1)}(\lambda_j; \lambda_j; \lambda_i; \beta)/\lambda_j$. We wish to show that ξ_{31} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{31} with respect to λ_j gives

$$\begin{aligned} \xi_{31}(\lambda_j; \lambda_i; \beta) &= (420480\beta + 1446912 + 31968\beta^2 + 432\beta^3)\lambda_j^4 \\ &\quad + (472320\beta - 1532160 + 185760\beta^2 + 6480\beta^3)\lambda_i\lambda_j^3 \\ &\quad + (-17280\beta + 414720 + 190080\beta^2 + 12960\beta^3)\lambda_i^2\lambda_j^2 \\ &\quad + (334080 + 47520\beta^2 + 4320\beta^3 + 342720\beta)\lambda_i^3\lambda_j \\ &\quad + (72000\beta + 46080 + 4320\beta^2)\lambda_i^4, \\ \xi_{31}^{(1)}(\lambda_j; \lambda_i; \beta) &= (1681920\beta + 5787648 + 127872\beta^2 + 1728\beta^3)\lambda_j^3 \\ &\quad + (1416960\beta - 4596480 + 557280\beta^2 + 19440\beta^3)\lambda_i\lambda_j^2 \\ &\quad + (-34560\beta + 829440 + 380160\beta^2 + 25920\beta^3)\lambda_i^2\lambda_j \\ &\quad + (334080 + 47520\beta^2 + 4320\beta^3 + 342720\beta)\lambda_i^3, \\ \xi_{31}^{(2)}(\lambda_j; \lambda_i; \beta) &= (5045760\beta + 17362944 + 383616\beta^2 + 5184\beta^3)\lambda_j^2 \\ &\quad + (2833920\beta - 9192960 + 1114560\beta^2 + 38880\beta^3)\lambda_i\lambda_j \\ &\quad + (-34560\beta + 829440 + 380160\beta^2 + 25920\beta^3)\lambda_i^2, \\ \xi_{31}^{(3)}(\lambda_j; \lambda_i; \beta) &= (10091520\beta + 34725888 + 767232\beta^2 + 10368\beta^3)\lambda_j \\ &\quad + (2833920\beta - 9192960 + 1114560\beta^2 + 38880\beta^3)\lambda_i, \\ \xi_{31}^{(4)}(\lambda_j; \lambda_i; \beta) &= 10091520\beta + 34725888 + 767232\beta^2 + 10368\beta^3 \geq 0 \text{ for all } \beta \geq -5.7919. \end{aligned}$$

Now

$$\begin{aligned} \xi_{31}(\lambda_i; \lambda_i; \beta) &= 8064\lambda_i^4(160\beta + 88 + 57\beta^2 + 3\beta^3) \geq 0 \text{ for all } \beta \geq -0.7350, \\ \xi_{31}^{(1)}(\lambda_i; \lambda_i; \beta) &= 1008\lambda_i^3(3380\beta + 2336 + 1104\beta^2 + 51\beta^3) \geq 0 \text{ for all } \beta \geq -1.0068, \\ \xi_{31}^{(2)}(\lambda_i; \lambda_i; \beta) &= 864\lambda_i^2(9080\beta + 10416 + 2174\beta^2 + 81\beta^3) \geq 0 \text{ for all } \beta \geq -22.0124, \\ \xi_{31}^{(3)}(\lambda_i; \lambda_i; \beta) &= 864\lambda_i(14960\beta + 29552 + 2178\beta^2 + 57\beta^3) \geq 0 \text{ for all } \beta \geq -30.0509. \end{aligned}$$

Hence, $\xi_3^{(1)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{30}(\lambda_j; \lambda_i; \beta) = \xi_3(\lambda_j; \lambda_j; \lambda_i; \beta)/\lambda_j^2$. We wish to show that ξ_{30} is non-negative for

all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{30} with respect to λ_j gives

$$\begin{aligned}\xi_{30}(\lambda_j; \lambda_i; \beta) &= (124992\beta + 436608 + 9504\beta^2 + 144\beta^3)\lambda_j^4 \\ &\quad + (63936\beta^2 - 543744 + 168192\beta + 2592\beta^3)\lambda_i\lambda_j^3 \\ &\quad + (77760\beta^2 - 8640\beta + 172800 + 6480\beta^3)\lambda_i^2\lambda_j^2 \\ &\quad + (25920\beta^2 + 184320 + 2880\beta^3 + 190080\beta)\lambda_i^3\lambda_j \\ &\quad + (40320 + 4320\beta^2 + 57600\beta)\lambda_i^4, \\ \xi_{30}^{(1)}(\lambda_j; \lambda_i; \beta) &= (499968\beta + 1746432 + 38016\beta^2 + 576\beta^3)\lambda_j^3 \\ &\quad + (191808\beta^2 - 1631232 + 504576\beta + 7776\beta^3)\lambda_i\lambda_j^2 \\ &\quad + (155520\beta^2 - 17280\beta + 345600 + 12960\beta^3)\lambda_i^2\lambda_j \\ &\quad + (25920\beta^2 + 184320 + 2880\beta^3 + 190080\beta)\lambda_i^3, \\ \xi_{30}^{(2)}(\lambda_j; \lambda_i; \beta) &= (1499904\beta + 5239296 + 114048\beta^2 + 1728\beta^3)\lambda_j^2 \\ &\quad + (383616\beta^2 - 3262464 + 1009152\beta + 15552\beta^3)\lambda_i\lambda_j \\ &\quad + (155520\beta^2 - 17280\beta + 345600 + 12960\beta^3)\lambda_i^2, \\ \xi_{30}^{(3)}(\lambda_j; \lambda_i; \beta) &= (2999808\beta + 10478592 + 228096\beta^2 + 3456\beta^3)\lambda_j \\ &\quad + (383616\beta^2 - 3262464 + 1009152\beta + 15552\beta^3)\lambda_i, \\ \xi_{30}^{(4)}(\lambda_j; \lambda_i; \beta) &= 2999808\beta + 10478592 + 228096\beta^2 + 3456\beta^3 \geq 0 \text{ for all } \beta \geq -5.9149.\end{aligned}$$

Now

$$\begin{aligned}\xi_{30}(\lambda_i; \lambda_i; \beta) &= 12096\lambda_i^4(\beta + 3)(\beta^2 + 12\beta + 8) > 0 \text{ for all } \beta > -0.7085, \\ \xi_{30}^{(1)}(\lambda_i; \lambda_i; \beta) &= 8064\lambda_i^3(146\beta + 80 + 51\beta^2 + 3\beta^3) > 0 \text{ for all } \beta > -0.7226, \\ \xi_{30}^{(2)}(\lambda_i; \lambda_i; \beta) &= 6048\lambda_i^2(412\beta + 384 + 108\beta^2 + 5\beta^3) > 0 \text{ for all } \beta > -1.4380, \\ \xi_{30}^{(3)}(\lambda_i; \lambda_i; \beta) &= 1728\lambda_i(2320\beta + 4176 + 354\beta^2 + 11\beta^3) > 0 \text{ for all } \beta > -3.2433.\end{aligned}$$

Hence, ξ_3 is positive for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$. This implies that $\xi(\lambda_l; \lambda_l; \lambda_j; \lambda_i; \beta)$ is positive for all $\lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Now

$$\begin{aligned}\xi_4(\lambda_l; \lambda_j; \lambda_i; \beta) &= (131328 + 31104\beta + 3456\beta^2)(\lambda_j^2 + \lambda_i^2)\lambda_l^3 \\ &\quad + (52992\beta - 147456 + 10368\beta^2)\lambda_i\lambda_j\lambda_l^3 \\ &\quad + (14400\beta + 5760)(\lambda_j + \lambda_i)\lambda_l^4 \\ &\quad + (52992\beta - 147456 + 10368\beta^2)(\lambda_i\lambda_j^3 + \lambda_i^3\lambda_j)\lambda_l \\ &\quad + (14400\beta + 5760)(\lambda_j^4 + \lambda_i^4)\lambda_l \\ &\quad + (-62208 + 31104\beta^2 - 31104\beta)\lambda_i^2\lambda_j^2\lambda_l \\ &\quad + (-62208 + 31104\beta^2 - 31104\beta)(\lambda_i^2\lambda_j + \lambda_i\lambda_j^2)\lambda_l^2 \\ &\quad + (131328 + 31104\beta + 3456\beta^2)(\lambda_i^3 + \lambda_j^3)\lambda_l^2 \\ &\quad + (14400\beta + 5760)(\lambda_i^4\lambda_j + \lambda_i\lambda_j^4) \\ &\quad + (131328 + 31104\beta + 3456\beta^2)(\lambda_i^2\lambda_j^3 + \lambda_i^3\lambda_j^2),\end{aligned}$$

and, hence,

$$\xi_4^{(4)}(\lambda_i; \lambda_j; \lambda_i; \beta) = (345600\beta + 138240)(\lambda_j + \lambda_i) \geq 0 \text{ for all } \beta \geq -\frac{2}{5}.$$

Define $\xi_{43}(\lambda_j; \lambda_i; \beta) = \xi_4^{(3)}(\lambda_j; \lambda_j; \lambda_i; \beta)$. We wish to show that ξ_{43} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{43} with respect to λ_j gives

$$\begin{aligned} \xi_{43}(\lambda_j; \lambda_i; \beta) &= (532224\beta + 926208 + 20736\beta^2)\lambda_j^2 \\ &\quad + (663552\beta - 746496 + 62208\beta^2)\lambda_i\lambda_j \\ &\quad + (186624\beta + 20736\beta^2 + 787968)\lambda_i^2, \\ \xi_{43}^{(1)}(\lambda_j; \lambda_i; \beta) &= (1064448\beta + 1852416 + 41472\beta^2)\lambda_j \\ &\quad + (663552\beta - 746496 + 62208\beta^2)\lambda_i, \\ \xi_{43}^{(2)}(\lambda_j; \lambda_i; \beta) &= 1064448\beta + 1852416 + 41472\beta^2 \geq 0 \text{ for all } \beta \geq -1.8776. \end{aligned}$$

Now

$$\begin{aligned} \xi_{43}(\lambda_i; \lambda_i; \beta) &= 34560\lambda_i^2(28 + 40\beta + 3\beta^2) \geq 0 \text{ for all } \beta \geq -0.7412, \\ \xi_{43}^{(1)}(\lambda_i; \lambda_i; \beta) &= 34560\lambda_i(\beta + 16)(3\beta + 2) \geq 0 \text{ for all } \beta \geq -\frac{2}{3}. \end{aligned}$$

Hence, $\xi_4^{(3)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{42}(\lambda_j; \lambda_i; \beta) = \xi_4^{(2)}(\lambda_j; \lambda_j; \lambda_i; \beta)$. We wish to show that ξ_{42} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{42} with respect to λ_j gives

$$\begin{aligned} \xi_{42}(\lambda_j; \lambda_i; \beta) &= (421632\beta + 1119744 + 27648\beta^2)\lambda_j^3 \\ &\quad + (428544\beta - 940032 + 124416\beta^2)\lambda_i\lambda_j^2 \\ &\quad + (124416\beta + 82944\beta^2 + 663552)\lambda_i^2\lambda_j \\ &\quad + (6912\beta^2 + 262656 + 62208\beta)\lambda_i^3, \\ \xi_{42}^{(1)}(\lambda_j; \lambda_i; \beta) &= (1264896\beta + 3359232 + 82944\beta^2)\lambda_j^2 \\ &\quad + (857088\beta - 1880064 + 248832\beta^2)\lambda_i\lambda_j \\ &\quad + (124416\beta + 82944\beta^2 + 663552)\lambda_i^2, \\ \xi_{42}^{(2)}(\lambda_j; \lambda_i; \beta) &= (2529792\beta + 6718464 + 165888\beta^2)\lambda_j \\ &\quad + (857088\beta - 1880064 + 248832\beta^2)\lambda_i, \\ \xi_{42}^{(3)}(\lambda_j; \lambda_i; \beta) &= 2529792\beta + 6718464 + 165888\beta^2 \geq 0 \text{ for all } \beta \geq -3.4249. \end{aligned}$$

Now

$$\begin{aligned} \xi_{42}(\lambda_i; \lambda_i; \beta) &= 34560\lambda_i^3(7\beta + 16)(\beta + 2) \geq 0 \text{ for all } \beta \geq -2, \\ \xi_{42}^{(1)}(\lambda_i; \lambda_i; \beta) &= 34560\lambda_i^2(65\beta + 62 + 12\beta^2) \geq 0 \text{ for all } \beta \geq -1.2358, \\ \xi_{42}^{(2)}(\lambda_i; \lambda_i; \beta) &= 69120\lambda_i(49\beta + 70 + 6\beta^2) \geq 0 \text{ for all } \beta \geq -1.8457. \end{aligned}$$

Hence, $\xi_4^{(2)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{41}(\lambda_j; \lambda_i; \beta) = \xi_4^{(1)}(\lambda_j; \lambda_j; \lambda_i; \beta)$. We wish to show that ξ_{41} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{41} with respect to λ_j gives

$$\begin{aligned} \xi_{41}(\lambda_j; \lambda_i; \beta) &= (685440 + 227520\beta + 17280\beta^2)\lambda_j^4 \\ &\quad + (103680\beta^2 + 207360\beta - 691200)\lambda_i\lambda_j^3 \\ &\quad + (207360 + 103680\beta^2)\lambda_i^2\lambda_j^2 \\ &\quad + (17280\beta^2 + 115200 + 115200\beta)\lambda_i^3\lambda_j \\ &\quad + (14400\beta + 5760)\lambda_i^4, \\ \xi_{41}^{(1)}(\lambda_j; \lambda_i; \beta) &= (2741760 + 910080\beta + 69120\beta^2)\lambda_j^3 \\ &\quad + (311040\beta^2 + 622080\beta - 2073600)\lambda_i\lambda_j^2 \\ &\quad + (414720 + 207360\beta^2)\lambda_i^2\lambda_j + (17280\beta^2 + 115200 + 115200\beta)\lambda_i^3, \\ \xi_{41}^{(2)}(\lambda_j; \lambda_i; \beta) &= (8225280 + 2730240\beta + 207360\beta^2)\lambda_j^2 \\ &\quad + (622080\beta^2 + 1244160\beta - 4147200)\lambda_i\lambda_j \\ &\quad + (414720 + 207360\beta^2)\lambda_i^2, \\ \xi_{41}^{(3)}(\lambda_j; \lambda_i; \beta) &= (16450560 + 5460480\beta + 414720\beta^2)\lambda_j \\ &\quad + (622080\beta^2 + 1244160\beta - 4147200)\lambda_i, \\ \xi_{41}^{(4)}(\lambda_j; \lambda_i; \beta) &= 69120(2\beta + 17)(3\beta + 14) \geq 0 \text{ for all } \beta \geq -\frac{14}{3}. \end{aligned}$$

Now

$$\begin{aligned} \xi_{41}(\lambda_i; \lambda_i; \beta) &= 80640\lambda_i^4(3\beta + 4)(\beta + 1) \geq 0 \text{ for all } \beta \geq -1, \\ \xi_{41}^{(1)}(\lambda_i; \lambda_i; \beta) &= 5760\lambda_i^3(105\beta^2 + 208 + 286\beta) \geq 0 \text{ for all } \beta, \\ \xi_{41}^{(2)}(\lambda_i; \lambda_i; \beta) &= 172800\lambda_i^2(26 + 23\beta + 6\beta^2) \geq 0 \text{ for all } \beta, \\ \xi_{41}^{(3)}(\lambda_i; \lambda_i; \beta) &= 69120\lambda_i(178 + 97\beta + 15\beta^2) \geq 0 \text{ for all } \beta. \end{aligned}$$

Hence, $\xi_4^{(1)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{40}(\lambda_j; \lambda_i; \beta) = \xi_4(\lambda_j; \lambda_j; \lambda_i; \beta)/\lambda_j$. We wish to show that ξ_{40} is non-negative for

all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{40} with respect to λ_j gives

$$\begin{aligned} \xi_{40}(\lambda_j; \lambda_i; \beta) &= (274176 + 91008\beta + 6912\beta^2)\lambda_j^4 \\ &\quad + (103680\beta - 345600 + 51840\beta^2)\lambda_i\lambda_j^3 \\ &\quad + (138240 + 69120\beta^2)\lambda_i^2\lambda_j^2 \\ &\quad + (115200 + 115200\beta + 17280\beta^2)\lambda_i^3\lambda_j \\ &\quad + (28800\beta + 11520)\lambda_i^4, \\ \xi_{40}^{(1)}(\lambda_j; \lambda_i; \beta) &= (1096704 + 364032\beta + 27648\beta^2)\lambda_j^3 \\ &\quad + (311040\beta - 1036800 + 155520\beta^2)\lambda_i\lambda_j^2 \\ &\quad + (276480 + 138240\beta^2)\lambda_i^2\lambda_j \\ &\quad + (115200 + 115200\beta + 17280\beta^2)\lambda_i^3, \\ \xi_{40}^{(2)}(\lambda_j; \lambda_i; \beta) &= (3290112 + 1092096\beta + 82944\beta^2)\lambda_j^2 \\ &\quad + (622080\beta - 2073600 + 311040\beta^2)\lambda_i\lambda_j \\ &\quad + (276480 + 138240\beta^2)\lambda_i^2, \\ \xi_{40}^{(3)}(\lambda_j; \lambda_i; \beta) &= (6580224 + 2184192\beta + 165888\beta^2)\lambda_j \\ &\quad + (622080\beta - 2073600 + 311040\beta^2)\lambda_i, \\ \xi_{40}^{(4)}(\lambda_j; \lambda_i; \beta) &= 27648(2\beta + 17)(3\beta + 14) \geq 0 \text{ for all } \beta \geq -\frac{14}{3}. \end{aligned}$$

Now

$$\begin{aligned} \xi_{40}(\lambda_i; \lambda_i; \beta) &= 48384\lambda_i^4(3\beta + 4)(\beta + 1) > 0 \text{ for all } \beta > -1, \\ \xi_{40}^{(1)}(\lambda_i; \lambda_i; \beta) &= 112896\lambda_i^3(3\beta + 4)(\beta + 1) > 0 \text{ for all } \beta > -1, \\ \xi_{40}^{(2)}(\lambda_i; \lambda_i; \beta) &= 6912\lambda_i^2(216 + 248\beta + 77\beta^2) > 0 \text{ for all } \beta, \\ \xi_{40}^{(3)}(\lambda_i; \lambda_i; \beta) &= 6912\lambda_i(652 + 406\beta + 69\beta^2) > 0 \text{ for all } \beta. \end{aligned}$$

Hence, ξ_4 is positive for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$. This implies that $\xi(\lambda_l; \lambda_l; \lambda_j; \lambda_i; \beta)$ is positive for all $\lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

□

Lemma B.15 Let

$$\begin{aligned} \xi(\lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta) = & (1200\beta + 480)t(0, 0, 4, 4) + (7200\beta + 720\beta^2 + 5760)t(0, 1, 3, 4) \\ & + (30240 + 6480\beta + 2160\beta^2)t(0, 2, 2, 4) \\ & + (-28800 + 6480\beta^2 + 1440\beta)t(1, 1, 2, 4) \\ & + (2160\beta^2 + 288\beta^3 + 81792 + 22752\beta)t(0, 2, 3, 3) \\ & + (30336\beta + 864\beta^3 + 5760\beta^2 - 90624)t(1, 1, 3, 3) \\ & + (2592\beta^3 + 8640\beta^2 - 41472 - 6912\beta)t(1, 2, 2, 3) \\ & + (7776\beta^3 + 186624 - 124416\beta)t(2, 2, 2, 2), \end{aligned}$$

where

$$t(a, b, c, d) = \sum_{\substack{(n_1, n_2, n_3, n_4) \\ \in \text{perm}(a, b, c, d)}} \lambda_i^{n_1} \lambda_j^{n_2} \lambda_l^{n_3} \lambda_k^{n_4}.$$

The expression $\xi(\lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta)$ is positive for all $\lambda_k \geq \lambda_l \geq \lambda_j \geq \lambda_i$, and $\beta \geq -\frac{2}{5}$.

Proof.

Let

$$\hat{s}(a, b, c) = \sum_{\substack{(n_1, n_2, n_3) \\ \in \text{perm}(a, b, c)}} \lambda_i^{n_1} \lambda_j^{n_2} \lambda_l^{n_3}.$$

Differentiating $\xi(\lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta)$ with respect to λ_k we obtain

$$\begin{aligned}
\xi^{(1)}(\lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta) = & 4(1200\beta + 480)\lambda_k^3\hat{s}(0, 0, 4) \\
& +4(7200\beta + 720\beta^2 + 5760)\lambda_k^3\hat{s}(0, 1, 3) \\
& +4(-28800 + 6480\beta^2 + 1440\beta)\lambda_k^3\hat{s}(1, 1, 2) \\
& +4(30240 + 6480\beta + 2160\beta^2)\lambda_k^3\hat{s}(0, 2, 2) \\
& +3(2160\beta^2 + 288\beta^3 + 81792 + 22752\beta)\lambda_k^2\hat{s}(0, 2, 3) \\
& +3(30336\beta + 864\beta^3 + 5760\beta^2 - 90624)\lambda_k^2\hat{s}(1, 1, 3) \\
& +3(7200\beta + 720\beta^2 + 5760)\lambda_k^2\hat{s}(0, 1, 4) \\
& +3(2592\beta^3 + 8640\beta^2 - 41472 - 6912\beta)\lambda_k^2\hat{s}(1, 2, 2) \\
& +2(2160\beta^2 + 288\beta^3 + 81792 + 22752\beta)\lambda_k\hat{s}(0, 3, 3) \\
& +2(7776\beta^3 + 186624 - 124416\beta)\lambda_k\hat{s}(2, 2, 2) \\
& +2(-28800 + 6480\beta^2 + 1440\beta)\lambda_k\hat{s}(1, 1, 4) \\
& +2(2592\beta^3 + 8640\beta^2 - 41472 - 6912\beta)\lambda_k\hat{s}(1, 2, 3) \\
& +2(30240 + 6480\beta + 2160\beta^2)\lambda_k\hat{s}(0, 2, 4) \\
& +(7200\beta + 720\beta^2 + 5760)\hat{s}(0, 3, 4) \\
& +(2592\beta^3 + 8640\beta^2 - 41472 - 6912\beta)\hat{s}(2, 2, 3) \\
& +(30336\beta + 864\beta^3 + 5760\beta^2 - 90624)\hat{s}(1, 3, 3) \\
& +(-28800 + 6480\beta^2 + 1440\beta)\hat{s}(1, 2, 4),
\end{aligned}$$

$$\begin{aligned}
\xi^{(2)}(\lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta) = & 12(1200\beta + 480)\lambda_k^2\hat{s}(0, 0, 4) \\
& +12(7200\beta + 720\beta^2 + 5760)\lambda_k^2\hat{s}(0, 1, 3) \\
& +12(-28800 + 6480\beta^2 + 1440\beta)\lambda_k^2\hat{s}(1, 1, 2) \\
& +12(30240 + 6480\beta + 2160\beta^2)\lambda_k^2\hat{s}(0, 2, 2) \\
& +6(2160\beta^2 + 288\beta^3 + 81792 + 22752\beta)\lambda_k\hat{s}(0, 2, 3) \\
& +6(30336\beta + 864\beta^3 + 5760\beta^2 - 90624)\lambda_k\hat{s}(1, 1, 3) \\
& +6(7200\beta + 720\beta^2 + 5760)\lambda_k\hat{s}(0, 1, 4) \\
& +6(2592\beta^3 + 8640\beta^2 - 41472 - 6912\beta)\lambda_k\hat{s}(1, 2, 2) \\
& +2(2160\beta^2 + 288\beta^3 + 81792 + 22752\beta)\hat{s}(0, 3, 3) \\
& +2(7776\beta^3 + 186624 - 124416\beta)\hat{s}(2, 2, 2) \\
& +2(-28800 + 6480\beta^2 + 1440\beta)\hat{s}(1, 1, 4) \\
& +2(2592\beta^3 + 8640\beta^2 - 41472 - 6912\beta)\hat{s}(1, 2, 3) \\
& +2(30240 + 6480\beta + 2160\beta^2)\hat{s}(0, 2, 4),
\end{aligned}$$

$$\begin{aligned}
\xi^{(3)}(\lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta) = & 24(1200\beta + 480)\lambda_k\hat{s}(0, 0, 4) \\
& +24(7200\beta + 720\beta^2 + 5760)\lambda_k\hat{s}(0, 1, 3) \\
& +24(-28800 + 6480\beta^2 + 1440\beta)\lambda_k\hat{s}(1, 1, 2) \\
& +24(30240 + 6480\beta + 2160\beta^2)\lambda_k\hat{s}(0, 2, 2) \\
& +6(2160\beta^2 + 288\beta^3 + 81792 + 22752\beta)\hat{s}(0, 2, 3) \\
& +6(30336\beta + 864\beta^3 + 5760\beta^2 - 90624)\hat{s}(1, 1, 3) \\
& +6(7200\beta + 720\beta^2 + 5760)\hat{s}(0, 1, 4) \\
& +6(2592\beta^3 + 8640\beta^2 - 41472 - 6912\beta)\hat{s}(1, 2, 2),
\end{aligned}$$

$$\begin{aligned}
\xi^{(4)}(\lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta) = & 24(1200\beta + 480)\hat{s}(0, 0, 4) \\
& +24(7200\beta + 720\beta^2 + 5760)\hat{s}(0, 1, 3) \\
& +24(-28800 + 6480\beta^2 + 1440\beta)\hat{s}(1, 1, 2) \\
& +24(30240 + 6480\beta + 2160\beta^2)\hat{s}(0, 2, 2).
\end{aligned}$$

Define

$$\begin{aligned}
\xi_0(\lambda_l; \lambda_j; \lambda_i; \beta) & \stackrel{\text{def}}{=} \xi(\lambda_l; \lambda_l; \lambda_j; \lambda_i; \beta), \\
\xi_1(\lambda_l; \lambda_j; \lambda_i; \beta) & \stackrel{\text{def}}{=} \xi^{(1)}(\lambda_l; \lambda_l; \lambda_j; \lambda_i; \beta), \\
\xi_2(\lambda_l; \lambda_j; \lambda_i; \beta) & \stackrel{\text{def}}{=} \xi^{(2)}(\lambda_l; \lambda_l; \lambda_j; \lambda_i; \beta), \\
\xi_3(\lambda_l; \lambda_j; \lambda_i; \beta) & \stackrel{\text{def}}{=} \xi^{(3)}(\lambda_l; \lambda_l; \lambda_j; \lambda_i; \beta), \\
\xi_4(\lambda_l; \lambda_j; \lambda_i; \beta) & \stackrel{\text{def}}{=} \xi^{(4)}(\lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta).
\end{aligned}$$

We will prove that the above equations are non-negative for all $\lambda_k \geq \lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$ and, hence, deduce that $\xi(\lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta)$ is positive for all $\lambda_k \geq \lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Now

$$\begin{aligned}
 \xi_0(\lambda_i; \lambda_j; \lambda_i; \beta) = & (1200\beta + 480)\lambda_i^8 \\
 & + (14400\beta + 11520 + 1440\beta^2)(\lambda_j + \lambda_i)\lambda_i^7 \\
 & + (142272 + 6480\beta^2 + 288\beta^3 + 35712\beta)(\lambda_j^2 + \lambda_i^2)\lambda_i^6 \\
 & + (-148224 + 33216\beta + 18720\beta^2 + 864\beta^3)\lambda_i\lambda_j\lambda_i^6 \\
 & + (-140544 + 5184\beta^3 - 10944\beta + 30240\beta^2)(\lambda_i\lambda_j^2 + \lambda_i^2\lambda_j)\lambda_i^5 \\
 & + (175104 + 576\beta^3 + 59904\beta + 5760\beta^2)(\lambda_j^3 + \lambda_i^3)\lambda_i^5 \\
 & + (-211200 + 21600\beta^2 + 68160\beta + 4320\beta^3)(\lambda_i\lambda_j^3 + \lambda_i^3\lambda_j)\lambda_i^4 \\
 & + (23280\beta + 42720 + 3600\beta^2)(\lambda_j^4 + \lambda_i^4)\lambda_i^4 \\
 & + (-125280\beta + 12960\beta^3 + 21600\beta^2 + 164160)\lambda_i^2\lambda_j^2\lambda_i^4 \\
 & + (5760\beta^3 + 31680\beta + 80640 + 21600\beta^2)(\lambda_i^2\lambda_j^3 + \lambda_i^3\lambda_j^2)\lambda_i^3 \\
 & + (-46080 + 14400\beta^2 + 17280\beta)(\lambda_i\lambda_j^4 + \lambda_i^4\lambda_j)\lambda_i^3 \\
 & + (31680 + 14400\beta + 10800\beta^2)(\lambda_i^2\lambda_j^4 + \lambda_i^4\lambda_j^2)\lambda_i^2 \\
 & + (10080\beta^2 + 72960 + 75840\beta + 1440\beta^3)\lambda_i^3\lambda_j^3\lambda_i^2 \\
 & + (14400\beta + 11520 + 1440\beta^2)(\lambda_i^3\lambda_j^4 + \lambda_i^4\lambda_j^3)\lambda_i \\
 & + (1200\beta + 480)\lambda_i^4\lambda_j^4,
 \end{aligned}$$

and, hence,

$$\begin{aligned}
 \xi_0^{(7)}(\lambda_i; \lambda_j; \lambda_i; \beta) = & (48384000\beta + 19353600)\lambda_i \\
 & + (72576000\beta + 58060800 + 7257600\beta^2)(\lambda_i + \lambda_j) \geq 0 \text{ for all } \beta \geq -\frac{2}{5}.
 \end{aligned}$$

Define $\xi_{06}(\lambda_j; \lambda_i; \beta) = \xi_0^{(6)}(\lambda_j; \lambda_j; \lambda_i; \beta)$. We wish to show that ξ_{06} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{06} with respect to λ_j gives

$$\begin{aligned}
 \xi_{06}(\lambda_j; \lambda_i; \beta) = & (122480640\beta + 170173440 + 11923200\beta^2 + 207360\beta^3)\lambda_j^2 \\
 & + (96491520\beta - 48660480 + 20736000\beta^2 + 622080\beta^3)\lambda_i\lambda_j \\
 & + (102435840 + 4665600\beta^2 + 207360\beta^3 + 25712640\beta)\lambda_i^2, \\
 \xi_{06}^{(1)}(\lambda_j; \lambda_i; \beta) = & (244961280\beta + 340346880 + 23846400\beta^2 + 414720\beta^3)\lambda_j \\
 & + (96491520\beta - 48660480 + 20736000\beta^2 + 622080\beta^3)\lambda_i, \\
 \xi_{06}^{(2)}(\lambda_j; \lambda_i; \beta) = & 244961280\beta + 340346880 + 23846400\beta^2 + 414720\beta^3 \\
 \geq & 0 \text{ for all } \beta \geq -1.6454.
 \end{aligned}$$

Now

$$\begin{aligned}
 \xi_{06}(\lambda_i; \lambda_i; \beta) = & 1036800\lambda_i^2(236\beta + 216 + 36\beta^2\beta^3) \geq 0 \text{ for all } \beta \geq -1.0915, \\
 \xi_{06}^{(1)}(\lambda_i; \lambda_i; \beta) = & 345600\lambda_i(988\beta + 844 + 129\beta^2 + 3\beta^3) \geq 0 \text{ for all } \beta \geq -0.9757, \\
 \xi_{06}^{(2)}(\lambda_i; \lambda_i; \beta) = & \geq 0 \text{ for all } \beta \geq -1.6454.
 \end{aligned}$$

Hence, $\xi_0^{(6)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{05}(\lambda_j; \lambda_i; \beta) = \xi_0^{(5)}(\lambda_j; \lambda_j; \lambda_i; \beta)$. We wish to show that ξ_{05} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{05} with respect to λ_j gives

$$\begin{aligned} \xi_{05}(\lambda_j; \lambda_i; \beta) &= (77253120\beta + 155704320 + 8985600\beta^2 + 276480\beta^3)\lambda_j^3 \\ &\quad + (-94556160 + 58890240\beta + 20736000\beta^2 + 1244160\beta^3)\lambda_i\lambda_j^2 \\ &\quad + (85570560 + 8294400\beta^2 + 829440\beta^3 + 24399360\beta)\lambda_i^2\lambda_j \\ &\quad + (21012480 + 69120\beta^3 + 7188480\beta + 691200\beta^2)\lambda_i^3, \\ \xi_{05}^{(1)}(\lambda_j; \lambda_i; \beta) &= (231759360\beta + 467112960 + 26956800\beta^2 + 829440\beta^3)\lambda_j^2 \\ &\quad + (-189112320 + 117780480\beta + 41472000\beta^2 + 2488320\beta^3)\lambda_i\lambda_j \\ &\quad + (85570560 + 8294400\beta^2 + 829440\beta^3 + 24399360\beta)\lambda_i^2, \\ \xi_{05}^{(2)}(\lambda_j; \lambda_i; \beta) &= (463518720\beta + 934225920 + 53913600\beta^2 + 1658880\beta^3)\lambda_j \\ &\quad + (-189112320 + 117780480\beta + 41472000\beta^2 + 2488320\beta^3)\lambda_i, \\ \xi_{05}^{(3)}(\lambda_j; \lambda_i; \beta) &= 463518720\beta + 934225920 + 53913600\beta^2 + 1658880\beta^3 \\ &\geq 0 \text{ for all } \beta \geq -2.9155. \end{aligned}$$

Now

$$\begin{aligned} \xi_{05}(\lambda_i; \lambda_i; \beta) &= 806400\lambda_i^3(208\beta + 208 + 48\beta^2 + 3\beta^3) \geq 0 \text{ for all } \beta \geq -1.4294, \\ \xi_{05}^{(1)}(\lambda_i; \lambda_i; \beta) &= 691200\lambda_i^2(541\beta + 526 + 111\beta^2 + 6\beta^3) \geq 0 \text{ for all } \beta \geq -1.2898, \\ \xi_{05}^{(2)}(\lambda_i; \lambda_i; \beta) &= 691200\lambda_i(841\beta + 1078 + 138\beta^2 + 6\beta^3) \geq 0 \text{ for all } \beta \geq -1.7421. \end{aligned}$$

Hence, $\xi_0^{(5)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{04}(\lambda_j; \lambda_i; \beta) = \xi_0^{(4)}(\lambda_j; \lambda_j; \lambda_i; \beta)$. We wish to show that ξ_{04} is non-negative for

all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{04} with respect to λ_j gives

$$\begin{aligned} \xi_{04}(\lambda_j; \lambda_i; \beta) &= (34715520\beta + 83738880 + 4320000\beta^2 + 172800\beta^3)\lambda_j^4 \\ &\quad + (-65617920 + 24376320\beta + 12096000\beta^2 + 1036800\beta^3)\lambda_i\lambda_j^3 \\ &\quad + (38292480 + 6480000\beta^2 + 1036800\beta^3 + 8536320\beta)\lambda_i^2\lambda_j^2 \\ &\quad + (15943680 + 1209600\beta^2 + 8824320\beta + 172800\beta^3)\lambda_i^3\lambda_j \\ &\quad + (558720\beta + 1025280 + 86400\beta^2)\lambda_i^4, \\ \xi_{04}^{(1)}(\lambda_j; \lambda_i; \beta) &= (138862080\beta + 334955520 + 17280000\beta^2 + 691200\beta^3)\lambda_j^3 \\ &\quad + (-196853760 + 73128960\beta + 36288000\beta^2 + 3110400\beta^3)\lambda_i\lambda_j^2 \\ &\quad + (76584960 + 12960000\beta^2 + 2073600\beta^3 + 17072640\beta)\lambda_i^2\lambda_j \\ &\quad + (15943680 + 1209600\beta^2 + 8824320\beta + 172800\beta^3)\lambda_i^3, \\ \xi_{04}^{(2)}(\lambda_j; \lambda_i; \beta) &= (416586240\beta + 1004866560 + 51840000\beta^2 + 2073600\beta^3)\lambda_j^2 \\ &\quad + (-393707520 + 146257920\beta + 72576000\beta^2 + 6220800\beta^3)\lambda_i\lambda_j \\ &\quad + (76584960 + 12960000\beta^2 + 2073600\beta^3 + 17072640\beta)\lambda_i^2, \\ \xi_{04}^{(3)}(\lambda_j; \lambda_i; \beta) &= (833172480\beta + 2009733120 + 103680000\beta^2 + 4147200\beta^3)\lambda_j \\ &\quad + (-393707520 + 146257920\beta + 72576000\beta^2 + 6220800\beta^3)\lambda_i, \\ \xi_{04}^{(4)}(\lambda_j; \lambda_i; \beta) &= 833172480\beta + 2009733120 + 103680000\beta^2 + 4147200\beta^3 \\ &\geq 0 \text{ for all } \beta \geq -4.3861. \end{aligned}$$

Now

$$\begin{aligned} \xi_{04}(\lambda_i; \lambda_i; \beta) &= 403200\lambda_i^4(191\beta + 182 + 60\beta^2 + 6\beta^3) \geq 0 \text{ for all } \beta \geq -1.7321, \\ \xi_{04}^{(1)}(\lambda_i; \lambda_i; \beta) &= 403200\lambda_i^3(590\beta + 572 + 168\beta^2 + 15\beta^3) \geq 0 \text{ for all } \beta \geq -1.5802, \\ \xi_{04}^{(2)}(\lambda_i; \lambda_i; \beta) &= 172800\lambda_i^2(3356\beta + 3980 + 795\beta^2 + 60\beta^3) \geq 0 \text{ for all } \beta \geq -1.9649, \\ \xi_{04}^{(3)}(\lambda_i; \lambda_i; \beta) &= 691200\lambda_i(1417\beta + 2338 + 255\beta^2 + 15\beta^3) \geq 0 \text{ for all } \beta \geq -2.9237. \end{aligned}$$

Hence, $\xi_0^{(4)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{03}(\lambda_j; \lambda_i; \beta) = \xi_0^{(3)}(\lambda_j; \lambda_j; \lambda_i; \beta)/\lambda_j$. We wish to show that ξ_{03} is non-negative

for all $\lambda_j \geq \lambda_i \geq 0$ for all $\beta \geq .$ Differentiating ξ_{03} with respect to λ_j gives

$$\begin{aligned}
\xi_{03}(\lambda_j; \lambda_i; \beta) &= (11865600\beta + 31184640 + 1512000\beta^2 + 69120\beta^3)\lambda_j^4 \\
&\quad + (-29145600 + 8092800\beta + 4968000\beta^2 + 518400\beta^3)\lambda_i\lambda_j^3 \\
&\quad + (13063680 + 3240000\beta^2 + 691200\beta^3 + 812160\beta)\lambda_i^2\lambda_j^2 \\
&\quad + (5921280 + 172800\beta^3 + 5420160\beta + 993600\beta^2)\lambda_i^3\lambda_j \\
&\quad + (662400\beta + 748800 + 172800\beta^2)\lambda_i^4, \\
\xi_{03}^{(1)}(\lambda_j; \lambda_i; \beta) &= (47462400\beta + 124738560 + 6048000\beta^2 + 276480\beta^3)\lambda_j^3 \\
&\quad + (-87436800 + 24278400\beta + 14904000\beta^2 + 1555200\beta^3)\lambda_i\lambda_j^2 \\
&\quad + (26127360 + 6480000\beta^2 + 1382400\beta^3 + 1624320\beta)\lambda_i^2\lambda_j \\
&\quad + (5921280 + 172800\beta^3 + 5420160\beta + 993600\beta^2)\lambda_i^3, \\
\xi_{03}^{(2)}(\lambda_j; \lambda_i; \beta) &= (142387200\beta + 374215680 + 18144000\beta^2 + 829440\beta^3)\lambda_j^2 \\
&\quad + (-174873600 + 48556800\beta + 29808000\beta^2 + 3110400\beta^3)\lambda_i\lambda_j \\
&\quad + (26127360 + 6480000\beta^2 + 1382400\beta^3 + 1624320\beta)\lambda_i^2, \\
\xi_{03}^{(3)}(\lambda_j; \lambda_i; \beta) &= (284774400\beta + 748431360 + 36288000\beta^2 + 1658880\beta^3)\lambda_j \\
&\quad + (-174873600 + 48556800\beta + 29808000\beta^2 + 3110400\beta^3)\lambda_i, \\
\xi_{03}^{(4)}(\lambda_j; \lambda_i; \beta) &= 284774400\beta + 748431360 + 36288000\beta^2 + 1658880\beta^3 \\
&\geq 0 \text{ for all } \beta \geq -5.6105.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{03}(\lambda_i; \lambda_i; \beta) &= 725760\lambda_i^4(2\beta + 5)(\beta + 3)(\beta + 2) \geq 0 \text{ for all } \beta \geq -2, \\
\xi_{03}^{(1)}(\lambda_i; \lambda_i; \beta) &= 40320\lambda_i^3(1954\beta + 1720 + 705\beta^2 + 84\beta^3) \geq 0 \text{ for all } \beta \geq -1.7877, \\
\xi_{03}^{(2)}(\lambda_i; \lambda_i; \beta) &= 241920\lambda_i^2(796\beta + 932 + 225\beta^2 + 22\beta^3) \geq 0 \text{ for all } \beta \geq -2.5561, \\
\xi_{03}^{(3)}(\lambda_i; \lambda_i; \beta) &= 51840\lambda_i(6430\beta + 11064 + 1275\beta^2 + 92\beta^3) \geq 0 \text{ for all } \beta \geq -3.7901.
\end{aligned}$$

Hence, $\xi_0^{(3)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{02}(\lambda_j; \lambda_i; \beta) = \xi_0^{(2)}(\lambda_j; \lambda_j; \lambda_i; \beta)/\lambda_j^2$. We wish to show that ξ_{02} is non-negative

for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{02} with respect to λ_j gives

$$\begin{aligned} \xi_{02}(\lambda_j; \lambda_i; \beta) &= (3220800\beta + 8793600 + 413280\beta^2 + 20160\beta^3)\lambda_j^4 \\ &\quad + (-9584640 + 2304000\beta + 1572480\beta^2 + 181440\beta^3)\lambda_i\lambda_j^3 \\ &\quad + (3974400 + 1209600\beta^2 + 302400\beta^3 - 432000\beta)\lambda_i^2\lambda_j^2 \\ &\quad + (1597440 + 100800\beta^3 + 2357760\beta + 524160\beta^2)\lambda_i^3\lambda_j \\ &\quad + (411840\beta + 299520 + 151200\beta^2)\lambda_i^4, \\ \xi_{02}^{(1)}(\lambda_j; \lambda_i; \beta) &= (12883200\beta + 35174400 + 1653120\beta^2 + 80640\beta^3)\lambda_j^3 \\ &\quad + (-28753920 + 6912000\beta + 4717440\beta^2 + 544320\beta^3)\lambda_i\lambda_j^2 \\ &\quad + (7948800 + 2419200\beta^2 + 604800\beta^3 - 864000\beta)\lambda_i^2\lambda_j \\ &\quad + (1597440 + 100800\beta^3 + 2357760\beta + 524160\beta^2)\lambda_i^3, \\ \xi_{02}^{(2)}(\lambda_j; \lambda_i; \beta) &= (38649600\beta + 105523200 + 4959360\beta^2 + 241920\beta^3)\lambda_j^2 \\ &\quad + (-57507840 + 13824000\beta + 9434880\beta^2 + 1088640\beta^3)\lambda_i\lambda_j \\ &\quad + (7948800 + 2419200\beta^2 + 604800\beta^3 - 864000\beta)\lambda_i^2, \\ \xi_{02}^{(3)}(\lambda_j; \lambda_i; \beta) &= (77299200\beta + 211046400 + 9918720\beta^2 + 483840\beta^3)\lambda_j \\ &\quad + (-57507840 + 13824000\beta + 9434880\beta^2 + 1088640\beta^3)\lambda_i, \\ \xi_{02}^{(4)}(\lambda_j; \lambda_i; \beta) &= 77299200\beta + 211046400 + 9918720\beta^2 + 483840\beta^3 \\ &\geq 0 \text{ for all } \beta \geq -5.9825. \end{aligned}$$

Now

$$\begin{aligned} \xi_{02}(\lambda_i; \lambda_i; \beta) &= 120960\lambda_i^4(5\beta + 7)(\beta + 3)(\beta + 2) \geq 0 \text{ for all } \beta \geq -2, \\ \xi_{02}^{(1)}(\lambda_i; \lambda_i; \beta) &= 1330560\lambda_i^3(\beta + 3)(\beta + 2)^2 \geq 0 \text{ for all } \beta \geq -2, \\ \xi_{02}^{(2)}(\lambda_i; \lambda_i; \beta) &= 40320\lambda_i^2(1280\beta + 1388 + 417\beta^2 + 48\beta^3) \geq 0 \text{ for all } \beta \geq -3.0665, \\ \xi_{02}^{(3)}(\lambda_i; \lambda_i; \beta) &= 40320\lambda_i(2260\beta + 3808 + 480\beta^2 + 39\beta^3) \geq 0 \text{ for all } \beta \geq -3.8373. \end{aligned}$$

Hence, $\xi_0^{(2)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{01}(\lambda_j; \lambda_i; \beta) = \xi_0^{(1)}(\lambda_j; \lambda_j; \lambda_i; \beta)/\lambda_j^3$. We wish to show that ξ_{01} is non-negative

for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{01} with respect to λ_j gives

$$\begin{aligned}
\xi_{01}(\lambda_j; \lambda_i; \beta) &= (717312\beta + 1984512 + 92160\beta^2 + 4608\beta^3)\lambda_j^4 \\
&\quad + (-2494464 + 569856\beta + 403200\beta^2 + 48384\beta^3)\lambda_i\lambda_j^3 \\
&\quad + (1112832 + 362880\beta^2 + 96768\beta^3 - 217728\beta)\lambda_i^2\lambda_j^2 \\
&\quad + (430080 + 40320\beta^3 + 833280\beta + 201600\beta^2)\lambda_i^3\lambda_j \\
&\quad + (188160\beta + 107520 + 80640\beta^2)\lambda_i^4, \\
\xi_{01}^{(1)}(\lambda_j; \lambda_i; \beta) &= (2869248\beta + 7938048 + 368640\beta^2 + 18432\beta^3)\lambda_j^3 \\
&\quad + (-7483392 + 1709568\beta + 1209600\beta^2 + 145152\beta^3)\lambda_i\lambda_j^2 \\
&\quad + (2225664 + 725760\beta^2 + 193536\beta^3 - 435456\beta)\lambda_i^2\lambda_j \\
&\quad + (430080 + 40320\beta^3 + 833280\beta + 201600\beta^2)\lambda_i^3, \\
\xi_{01}^{(2)}(\lambda_j; \lambda_i; \beta) &= (8607744\beta + 23814144 + 1105920\beta^2 + 55296\beta^3)\lambda_j^2 \\
&\quad + (-14966784 + 3419136\beta + 2419200\beta^2 + 290304\beta^3)\lambda_i\lambda_j \\
&\quad + (2225664 + 725760\beta^2 + 193536\beta^3 - 435456\beta)\lambda_i^2, \\
\xi_{01}^{(3)}(\lambda_j; \lambda_i; \beta) &= (17215488\beta + 47628288 + 2211840\beta^2 + 110592\beta^3)\lambda_j \\
&\quad + (-14966784 + 3419136\beta + 2419200\beta^2 + 290304\beta^3)\lambda_i, \\
\xi_{01}^{(4)}(\lambda_j; \lambda_i; \beta) &= 17215488\beta + 47628288 + 2211840\beta^2 + 110592\beta^3 \\
&\geq 0 \text{ for all } \beta \geq -6.0282.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{01}(\lambda_i; \lambda_i; \beta) &= 190080\lambda_i^4(\beta + 3)(\beta + 2)(\beta + 1) \geq 0 \text{ for all } \beta \geq -1, \\
\xi_{01}^{(1)}(\lambda_i; \lambda_i; \beta) &= 17280\lambda_i^3(23\beta + 30)(\beta + 3)(\beta + 2) \geq 0 \text{ for all } \beta \geq -\frac{30}{23}, \\
\xi_{01}^{(2)}(\lambda_i; \lambda_i; \beta) &= 20736\lambda_i^2(\beta + 3)(26\beta^2 + 127\beta + 178) \geq 0 \text{ for all } \beta \geq -3, \\
\xi_{01}^{(3)}(\lambda_i; \lambda_i; \beta) &= 4608\lambda_i(4478\beta + 7088 + 1005\beta^2 + 87\beta^3) \\
&\geq 0 \text{ for all } \beta \geq -3.4940.
\end{aligned}$$

Hence, $\xi_0^{(1)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{00}(\lambda_j; \lambda_i; \beta) = \xi_0(\lambda_j; \lambda_j; \lambda_i; \beta)/\lambda_j^4$. We wish to show that ξ_{00} is non-negative for

all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{00} with respect to λ_j gives

$$\begin{aligned} \xi_{00}(\lambda_j; \lambda_i; \beta) &= (134496\beta + 372096 + 17280\beta^2 + 864\beta^3)\lambda_j^4 \\ &\quad + (-534528 + 122112\beta + 86400\beta^2 + 10368\beta^3)\lambda_i\lambda_j^3 \\ &\quad + (278208 + 90720\beta^2 + 24192\beta^3 - 54432\beta)\lambda_i^2\lambda_j^2 \\ &\quad + (129024 + 12096\beta^3 + 249984\beta + 60480\beta^2)\lambda_i^3\lambda_j \\ &\quad + (70560\beta + 40320 + 30240\beta^2)\lambda_i^4, \\ \xi_{00}^{(1)}(\lambda_j; \lambda_i; \beta) &= (537984\beta + 1488384 + 69120\beta^2 + 3456\beta^3)\lambda_j^3 \\ &\quad + (-1603584 + 366336\beta + 259200\beta^2 + 31104\beta^3)\lambda_i\lambda_j^2 \\ &\quad + (556416 + 181440\beta^2 + 48384\beta^3 - 108864\beta)\lambda_i^2\lambda_j \\ &\quad + (129024 + 12096\beta^3 + 249984\beta + 60480\beta^2)\lambda_i^3, \\ \xi_{00}^{(2)}(\lambda_j; \lambda_i; \beta) &= (1613952\beta + 4465152 + 207360\beta^2 + 10368\beta^3)\lambda_j^2 \\ &\quad + (-3207168 + 732672\beta + 518400\beta^2 + 62208\beta^3)\lambda_i\lambda_j \\ &\quad + (556416 + 181440\beta^2 + 48384\beta^3 - 108864\beta)\lambda_i^2, \\ \xi_{00}^{(3)}(\lambda_j; \lambda_i; \beta) &= (3227904\beta + 8930304 + 414720\beta^2 + 20736\beta^3)\lambda_j \\ &\quad + (-3207168 + 732672\beta + 518400\beta^2 + 62208\beta^3)\lambda_i, \\ \xi_{00}^{(4)}(\lambda_j; \lambda_i; \beta) &= 3227904\beta + 8930304 + 414720\beta^2 + 20736\beta^3 \\ &\geq 0 \text{ for all } \beta \geq -6.0282. \end{aligned}$$

Now

$$\begin{aligned} \xi_{00}(\lambda_i; \lambda_i; \beta) &= 47520\lambda_i^4(\beta + 3)(\beta + 2)(\beta + 1) > 0 \text{ for all } \beta > -1, \\ \xi_{00}^{(1)}(\lambda_i; \lambda_i; \beta) &= 95040\lambda_i^3(\beta + 3)(\beta + 2)(\beta + 1) > 0 \text{ for all } \beta > -1, \\ \xi_{00}^{(2)}(\lambda_i; \lambda_i; \beta) &= 60480\lambda_i^2(2\beta + 5)(\beta + 3)(\beta + 2) > 0 \text{ for all } \beta > -2, \\ \xi_{00}^{(3)}(\lambda_i; \lambda_i; \beta) &= 20736\lambda_i(\beta + 3)(4\beta^2 + 33\beta + 92) > 0 \text{ for all } \beta > -3. \end{aligned}$$

Hence, ξ_0 is positive for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$. This implies that $\xi(\lambda_l; \lambda_l; \lambda_j; \lambda_i; \beta)$ is positive for all $\lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Now

$$\begin{aligned}
\xi_1(\lambda_i; \lambda_j; \lambda_i; \beta) &= (4800\beta + 1920)\lambda_i^7 \\
&+ (50400\beta + 40320 + 5040\beta^2)(\lambda_i + \lambda_j)\lambda_i^6 \\
&+ (19440\beta^2 + 107136\beta + 864\beta^3 + 426816)(\lambda_j^2 + \lambda_i^2)\lambda_i^5 \\
&+ (56160\beta^2 - 444672 + 99648\beta + 2592\beta^3)\lambda_i\lambda_j\lambda_i^5 \\
&+ (-27360\beta - 351360 + 12960\beta^3 + 75600\beta^2)(\lambda_i\lambda_j^2 + \lambda_i^2\lambda_j)\lambda_i^4 \\
&+ (437760 + 14400\beta^2 + 1440\beta^3 + 149760\beta)(\lambda_j^3 + \lambda_i^3)\lambda_i^4 \\
&+ (8640\beta^3 + 43200\beta^2 - 422400 + 136320\beta)(\lambda_i\lambda_j^3 + \lambda_i^3\lambda_j)\lambda_i^3 \\
&+ (7200\beta^2 + 46560\beta + 85440)(\lambda_j^4 + \lambda_i^4)\lambda_i^3 \\
&+ (-250560\beta + 25920\beta^3 + 328320 + 43200\beta^2)\lambda_i^2\lambda_j^2\lambda_i^3 \\
&+ (120960 + 32400\beta^2 + 47520\beta + 8640\beta^3)(\lambda_i^2\lambda_j^3 + \lambda_i^3\lambda_j^2)\lambda_i^2 \\
&+ (25920\beta + 21600\beta^2 - 69120)(\lambda_i\lambda_j^4 + \lambda_i^4\lambda_j)\lambda_i^2 \\
&+ (31680 + 10800\beta^2 + 14400\beta)(\lambda_i^2\lambda_j^4 + \lambda_i^4\lambda_j^2)\lambda_i \\
&+ (72960 + 75840\beta + 1440\beta^3 + 10080\beta^2)\lambda_i^3\lambda_j^3\lambda_i \\
&+ (7200\beta + 720\beta^2 + 5760)(\lambda_i^3\lambda_j^4 + \lambda_i^4\lambda_j^3),
\end{aligned}$$

and, hence,

$$\begin{aligned}
\xi_1^{(6)}(\lambda_i; \lambda_j; \lambda_i; \beta) &= (24192000\beta + 9676800)\lambda_i + (36288000\beta + 29030400 + 3628800\beta^2)(\lambda_i + \lambda_j) \\
&\geq 0 \text{ for all } \beta \geq -\frac{2}{5}.
\end{aligned}$$

Define $\xi_{15}(\lambda_j; \lambda_i; \beta) = \xi_1^{(5)}(\lambda_i; \lambda_j; \lambda_i; \beta)$. We wish to show that ξ_{15} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{15} with respect to λ_j gives

$$\begin{aligned}
\xi_{15}(\lambda_j; \lambda_i; \beta) &= (61240320\beta + 85086720 + 5961600\beta^2 + 103680\beta^3)\lambda_j^2 \\
&+ (48245760\beta - 24330240 + 10368000\beta^2 + 311040\beta^3)\lambda_i\lambda_j \\
&+ (2332800\beta^2 + 12856320\beta + 103680\beta^3 + 51217920)\lambda_i^2, \\
\xi_{15}^{(1)}(\lambda_j; \lambda_i; \beta) &= (122480640\beta + 170173440 + 11923200\beta^2 + 207360\beta^3)\lambda_j \\
&+ (48245760\beta - 24330240 + 10368000\beta^2 + 311040\beta^3)\lambda_i, \\
\xi_{15}^{(2)}(\lambda_j; \lambda_i; \beta) &= 122480640\beta + 170173440 + 11923200\beta^2 + 207360\beta^3 \geq 0 \text{ for all } \beta \geq -1.6454.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{15}(\lambda_i; \lambda_i; \beta) &= 518400\lambda_i^2(236\beta + 216 + 36\beta^2\beta^3) \geq 0 \text{ for all } \beta \geq -1.0915, \\
\xi_{15}^{(1)}(\lambda_i; \lambda_i; \beta) &= 172800\lambda_i(988\beta + 844 + 129\beta^2 + 3\beta^3) \geq 0 \text{ for all } \beta \geq -0.9757.
\end{aligned}$$

Hence, $\xi_1^{(5)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{14}(\lambda_j; \lambda_i; \beta) = \xi_1^{(4)}(\lambda_i; \lambda_j; \lambda_i; \beta)$. We wish to show that ξ_{14} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{14} with respect to λ_j gives

$$\begin{aligned} \xi_{14}(\lambda_j; \lambda_i; \beta) &= (38626560\beta + 77852160 + 4492800\beta^2 + 138240\beta^3)\lambda_j^3 \\ &\quad + (10368000\beta^2 - 47278080 + 29445120\beta + 622080\beta^3)\lambda_i\lambda_j^2 \\ &\quad + (4147200\beta^2 + 12199680\beta + 414720\beta^3 + 42785280)\lambda_i^2\lambda_j \\ &\quad + (10506240 + 345600\beta^2 + 34560\beta^3 + 3594240\beta)\lambda_i^3, \\ \xi_{14}^{(1)}(\lambda_j; \lambda_i; \beta) &= (115879680\beta + 233556480 + 13478400\beta^2 + 414720\beta^3)\lambda_j^2 \\ &\quad + (20736000\beta^2 - 94556160 + 58890240\beta + 1244160\beta^3)\lambda_i\lambda_j \\ &\quad + (4147200\beta^2 + 12199680\beta + 414720\beta^3 + 42785280)\lambda_i^2, \\ \xi_{14}^{(2)}(\lambda_j; \lambda_i; \beta) &= (231759360\beta + 467112960 + 26956800\beta^2 + 829440\beta^3)\lambda_j \\ &\quad + (20736000\beta^2 - 94556160 + 58890240\beta + 1244160\beta^3)\lambda_i, \\ \xi_{14}^{(3)}(\lambda_j; \lambda_i; \beta) &= 231759360\beta + 467112960 + 26956800\beta^2 + 829440\beta^3 \\ &\geq 0 \text{ for all } \beta \geq -2.9155. \end{aligned}$$

Now

$$\begin{aligned} \xi_{14}(\lambda_i; \lambda_i; \beta) &= 403200\lambda_i^3(208\beta + 208 + 48\beta^2 + 3\beta^3) \geq 0 \text{ for all } \beta \geq -1.4294, \\ \xi_{14}^{(1)}(\lambda_i; \lambda_i; \beta) &= 345600\lambda_i^2(541\beta + 526 + 111\beta^2 + 6\beta^3) \geq 0 \text{ for all } \beta \geq -1.2898, \\ \xi_{14}^{(2)}(\lambda_i; \lambda_i; \beta) &= 345600\lambda_i(841\beta + 1078 + 138\beta^2 + 6\beta^3) \geq 0 \text{ for all } \beta \geq -1.7421. \end{aligned}$$

Hence, $\xi_1^{(4)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{13}(\lambda_j; \lambda_i; \beta) = \xi_1^{(3)}(\lambda_j; \lambda_j; \lambda_i; \beta)$. We wish to show that ξ_{13} is non-negative for

all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{13} with respect to λ_j gives

$$\begin{aligned}
\xi_{13}(\lambda_j; \lambda_i; \beta) &= (17357760\beta + 41869440 + 2160000\beta^2 + 86400\beta^3)\lambda_j^4 \\
&\quad + (6048000\beta^2 - 32808960 + 12188160\beta + 518400\beta^3)\lambda_i\lambda_j^3 \\
&\quad + (3240000\beta^2 + 4268160\beta + 518400\beta^3 + 19146240)\lambda_i^2\lambda_j^2 \\
&\quad + (86400\beta^3 + 604800\beta^2 + 7971840 + 4412160\beta)\lambda_i^3\lambda_j \\
&\quad + (43200\beta^2 + 279360\beta + 512640)\lambda_i^4, \\
\xi_{13}^{(1)}(\lambda_j; \lambda_i; \beta) &= (69431040\beta + 167477760 + 8640000\beta^2 + 345600\beta^3)\lambda_j^3 \\
&\quad + (18144000\beta^2 - 98426880 + 36564480\beta + 1555200\beta^3)\lambda_i\lambda_j^2 \\
&\quad + (6480000\beta^2 + 8536320\beta + 1036800\beta^3 + 38292480)\lambda_i^2\lambda_j \\
&\quad + (86400\beta^3 + 604800\beta^2 + 7971840 + 4412160\beta)\lambda_i^3, \\
\xi_{13}^{(2)}(\lambda_j; \lambda_i; \beta) &= (208293120\beta + 502433280 + 25920000\beta^2 + 1036800\beta^3)\lambda_j^2 \\
&\quad + (36288000\beta^2 - 196853760 + 73128960\beta + 3110400\beta^3)\lambda_i\lambda_j \\
&\quad + (6480000\beta^2 + 8536320\beta + 1036800\beta^3 + 38292480)\lambda_i^2, \\
\xi_{13}^{(3)}(\lambda_j; \lambda_i; \beta) &= (416586240\beta + 1004866560 + 51840000\beta^2 + 2073600\beta^3)\lambda_j \\
&\quad + (36288000\beta^2 - 196853760 + 73128960\beta + 3110400\beta^3)\lambda_i, \\
\xi_{13}^{(4)}(\lambda_j; \lambda_i; \beta) &= 416586240\beta + 1004866560 + 51840000\beta^2 + 2073600\beta^3 \\
&\geq 0 \text{ for all } \beta \geq -4.3861.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{13}(\lambda_i; \lambda_i; \beta) &= 201600\lambda_i^4(191\beta + 182 + 60\beta^2 + 6\beta^3) \geq 0 \text{ for all } \beta \geq -1.7321, \\
\xi_{13}^{(1)}(\lambda_i; \lambda_i; \beta) &= 201600\lambda_i^3(590\beta + 572 + 168\beta^2 + 15\beta^3) \geq 0 \text{ for all } \beta \geq -1.5802, \\
\xi_{13}^{(2)}(\lambda_i; \lambda_i; \beta) &= 86400\lambda_i^2(3356\beta + 3980 + 795\beta^2 + 60\beta^3) \geq 0 \text{ for all } \beta \geq -1.9649, \\
\xi_{13}^{(3)}(\lambda_i; \lambda_i; \beta) &= 345600\lambda_i(1417\beta + 2338 + 255\beta^2 + 15\beta^3) \geq 0 \text{ for all } \beta \geq -2.9237.
\end{aligned}$$

Hence, $\xi_1^{(3)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{12}(\lambda_j; \lambda_i; \beta) = \xi_1^{(2)}(\lambda_j; \lambda_j; \lambda_i; \beta)/\lambda_j$. We wish to show that ξ_{12} is non-negative

for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{12} with respect to λ_j gives

$$\begin{aligned} \xi_{12}(\lambda_j; \lambda_i; \beta) &= (5932800\beta + 15592320 + 756000\beta^2 + 34560\beta^3)\lambda_j^4 \\ &\quad + (2484000\beta^2 - 14572800 + 4046400\beta + 259200\beta^3)\lambda_i\lambda_j^3 \\ &\quad + (1620000\beta^2 + 406080\beta + 345600\beta^3 + 6531840)\lambda_i^2\lambda_j^2 \\ &\quad + (2960640 + 496800\beta^2 + 86400\beta^3 + 2710080\beta)\lambda_i^3\lambda_j \\ &\quad + (86400\beta^2 + 331200\beta + 374400)\lambda_i^4, \\ \xi_{12}^{(1)}(\lambda_j; \lambda_i; \beta) &= (23731200\beta + 62369280 + 3024000\beta^2 + 138240\beta^3)\lambda_j^3 \\ &\quad + (7452000\beta^2 - 43718400 + 12139200\beta + 777600\beta^3)\lambda_i\lambda_j^2 \\ &\quad + (3240000\beta^2 + 812160\beta + 691200\beta^3 + 13063680)\lambda_i^2\lambda_j \\ &\quad + (2960640 + 496800\beta^2 + 86400\beta^3 + 2710080\beta)\lambda_i^3, \\ \xi_{12}^{(2)}(\lambda_j; \lambda_i; \beta) &= (71193600\beta + 187107840 + 9072000\beta^2 + 414720\beta^3)\lambda_j^2 \\ &\quad + (14904000\beta^2 - 87436800 + 24278400\beta + 1555200\beta^3)\lambda_i\lambda_j \\ &\quad + (3240000\beta^2 + 812160\beta + 691200\beta^3 + 13063680)\lambda_i^2, \\ \xi_{12}^{(3)}(\lambda_j; \lambda_i; \beta) &= (142387200\beta + 374215680 + 18144000\beta^2 + 829440\beta^3)\lambda_j \\ &\quad + (14904000\beta^2 - 87436800 + 24278400\beta + 1555200\beta^3)\lambda_i, \\ \xi_{12}^{(4)}(\lambda_j; \lambda_i; \beta) &= 142387200\beta + 374215680 + 18144000\beta^2 + 829440\beta^3 \\ &\geq 0 \text{ for all } \beta \geq -5.6105. \end{aligned}$$

Now

$$\begin{aligned} \xi_{12}(\lambda_i; \lambda_i; \beta) &= 362880\lambda_i^4(2\beta + 5)(\beta + 3)(\beta + 2) \geq 0 \text{ for all } \beta \geq -2, \\ \xi_{12}^{(1)}(\lambda_i; \lambda_i; \beta) &= 20160\lambda_i^3(1954\beta + 1720 + 705\beta^2 + 84\beta^3) \geq 0 \text{ for all } \beta \geq -1.7877, \\ \xi_{12}^{(2)}(\lambda_i; \lambda_i; \beta) &= 120960\lambda_i^2(796\beta + 932 + 225\beta^2 + 22\beta^3) \geq 0 \text{ for all } \beta \geq -2.5561, \\ \xi_{12}^{(3)}(\lambda_i; \lambda_i; \beta) &= 25920\lambda_i(6430\beta + 11064 + 1275\beta^2 + 92\beta^3) \geq 0 \text{ for all } \beta \geq -3.7901. \end{aligned}$$

Hence, $\xi_1^{(2)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{11}(\lambda_j; \lambda_i; \beta) = \xi_1^{(1)}(\lambda_j; \lambda_j; \lambda_i; \beta)/\lambda_j^2$. We wish to show that ξ_{11} is non-negative

for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{11} with respect to λ_j gives

$$\begin{aligned}
\xi_{11}(\lambda_j; \lambda_i; \beta) &= (1610400\beta + 4396800 + 206640\beta^2 + 10080\beta^3)\lambda_j^4 \\
&\quad + (786240\beta^2 - 4792320 + 1152000\beta + 90720\beta^3)\lambda_i\lambda_j^3 \\
&\quad + (604800\beta^2 - 216000\beta + 151200\beta^3 + 1987200)\lambda_i^2\lambda_j^2 \\
&\quad + (798720 + 262080\beta^2 + 50400\beta^3 + 1178880\beta)\lambda_i^3\lambda_j \\
&\quad + (75600\beta^2 + 205920\beta + 149760)\lambda_i^4, \\
\xi_{11}^{(1)}(\lambda_j; \lambda_i; \beta) &= (6441600\beta + 17587200 + 826560\beta^2 + 40320\beta^3)\lambda_j^3 \\
&\quad + (2358720\beta^2 - 14376960 + 3456000\beta + 272160\beta^3)\lambda_i\lambda_j^2 \\
&\quad + (1209600\beta^2 - 432000\beta + 302400\beta^3 + 3974400)\lambda_i^2\lambda_j \\
&\quad + (798720 + 262080\beta^2 + 50400\beta^3 + 1178880\beta)\lambda_i^3, \\
\xi_{11}^{(2)}(\lambda_j; \lambda_i; \beta) &= (19324800\beta + 52761600 + 2479680\beta^2 + 120960\beta^3)\lambda_j^2 \\
&\quad + (4717440\beta^2 - 28753920 + 6912000\beta + 544320\beta^3)\lambda_i\lambda_j \\
&\quad + (1209600\beta^2 - 432000\beta + 302400\beta^3 + 3974400)\lambda_i^2, \\
\xi_{11}^{(3)}(\lambda_j; \lambda_i; \beta) &= (38649600\beta + 105523200 + 4959360\beta^2 + 241920\beta^3)\lambda_j \\
&\quad + (4717440\beta^2 - 28753920 + 6912000\beta + 544320\beta^3)\lambda_i, \\
\xi_{11}^{(4)}(\lambda_j; \lambda_i; \beta) &= 38649600\beta + 105523200 + 4959360\beta^2 + 241920\beta^3 \\
&\geq 0 \text{ for all } \beta \geq -5.9825.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{11}(\lambda_i; \lambda_i; \beta) &= 60480\lambda_i^4(5\beta + 7)(\beta + 3)(\beta + 2) \geq 0 \text{ for all } \beta \geq -\frac{7}{4}, \\
\xi_{11}^{(1)}(\lambda_i; \lambda_i; \beta) &= 665280\lambda_i^3(\beta + 3)(\beta + 2)^2 \geq 0 \text{ for all } \beta \geq -2, \\
\xi_{11}^{(2)}(\lambda_i; \lambda_i; \beta) &= 20160\lambda_i^2(1280\beta + 1388 + 417\beta^2 + 48\beta^3) \geq 0 \text{ for all } \beta \geq -3.0665, \\
\xi_{11}^{(3)}(\lambda_i; \lambda_i; \beta) &= 20160\lambda_i(2260\beta + 3808 + 480\beta^2 + 39\beta^3) \geq 0 \text{ for all } \beta \geq -3.8373.
\end{aligned}$$

Hence, $\xi_{11}^{(1)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{10}(\lambda_j; \lambda_i; \beta) = \xi_{11}(\lambda_j; \lambda_j; \lambda_i; \beta)/\lambda_j^3$. We wish to show that ξ_{10} is non-negative for

all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{10} with respect to λ_j gives

$$\begin{aligned} \xi_{10}(\lambda_j; \lambda_i; \beta) &= (358656\beta + 992256 + 46080\beta^2 + 2304\beta^3)\lambda_j^4 \\ &\quad + (201600\beta^2 - 1247232 + 284928\beta + 24192\beta^3)\lambda_i\lambda_j^3 \\ &\quad + (181440\beta^2 - 108864\beta + 48384\beta^3 + 556416)\lambda_i^2\lambda_j^2 \\ &\quad + (215040 + 100800\beta^2 + 20160\beta^3 + 416640\beta)\lambda_i^3\lambda_j \\ &\quad + (40320\beta^2 + 94080\beta + 53760)\lambda_i^4, \\ \xi_{10}^{(1)}(\lambda_j; \lambda_i; \beta) &= (1434624\beta + 3969024 + 184320\beta^2 + 9216\beta^3)\lambda_j^3 \\ &\quad + (604800\beta^2 - 3741696 + 854784\beta + 72576\beta^3)\lambda_i\lambda_j^2 \\ &\quad + (362880\beta^2 - 217728\beta + 96768\beta^3 + 1112832)\lambda_i^2\lambda_j \\ &\quad + (215040 + 100800\beta^2 + 20160\beta^3 + 416640\beta)\lambda_i^3, \\ \xi_{10}^{(2)}(\lambda_j; \lambda_i; \beta) &= (4303872\beta + 11907072 + 552960\beta^2 + 27648\beta^3)\lambda_j^2 \\ &\quad + (1209600\beta^2 - 7483392 + 1709568\beta + 145152\beta^3)\lambda_i\lambda_j \\ &\quad + (362880\beta^2 - 217728\beta + 96768\beta^3 + 1112832)\lambda_i^2, \\ \xi_{10}^{(3)}(\lambda_j; \lambda_i; \beta) &= (8607744\beta + 23814144 + 1105920\beta^2 + 55296\beta^3)\lambda_j \\ &\quad + (1209600\beta^2 - 7483392 + 1709568\beta + 145152\beta^3)\lambda_i, \\ \xi_{10}^{(4)}(\lambda_j; \lambda_i; \beta) &= 8607744\beta + 23814144 + 1105920\beta^2 + 55296\beta^3 \\ &\geq 0 \text{ for all } \beta \geq -6.0282. \end{aligned}$$

Now

$$\begin{aligned} \xi_{10}(\lambda_i; \lambda_i; \beta) &= 95040\lambda_i^4(\beta + 3)(\beta + 2)(\beta + 1) > 0 \text{ for all } \beta > -1, \\ \xi_{10}^{(1)}(\lambda_i; \lambda_i; \beta) &= 8640\lambda_i^3(23\beta + 30)(\beta + 3)(\beta + 2) > 0 \text{ for all } \beta > -\frac{30}{23}, \\ \xi_{10}^{(2)}(\lambda_i; \lambda_i; \beta) &= 10368\lambda_i^2(\beta + 3)(26\beta^2 + 127\beta + 178) > 0 \text{ for all } \beta > -3, \\ \xi_{10}^{(3)}(\lambda_i; \lambda_i; \beta) &= 2304\lambda_i(4478\beta + 7088 + 1005\beta^2 + 87\beta^3) > 0 \text{ for all } \beta > -3.4940. \end{aligned}$$

Hence, ξ_1 is positive for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$. This implies that $\xi(\lambda_l; \lambda_l; \lambda_j; \lambda_i; \beta)$ is positive for all $\lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Now

$$\begin{aligned}
\xi_2(\lambda_l; \lambda_j; \lambda_i; \beta) = & (14400\beta + 5760)\lambda_l^6 \\
& +(129600\beta + 12960\beta^2 + 103680)(\lambda_j + \lambda_i)\lambda_l^5 \\
& +(43200\beta^2 + 914112 + 227232\beta + 1728\beta^3)(\lambda_j^2 + \lambda_i^2)\lambda_l^4 \\
& +(125280\beta^2 - 946944 + 202176\beta + 5184\beta^3)\lambda_i\lambda_j\lambda_l^4 \\
& +(-677376 + 146880\beta^2 - 38016\beta + 20736\beta^3)(\lambda_i\lambda_j^2 + \lambda_i^2\lambda_j)\lambda_l^3 \\
& +(268416\beta + 723456 + 2304\beta^3 + 25920\beta^2)(\lambda_j^3 + \lambda_i^3)\lambda_l^3 \\
& +(10368\beta^3 + 254592\beta + 60480\beta^2 - 557568)(\lambda_i\lambda_j^3 + \lambda_i^3\lambda_j)\lambda_l^2 \\
& +(8640\beta^2 + 70560\beta + 100800)(\lambda_j^4 + \lambda_i^4)\lambda_l^2 \\
& +(487296 - 212544\beta + 31104\beta^3 + 77760\beta^2)\lambda_i^2\lambda_j^2\lambda_l^2 \\
& +(122688\beta + 6912\beta^3 + 407808 + 30240\beta^2)(\lambda_i^2\lambda_j^3 + \lambda_i^3\lambda_j^2)\lambda_l \\
& +(17280\beta^2 - 23040 + 46080\beta)(\lambda_i\lambda_j^4 + \lambda_i^4\lambda_j)\lambda_l \\
& +(4320\beta^2 + 576\beta^3 + 163584 + 45504\beta)\lambda_i^3\lambda_j^3 \\
& +(60480 + 12960\beta + 4320\beta^2)(\lambda_i^2\lambda_j^4 + \lambda_i^4\lambda_j^2),
\end{aligned}$$

and, hence,

$$\begin{aligned}
\xi_2^{(5)}(\lambda_l; \lambda_j; \lambda_i; \beta) = & (10368000\beta + 4147200)\lambda_l \\
& +(15552000\beta + 1555200\beta^2 + 12441600)(\lambda_i + \lambda_j) \geq 0 \text{ for all } \beta \geq -\frac{2}{5}.
\end{aligned}$$

Define $\xi_{24}(\lambda_j; \lambda_i; \beta) = \xi_2^{(4)}(\lambda_l; \lambda_j; \lambda_i; \beta)$. We wish to show that ξ_{24} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{24} with respect to λ_j gives

$$\begin{aligned}
\xi_{24}(\lambda_j; \lambda_i; \beta) = & (26189568\beta + 36453888 + 2592000\beta^2 + 41472\beta^3)\lambda_j^2 \\
& +(20404224\beta + 4561920\beta^2 - 10285056 + 124416\beta^3)\lambda_i\lambda_j \\
& +(1036800\beta^2 + 21938688 + 5453568\beta + 41472\beta^3)\lambda_i^2, \\
\xi_{24}^{(1)}(\lambda_j; \lambda_i; \beta) = & (52379136\beta + 72907776 + 5184000\beta^2 + 82944\beta^3)\lambda_j \\
& +(20404224\beta + 4561920\beta^2 - 10285056 + 124416\beta^3)\lambda_i, \\
\xi_{24}^{(2)}(\lambda_j; \lambda_i; \beta) = & 52379136\beta + 72907776 + 5184000\beta^2 + 82944\beta^3 \geq 0 \text{ for all } \beta \geq -1.6562.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{24}(\lambda_i; \lambda_i; \beta) = & 103680\lambda_i^2(502\beta + 464 + 79\beta^2 + 2\beta^3) \geq 0 \text{ for all } \beta \geq -1.1141, \\
\xi_{24}^{(1)}(\lambda_i; \lambda_i; \beta) = & 207360\lambda_i(351\beta + 302 + 47\beta^2\beta^3) \geq 0 \text{ for all } \beta \geq -0.9885.
\end{aligned}$$

Hence, $\xi_2^{(4)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{23}(\lambda_j; \lambda_i; \beta) = \xi_2^{(3)}(\lambda_j; \lambda_j; \lambda_i; \beta)$. We wish to show that ξ_{23} is non-negative for

all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{23} with respect to λ_j gives

$$\begin{aligned} \xi_{23}(\lambda_j; \lambda_i; \beta) &= (16568064\beta + 33191424 + 1969920\beta^2 + 55296\beta^3)\lambda_j^3 \\ &\quad + (4665600\beta^2 - 20570112 + 12400128\beta + 248832\beta^3)\lambda_i\lambda_j^2 \\ &\quad + (1918080\beta^2 + 17874432 + 5225472\beta + 165888\beta^3)\lambda_i^2\lambda_j \\ &\quad + (1610496\beta + 4340736 + 13824\beta^3 + 155520\beta^2)\lambda_i^3, \\ \xi_{23}^{(1)}(\lambda_j; \lambda_i; \beta) &= (49704192\beta + 99574272 + 5909760\beta^2 + 165888\beta^3)\lambda_j^2 \\ &\quad + (9331200\beta^2 - 41140224 + 24800256\beta + 497664\beta^3)\lambda_i\lambda_j \\ &\quad + (1918080\beta^2 + 17874432 + 5225472\beta + 165888\beta^3)\lambda_i^2, \\ \xi_{23}^{(2)}(\lambda_j; \lambda_i; \beta) &= (99408384\beta + 199148544 + 11819520\beta^2 + 331776\beta^3)\lambda_j \\ &\quad + (9331200\beta^2 - 41140224 + 24800256\beta + 497664\beta^3)\lambda_i, \\ \xi_{23}^{(3)}(\lambda_j; \lambda_i; \beta) &= 99408384\beta + 199148544 + 11819520\beta^2 + 331776\beta^3 \\ &\geq 0 \text{ for all } \beta \geq -2.9562. \end{aligned}$$

Now

$$\begin{aligned} \xi_{23}(\lambda_i; \lambda_i; \beta) &= 483840\lambda_i^3(\beta + 4)(\beta^2 + 14\beta + 18) \geq 0 \text{ for all } \beta \geq -1.4322, \\ \xi_{23}^{(1)}(\lambda_i; \lambda_i; \beta) &= 51840\lambda_i^2(1538\beta + 1472 + 331\beta^2 + 16\beta^3) \geq 0 \text{ for all } \beta \geq -1.2958, \\ \xi_{23}^{(2)}(\lambda_i; \lambda_i; \beta) &= 207360\lambda_i(599\beta + 762 + 102\beta^2 + 4\beta^3) \geq 0 \text{ for all } \beta \geq -1.7669. \end{aligned}$$

Hence, $\xi_2^{(3)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{22}(\lambda_j; \lambda_i; \beta) = \xi_2^{(2)}(\lambda_j; \lambda_j; \lambda_i; \beta)$. We wish to show that ξ_{22} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{22} with respect to λ_j gives

$$\begin{aligned} \xi_{22}(\lambda_j; \lambda_i; \beta) &= (7502400\beta + 17758080 + 950400\beta^2 + 34560\beta^3)\lambda_j^4 \\ &\quad + (2764800\beta^2 - 14469120 + 5299200\beta + 207360\beta^3)\lambda_i\lambda_j^3 \\ &\quad + (1555200\beta^2 + 7879680 + 2073600\beta + 207360\beta^3)\lambda_i^2\lambda_j^2 \\ &\quad + (34560\beta^3 + 2119680\beta + 276480\beta^2 + 3225600)\lambda_i^3\lambda_j \\ &\quad + (17280\beta^2 + 141120\beta + 201600)\lambda_i^4, \\ \xi_{22}^{(1)}(\lambda_j; \lambda_i; \beta) &= (30009600\beta + 71032320 + 3801600\beta^2 + 138240\beta^3)\lambda_j^3 \\ &\quad + (8294400\beta^2 - 43407360 + 15897600\beta + 622080\beta^3)\lambda_i\lambda_j^2 \\ &\quad + (3110400\beta^2 + 15759360 + 4147200\beta + 414720\beta^3)\lambda_i^2\lambda_j \\ &\quad + (34560\beta^3 + 2119680\beta + 276480\beta^2 + 3225600)\lambda_i^3, \\ \xi_{22}^{(2)}(\lambda_j; \lambda_i; \beta) &= (90028800\beta + 213096960 + 11404800\beta^2 + 414720\beta^3)\lambda_j^2 \\ &\quad + (16588800\beta^2 - 86814720 + 31795200\beta + 1244160\beta^3)\lambda_i\lambda_j \\ &\quad + (3110400\beta^2 + 15759360 + 4147200\beta + 414720\beta^3)\lambda_i^2, \\ \xi_{22}^{(3)}(\lambda_j; \lambda_i; \beta) &= (180057600\beta + 426193920 + 22809600\beta^2 + 829440\beta^3)\lambda_j \\ &\quad + (16588800\beta^2 - 86814720 + 31795200\beta + 1244160\beta^3)\lambda_i, \\ \xi_{22}^{(4)}(\lambda_j; \lambda_i; \beta) &= 180057600\beta + 426193920 + 22809600\beta^2 + 829440\beta^3 \\ &\geq 0 \text{ for all } \beta \geq -4.5933. \end{aligned}$$

Now

$$\begin{aligned}\xi_{22}(\lambda_i; \lambda_i; \beta) &= 40320\lambda_i^4(425\beta + 362 + 138\beta^2 + 12\beta^3) \geq 0 \text{ for all } \beta \geq -1.4432, \\ \xi_{22}^{(1)}(\lambda_i; \lambda_i; \beta) &= 80640\lambda_i^3(647\beta + 578 + 192\beta^2 + 15\beta^3) \geq 0 \text{ for all } \beta \geq -1.4382, \\ \xi_{22}^{(2)}(\lambda_i; \lambda_i; \beta) &= 518400\lambda_i^2(243\beta + 274 + 60\beta^2 + 4\beta^3) \geq 0 \text{ for all } \beta \geq -1.9255, \\ \xi_{22}^{(3)}(\lambda_i; \lambda_i; \beta) &= 345600\lambda_i(613\beta + 982 + 114\beta^2 + 6\beta^3) \geq 0 \text{ for all } \beta \geq -3.0812.\end{aligned}$$

Hence, $\xi_2^{(2)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{21}(\lambda_j; \lambda_i; \beta) = \xi_2^{(1)}(\lambda_j; \lambda_j; \lambda_i; \beta)/\lambda_j$. We wish to show that ξ_{21} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{21} with respect to λ_j gives

$$\begin{aligned}\xi_{21}(\lambda_j; \lambda_i; \beta) &= (2589696\beta + 6581376 + 332640\beta^2 + 13824\beta^3)\lambda_j^4 \\ &\quad + (1144800\beta^2 - 6439680 + 1897920\beta + 103680\beta^3)\lambda_i\lambda_j^3 \\ &\quad + (799200\beta^2 + 3006720 + 492480\beta + 138240\beta^3)\lambda_i^2\lambda_j^2 \\ &\quad + (1437120\beta + 1463040 + 34560\beta^3 + 228960\beta^2)\lambda_i^3\lambda_j \\ &\quad + (34560\beta^2 + 187200\beta + 178560)\lambda_i^4, \\ \xi_{21}^{(1)}(\lambda_j; \lambda_i; \beta) &= (10358784\beta + 26325504 + 1330560\beta^2 + 55296\beta^3)\lambda_j^3 \\ &\quad + (3434400\beta^2 - 19319040 + 5693760\beta + 311040\beta^3)\lambda_i\lambda_j^2 \\ &\quad + (1598400\beta^2 + 6013440 + 984960\beta + 276480\beta^3)\lambda_i^2\lambda_j \\ &\quad + (1437120\beta + 1463040 + 34560\beta^3 + 228960\beta^2)\lambda_i^3, \\ \xi_{21}^{(2)}(\lambda_j; \lambda_i; \beta) &= (31076352\beta + 78976512 + 3991680\beta^2 + 165888\beta^3)\lambda_j^2 \\ &\quad + (6868800\beta^2 - 38638080 + 11387520\beta + 622080\beta^3)\lambda_i\lambda_j \\ &\quad + (1598400\beta^2 + 6013440 + 984960\beta + 276480\beta^3)\lambda_i^2, \\ \xi_{21}^{(3)}(\lambda_j; \lambda_i; \beta) &= (62152704\beta + 157953024 + 7983360\beta^2 + 331776\beta^3)\lambda_j \\ &\quad + (6868800\beta^2 - 38638080 + 11387520\beta + 622080\beta^3)\lambda_i, \\ \xi_{21}^{(4)}(\lambda_j; \lambda_i; \beta) &= 62152704\beta + 157953024 + 7983360\beta^2 + 331776\beta^3 \\ &\geq 0 \text{ for all } \beta \geq -10.7897.\end{aligned}$$

Now

$$\begin{aligned}\xi_{21}(\lambda_i; \lambda_i; \beta) &= 72576\lambda_i^4(\beta + 3)(4\beta^2 + 23\beta + 22) \geq 0 \text{ for all } \beta \geq -1.2120, \\ \xi_{21}^{(1)}(\lambda_i; \lambda_i; \beta) &= 4032\lambda_i^3(4582\beta + 3592 + 1635\beta^2 + 168\beta^3) \geq 0 \text{ for all } \beta \geq -1.3254, \\ \xi_{21}^{(2)}(\lambda_i; \lambda_i; \beta) &= 24192\lambda_i^2(1796\beta + 1916 + 515\beta^2 + 44\beta^3) \geq 0 \text{ for all } \beta \geq -2.1392, \\ \xi_{21}^{(3)}(\lambda_i; \lambda_i; \beta) &= 5184\lambda_i(14186\beta + 23016 + 2865\beta^2 + 184\beta^3) \geq 0 \text{ for all } \beta \geq -7.5649.\end{aligned}$$

Hence, $\xi_2^{(1)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{20}(\lambda_j; \lambda_i; \beta) = \xi_2(\lambda_j; \lambda_j; \lambda_i; \beta)/\lambda_j^2$. We wish to show that ξ_{20} is non-negative for

all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{20} with respect to λ_j gives

$$\begin{aligned} \xi_{20}(\lambda_j; \lambda_i; \beta) &= (710208\beta + 1847808 + 90720\beta^2 + 4032\beta^3)\lambda_j^4 \\ &\quad + (362880\beta^2 - 2101248 + 594432\beta + 36288\beta^3)\lambda_i\lambda_j^3 \\ &\quad + (302400\beta^2 + 1192320 + 112320\beta + 60480\beta^3)\lambda_i^2\lambda_j^2 \\ &\quad + (691200\beta + 737280 + 20160\beta^3 + 120960\beta^2)\lambda_i^3\lambda_j \\ &\quad + (30240\beta^2 + 129600\beta + 138240)\lambda_i^4, \\ \xi_{20}^{(1)}(\lambda_j; \lambda_i; \beta) &= (2840832\beta + 7391232 + 362880\beta^2 + 16128\beta^3)\lambda_j^3 \\ &\quad + (1088640\beta^2 - 6303744 + 1783296\beta + 108864\beta^3)\lambda_i\lambda_j^2 \\ &\quad + (604800\beta^2 + 2384640 + 224640\beta + 120960\beta^3)\lambda_i^2\lambda_j \\ &\quad + (691200\beta + 737280 + 20160\beta^3 + 120960\beta^2)\lambda_i^3, \\ \xi_{20}^{(2)}(\lambda_j; \lambda_i; \beta) &= (8522496\beta + 22173696 + 1088640\beta^2 + 48384\beta^3)\lambda_j^2 \\ &\quad + (2177280\beta^2 - 12607488 + 3566592\beta + 217728\beta^3)\lambda_i\lambda_j \\ &\quad + (604800\beta^2 + 2384640 + 224640\beta + 120960\beta^3)\lambda_i^2, \\ \xi_{20}^{(3)}(\lambda_j; \lambda_i; \beta) &= (17044992\beta + 44347392 + 2177280\beta^2 + 96768\beta^3)\lambda_j \\ &\quad + (2177280\beta^2 - 12607488 + 3566592\beta + 217728\beta^3)\lambda_i, \\ \xi_{20}^{(4)}(\lambda_j; \lambda_i; \beta) &= 17044992\beta + 44347392 + 2177280\beta^2 + 96768\beta^3 \\ &\geq 0 \text{ for all } \beta \geq -5.7061. \end{aligned}$$

Now

$$\begin{aligned} \xi_{20}(\lambda_i; \lambda_i; \beta) &= 60480\lambda_i^4(2\beta + 5)(\beta + 3)(\beta + 2) > 0 \text{ for all } \beta > -2, \\ \xi_{20}^{(1)}(\lambda_i; \lambda_i; \beta) &= 24192\lambda_i^3(\beta + 3)(11\beta^2 + 57\beta + 58) > 0 \text{ for all } \beta > -1.3909, \\ \xi_{20}^{(2)}(\lambda_i; \lambda_i; \beta) &= 24192\lambda_i^2(509\beta + 494 + 160\beta^2 + 16\beta^3) > 0 \text{ for all } \beta > -1.8336, \\ \xi_{20}^{(3)}(\lambda_i; \lambda_i; \beta) &= 24192\lambda_i(852\beta + 1312 + 180\beta^2 + 13\beta^3) > 0 \text{ for all } \beta > -3.2288. \end{aligned}$$

Hence, ξ_2 is positive for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$. This implies that $\xi(\lambda_l; \lambda_l; \lambda_j; \lambda_i; \beta)$ is positive for all $\lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Now

$$\begin{aligned}
\xi_3(\lambda_l; \lambda_j; \lambda_i; \beta) = & (28800\beta + 11520)\lambda_l^5 \\
& +(172800 + 21600\beta^2 + 216000\beta)(\lambda_j + \lambda_i)\lambda_l^4 \\
& +(64800\beta^2 + 1728\beta^3 + 292032\beta + 1216512)(\lambda_j^2 + \lambda_i^2)\lambda_l^3 \\
& +(-1234944 + 190080\beta^2 + 216576\beta + 5184\beta^3)\lambda_i\lambda_j\lambda_l^3 \\
& +(-6912\beta - 940032 + 207360\beta^2 + 15552\beta^3)(\lambda_i\lambda_j^2 + \lambda_i^2\lambda_j)\lambda_l^2 \\
& +(628992 + 309312\beta + 30240\beta^2 + 1728\beta^3)(\lambda_j^3 + \lambda_i^3)\lambda_l^2 \\
& +(4320\beta^2 + 72000\beta + 46080)(\lambda_j^4 + \lambda_i^4)\lambda_l \\
& +(354816\beta - 405504 + 51840\beta^2 + 5184\beta^3)(\lambda_i\lambda_j^3 + \lambda_i^3\lambda_j)\lambda_l \\
& +(476928 + 103680\beta^2 + 15552\beta^3 + 114048\beta)\lambda_i^2\lambda_j^2\lambda_l \\
& +(12960\beta^2 + 1728\beta^3 + 490752 + 136512\beta)(\lambda_i^3\lambda_j^2 + \lambda_i^2\lambda_j^3) \\
& +(43200\beta + 4320\beta^2 + 34560)(\lambda_i\lambda_j^4 + \lambda_i^4\lambda_j),
\end{aligned}$$

and, hence,

$$\begin{aligned}
\xi_3^{(4)}(\lambda_l; \lambda_j; \lambda_i; \beta) = & (3456000\beta + 1382400)\lambda_l \\
& +(4147200 + 518400\beta^2 + 5184000\beta)(\lambda_i + \lambda_j) \geq 0 \text{ for all } \beta \geq -\frac{2}{5}.
\end{aligned}$$

Define $\xi_{33}(\lambda_j; \lambda_i; \beta) = \xi_3^{(3)}(\lambda_j; \lambda_j; \lambda_i; \beta)$. We wish to show that ξ_{33} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{33} with respect to λ_j gives

$$\begin{aligned}
\xi_{33}(\lambda_j; \lambda_i; \beta) = & (8664192\beta + 12137472 + 907200\beta^2 + 10368\beta^3)\lambda_j^2 \\
& +(-3262464 + 1658880\beta^2 + 6483456\beta + 31104\beta^3)\lambda_i\lambda_j \\
& +(388800\beta^2 + 10368\beta^3 + 1752192\beta + 7299072)\lambda_i^2, \\
\xi_{33}^{(1)}(\lambda_j; \lambda_i; \beta) = & (17328384\beta + 24274944 + 1814400\beta^2 + 20736\beta^3)\lambda_j \\
& +(-3262464 + 1658880\beta^2 + 6483456\beta + 31104\beta^3)\lambda_i, \\
\xi_{33}^{(2)}(\lambda_j; \lambda_i; \beta) = & 17328384\beta + 24274944 + 1814400\beta^2 + 20736\beta^3 \geq 0 \text{ for all } \beta \geq -1.6963.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{33}(\lambda_i; \lambda_i; \beta) = & 51840\lambda_i^2(326\beta + 312 + 57\beta^2\beta^3) \geq 0 \text{ for all } \beta \geq -1.2060, \\
\xi_{33}^{(1)}(\lambda_i; \lambda_i; \beta) = & 17280\lambda_i(1378\beta + 1216 + 201\beta^2 + 3\beta^3) \geq 0 \text{ for all } \beta \geq -1.0368.
\end{aligned}$$

Hence, $\xi_3^{(3)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{32}(\lambda_j; \lambda_i; \beta) = \xi_3^{(2)}(\lambda_j; \lambda_j; \lambda_i; \beta)$. We wish to show that ξ_{32} is non-negative for

all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{32} with respect to λ_j gives

$$\begin{aligned} \xi_{32}(\lambda_j; \lambda_i; \beta) &= (5538816\beta + 10861056 + 708480\beta^2 + 13824\beta^3)\lambda_j^3 \\ &\quad + (-7216128 + 1814400\beta^2 + 3877632\beta + 62208\beta^3)\lambda_i\lambda_j^2 \\ &\quad + (803520\beta^2 + 41472\beta^3 + 1738368\beta + 5419008)\lambda_i^2\lambda_j \\ &\quad + (1257984 + 618624\beta + 60480\beta^2 + 3456\beta^3)\lambda_i^3, \\ \xi_{32}^{(1)}(\lambda_j; \lambda_i; \beta) &= (16616448\beta + 32583168 + 2125440\beta^2 + 41472\beta^3)\lambda_j^2 \\ &\quad + (-14432256 + 3628800\beta^2 + 7755264\beta + 124416\beta^3)\lambda_i\lambda_j \\ &\quad + (803520\beta^2 + 41472\beta^3 + 1738368\beta + 5419008)\lambda_i^2, \\ \xi_{32}^{(2)}(\lambda_j; \lambda_i; \beta) &= (33232896\beta + 65166336 + 4250880\beta^2 + 82944\beta^3)\lambda_j \\ &\quad + (-14432256 + 3628800\beta^2 + 7755264\beta + 124416\beta^3)\lambda_i, \\ \xi_{32}^{(3)}(\lambda_j; \lambda_i; \beta) &= 33232896\beta + 65166336 + 4250880\beta^2 + 82944\beta^3 \geq 0 \text{ for all } \beta \geq -3.1578. \end{aligned}$$

Now

$$\begin{aligned} \xi_{32}(\lambda_i; \lambda_i; \beta) &= 40320\lambda_i^3(292\beta + 256 + 84\beta^2 + 3\beta^3) \geq 0 \text{ for all } \beta \geq -1.4510, \\ \xi_{32}^{(1)}(\lambda_i; \lambda_i; \beta) &= 8640\lambda_i^2(3022\beta + 2728 + 759\beta^2 + 24\beta^3) \geq 0 \text{ for all } \beta \geq -1.3255, \\ \xi_{32}^{(2)}(\lambda_i; \lambda_i; \beta) &= 69120\lambda_i(593\beta + 734 + 114\beta^2 + 3\beta^3) \geq 0 \text{ for all } \beta \geq -1.8910. \end{aligned}$$

Hence, $\xi_3^{(2)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{31}(\lambda_j; \lambda_i; \beta) = \xi_3^{(1)}(\lambda_j; \lambda_j; \lambda_i; \beta)$. We wish to show that ξ_{31} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{31} with respect to λ_j gives

$$\begin{aligned} \xi_{31}(\lambda_j; \lambda_i; \beta) &= (2574720\beta + 5702400 + 345600\beta^2 + 8640\beta^3)\lambda_j^4 \\ &\quad + (-5299200 + 1123200\beta^2 + 1854720\beta + 51840\beta^3)\lambda_i\lambda_j^3 \\ &\quad + (712800\beta^2 + 51840\beta^3 + 976320\beta + 2246400)\lambda_i^2\lambda_j^2 \\ &\quad + (852480 + 973440\beta + 112320\beta^2 + 8640\beta^3)\lambda_i^3\lambda_j \\ &\quad + (4320\beta^2 + 72000\beta + 46080)\lambda_i^4, \\ \xi_{31}^{(1)}(\lambda_j; \lambda_i; \beta) &= (10298880\beta + 22809600 + 1382400\beta^2 + 34560\beta^3)\lambda_j^3 \\ &\quad + (-15897600 + 3369600\beta^2 + 5564160\beta + 155520\beta^3)\lambda_i\lambda_j^2 \\ &\quad + (1425600\beta^2 + 103680\beta^3 + 1952640\beta + 4492800)\lambda_i^2\lambda_j \\ &\quad + (852480 + 973440\beta + 112320\beta^2 + 8640\beta^3)\lambda_i^3, \\ \xi_{31}^{(2)}(\lambda_j; \lambda_i; \beta) &= (30896640\beta + 68428800 + 4147200\beta^2 + 103680\beta^3)\lambda_j^2 \\ &\quad + (-31795200 + 6739200\beta^2 + 11128320\beta + 311040\beta^3)\lambda_i\lambda_j \\ &\quad + (1425600\beta^2 + 103680\beta^3 + 1952640\beta + 4492800)\lambda_i^2, \\ \xi_{31}^{(3)}(\lambda_j; \lambda_i; \beta) &= (61793280\beta + 136857600 + 8294400\beta^2 + 207360\beta^3)\lambda_j \\ &\quad + (-31795200 + 6739200\beta^2 + 11128320\beta + 311040\beta^3)\lambda_i, \\ \xi_{31}^{(4)}(\lambda_j; \lambda_i; \beta) &= 61793280\beta + 136857600 + 8294400\beta^2 + 207360\beta^3 \\ &\geq 0 \text{ for all } \beta \geq -31.1005. \end{aligned}$$

Now

$$\begin{aligned}\xi_{31}(\lambda_i; \lambda_i; \beta) &= 40320\lambda_i^4(160\beta + 88 + 57\beta^2 + 3\beta^3) \geq 0 \text{ for all } \beta \geq -0.7350, \\ \xi_{31}^{(1)}(\lambda_i; \lambda_i; \beta) &= 20160\lambda_i^3(932\beta + 608 + 312\beta^2 + 15\beta^3) \geq 0 \text{ for all } \beta \geq -0.9275, \\ \xi_{31}^{(2)}(\lambda_i; \lambda_i; \beta) &= 43200\lambda_i^2(1018\beta + 952 + 285\beta^2 + 12\beta^3) \geq 0 \text{ for all } \beta \geq -1.6188, \\ \xi_{31}^{(3)}(\lambda_i; \lambda_i; \beta) &= 172800\lambda_i(422\beta + 608 + 87\beta^2 + 3\beta^3) \geq 0 \text{ for all } \beta \geq -23.3467.\end{aligned}$$

Hence, $\xi_3^{(1)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{30}(\lambda_j; \lambda_i; \beta) = \xi_3(\lambda_j; \lambda_j; \lambda_i; \beta)/\lambda_j$. We wish to show that ξ_{30} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{30} with respect to λ_j gives

$$\begin{aligned}\xi_{30}(\lambda_j; \lambda_i; \beta) &= (918144\beta + 2075904 + 120960\beta^2 + 3456\beta^3)\lambda_j^4 \\ &\quad + (-2373120 + 475200\beta^2 + 823680\beta + 25920\beta^3)\lambda_i\lambda_j^3 \\ &\quad + (388800\beta^2 + 34560\beta^3 + 535680\beta + 1244160)\lambda_i^2\lambda_j^2 \\ &\quad + (714240 + 800640\beta + 95040\beta^2 + 8640\beta^3)\lambda_i^3\lambda_j \\ &\quad + (8640\beta^2 + 115200\beta + 80640)\lambda_i^4, \\ \xi_{30}^{(1)}(\lambda_j; \lambda_i; \beta) &= (3672576\beta + 8303616 + 483840\beta^2 + 13824\beta^3)\lambda_j^3 \\ &\quad + (-7119360 + 1425600\beta^2 + 2471040\beta + 77760\beta^3)\lambda_i\lambda_j^2 \\ &\quad + (777600\beta^2 + 69120\beta^3 + 1071360\beta + 2488320)\lambda_i^2\lambda_j \\ &\quad + (714240 + 800640\beta + 95040\beta^2 + 8640\beta^3)\lambda_i^3, \\ \xi_{30}^{(2)}(\lambda_j; \lambda_i; \beta) &= (11017728\beta + 24910848 + 1451520\beta^2 + 41472\beta^3)\lambda_j^2 \\ &\quad + (-14238720 + 2851200\beta^2 + 4942080\beta + 155520\beta^3)\lambda_i\lambda_j \\ &\quad + (777600\beta^2 + 69120\beta^3 + 1071360\beta + 2488320)\lambda_i^2, \\ \xi_{30}^{(3)}(\lambda_j; \lambda_i; \beta) &= (22035456\beta + 49821696 + 2903040\beta^2 + 82944\beta^3)\lambda_j \\ &\quad + (-14238720 + 2851200\beta^2 + 4942080\beta + 155520\beta^3)\lambda_i, \\ \xi_{30}^{(4)}(\lambda_j; \lambda_i; \beta) &= 22035456\beta + 49821696 + 2903040\beta^2 + 82944\beta^3 \\ &\geq 0 \text{ for all } \beta \geq -25.5082.\end{aligned}$$

Now

$$\begin{aligned}\xi_{30}(\lambda_i; \lambda_i; \beta) &= 72576\lambda_i^4(\beta + 3)(\beta^2 + 12\beta + 8) > 0 \text{ for all } \beta > -1, \\ \xi_{30}^{(1)}(\lambda_i; \lambda_i; \beta) &= 8064\lambda_i^3(994\beta + 544 + 345\beta^2 + 21\beta^3) > 0 \text{ for all } \beta > -0.7187, \\ \xi_{30}^{(2)}(\lambda_i; \lambda_i; \beta) &= 24192\lambda_i^2(704\beta + 544 + 210\beta^2 + 11\beta^3) > 0 \text{ for all } \beta > -1.1328, \\ \xi_{30}^{(3)}(\lambda_i; \lambda_i; \beta) &= 10368\lambda_i(\beta + 3)(23\beta^2 + 486\beta + 1144) > 0 \text{ for all } \beta > -2.6985.\end{aligned}$$

Hence, ξ_3 is positive for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$. This implies that $\xi(\lambda_l; \lambda_l; \lambda_j; \lambda_i; \beta)$ is positive for all $\lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Now,

$$\begin{aligned} \xi_4(\lambda_l; \lambda_j; \lambda_i; \beta) = & (28800\beta + 11520)\lambda_l^4 \\ & +(172800\beta + 17280\beta^2 + 138240)(\lambda_i + \lambda_j)\lambda_l^3 \\ & +(725760 + 155520\beta + 51840\beta^2)(\lambda_j^2 + \lambda_i^2)\lambda_l^2 \\ & +(-691200 + 34560\beta + 155520\beta^2)\lambda_i\lambda_j\lambda_l^2 \\ & +(-691200 + 34560\beta + 155520\beta^2)(\lambda_i\lambda_j^2 + \lambda_i^2\lambda_j)\lambda_l \\ & +(172800\beta + 17280\beta^2 + 138240)(\lambda_j^3 + \lambda_i^3)\lambda_l \\ & +(172800\beta + 17280\beta^2 + 138240)(\lambda_i\lambda_j^3 + \lambda_i^3\lambda_j) \\ & +(725760 + 155520\beta + 51840\beta^2)\lambda_i^2\lambda_j^2 \\ & +(28800\beta + 11520)(\lambda_j^4 + \lambda_i^4), \end{aligned}$$

and, hence,

$$\begin{aligned} \xi_4^{(3)}(\lambda_l; \lambda_j; \lambda_i; \beta) = & (691200\beta + 276480)\lambda_l \\ & +(1036800\beta + 103680\beta^2 + 829440)(\lambda_i + \lambda_j) \geq 0 \text{ for all } \beta \geq -\frac{2}{5}. \end{aligned}$$

Define $\xi_{42}(\lambda_j; \lambda_i; \beta) = \xi_4^{(2)}(\lambda_j; \lambda_j; \lambda_i; \beta)$. We wish to show that ξ_{42} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{42} with respect to λ_j gives

$$\begin{aligned} \xi_{42}(\lambda_j; \lambda_i; \beta) = & (1693440\beta + 2419200 + 207360\beta^2)\lambda_j^2 \\ & +(1105920\beta + 414720\beta^2 - 552960)\lambda_i\lambda_j \\ & +(1451520 + 311040\beta + 103680\beta^2)\lambda_i^2, \\ \xi_{42}^{(1)}(\lambda_j; \lambda_i; \beta) = & (3386880\beta + 4838400 + 414720\beta^2)\lambda_j \\ & +(1105920\beta + 414720\beta^2 - 552960)\lambda_i, \\ \xi_{42}^{(2)}(\lambda_j; \lambda_i; \beta) = & 3386880\beta + 4838400 + 414720\beta^2 \geq 0 \text{ for all } \beta \geq -1.8457. \end{aligned}$$

Now

$$\begin{aligned} \xi_{42}(\lambda_i; \lambda_i; \beta) = & 103680\lambda_i^2(7\beta + 16)(\beta + 2) \geq 0 \text{ for all } \beta \geq -2, \\ \xi_{42}^{(1)}(\lambda_i; \lambda_i; \beta) = & 69120\lambda_i(65\beta + 62 + 12\beta^2) \geq 0 \text{ for all } \beta \geq -1.2358. \end{aligned}$$

Hence, $\xi_4^{(2)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{41}(\lambda_j; \lambda_i; \beta) = \xi_4^{(1)}(\lambda_j; \lambda_j; \lambda_i; \beta)$. We wish to show that ξ_{41} is non-negative for

all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{41} with respect to λ_j gives

$$\begin{aligned}\xi_{41}(\lambda_j; \lambda_i; \beta) &= (1117440\beta + 2050560 + 172800\beta^2)\lambda_j^3 \\ &\quad + (622080\beta + 518400\beta^2 - 1658880)\lambda_i\lambda_j^2 \\ &\quad + (760320 + 345600\beta + 259200\beta^2)\lambda_i^2\lambda_j \\ &\quad + (172800\beta + 17280\beta^2 + 138240)\lambda_i^3, \\ \xi_{41}^{(1)}(\lambda_j; \lambda_i; \beta) &= (3352320\beta + 6151680 + 518400\beta^2)\lambda_j^2 \\ &\quad + (1244160\beta + 1036800\beta^2 - 3317760)\lambda_i\lambda_j \\ &\quad + (760320 + 345600\beta + 259200\beta^2)\lambda_i^2, \\ \xi_{41}^{(2)}(\lambda_j; \lambda_i; \beta) &= (6704640\beta + 12303360 + 1036800\beta^2)\lambda_j \\ &\quad + (1244160\beta + 1036800\beta^2 - 3317760)\lambda_i, \\ \xi_{41}^{(3)}(\lambda_j; \lambda_i; \beta) &= 6704640\beta + 12303360 + 1036800\beta^2 \geq 0 \text{ for all } \beta.\end{aligned}$$

Now

$$\begin{aligned}\xi_{41}(\lambda_i; \lambda_i; \beta) &= 322560\lambda_i^3(3\beta + 4)(\beta + 1) \geq 0 \text{ for all } \beta \geq -1, \\ \xi_{41}^{(1)}(\lambda_i; \lambda_i; \beta) &= 17280\lambda_i^2(105\beta^2 + 208 + 286\beta) \geq 0 \text{ for all } \beta, \\ \xi_{41}^{(2)}(\lambda_i; \lambda_i; \beta) &= 345600\lambda_i(23\beta + 26 + 6\beta^2) \geq 0 \text{ for all } \beta.\end{aligned}$$

Hence, $\xi_4^{(1)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{40}(\lambda_j; \lambda_i; \beta) = \xi_4(\lambda_j; \lambda_j; \lambda_i; \beta)$. We wish to show that ξ_{40} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{40} with respect to λ_j gives

$$\begin{aligned}\xi_{40}(\lambda_j; \lambda_i; \beta) &= (558720\beta + 1025280 + 86400\beta^2)\lambda_j^4 \\ &\quad + (-1105920 + 414720\beta + 345600\beta^2)\lambda_i\lambda_j^3 \\ &\quad + (760320 + 345600\beta + 259200\beta^2)\lambda_i^2\lambda_j^2 \\ &\quad + (345600\beta + 34560\beta^2 + 276480)\lambda_i^3\lambda_j \\ &\quad + (28800\beta + 11520)\lambda_i^4, \\ \xi_{40}^{(1)}(\lambda_j; \lambda_i; \beta) &= (2234880\beta + 4101120 + 345600\beta^2)\lambda_j^3 \\ &\quad + (1244160\beta + 1036800\beta^2 - 3317760)\lambda_i\lambda_j^2 \\ &\quad + (1520640 + 691200\beta + 518400\beta^2)\lambda_i^2\lambda_j \\ &\quad + (345600\beta + 34560\beta^2 + 276480)\lambda_i^3, \\ \xi_{40}^{(2)}(\lambda_j; \lambda_i; \beta) &= (6704640\beta + 12303360 + 1036800\beta^2)\lambda_j^2 \\ &\quad + (2488320\beta + 2073600\beta^2 - 6635520)\lambda_i\lambda_j \\ &\quad + (1520640 + 691200\beta + 518400\beta^2)\lambda_i^2, \\ \xi_{40}^{(3)}(\lambda_j; \lambda_i; \beta) &= (13409280\beta + 24606720 + 2073600\beta^2)\lambda_j \\ &\quad + (2488320\beta + 2073600\beta^2 - 6635520)\lambda_i, \\ \xi_{40}^{(4)}(\lambda_j; \lambda_i; \beta) &= 13409280\beta + 24606720 + 2073600\beta^2 \geq 0 \text{ for all } \beta.\end{aligned}$$

Now

$$\begin{aligned} \xi_{40}(\lambda_i; \lambda_i; \beta) &= 241920\lambda_i^4(3\beta + 4)(\beta + 1) > 0 \text{ for all } \beta > -1, \\ \xi_{40}^{(1)}(\lambda_i; \lambda_i; \beta) &= 645120\lambda_i^3(3\beta + 4)(\beta + 1) > 0 \text{ for all } \beta > -1, \\ \xi_{40}^{(2)}(\lambda_i; \lambda_i; \beta) &= 34560\lambda_i^2(105\beta^2 + 208 + 286\beta) > 0 \text{ for all } \beta, \\ \xi_{40}^{(3)}(\lambda_i; \lambda_i; \beta) &= 691200\lambda_i(23\beta + 26 + 6\beta^2) > 0 \text{ for all } \beta. \end{aligned}$$

Hence, ξ_4 is positive for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$. This implies that $\xi(\lambda_l; \lambda_l; \lambda_j; \lambda_i; \beta)$ is positive for all $\lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

□

Lemma B.16 Let

$$\begin{aligned} \xi(\lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta) &= (8640\beta + 38880\beta^2 - 172800)t(1, 1, 1, 4) \\ &+ (12960\beta^2 + 48960 + 36000\beta)t(0, 1, 2, 4) \\ &+ (1440\beta^2 + 19200\beta + 13440)t(0, 0, 3, 4) \\ &+ (15840\beta^2 + 119040 + 133440\beta + 1440\beta^3)t(0, 1, 3, 3) \\ &+ (12960\beta^3 - 552960 + 69120\beta + 77760\beta^2)t(1, 1, 2, 3) \\ &+ (30240\beta^2 + 4320\beta^3 + 570240 + 155520\beta)t(0, 2, 2, 3) \\ &+ (-311040\beta + 38880\beta^3 + 77760\beta^2)t(1, 2, 2, 2), \end{aligned}$$

where

$$t(a, b, c, d) = \sum_{\substack{(n_1, n_2, n_3, n_4) \\ \in \text{perm}(a, b, c, d)}} \lambda_i^{n_1} \lambda_j^{n_2} \lambda_l^{n_3} \lambda_k^{n_4}.$$

The expression $\xi(\lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta)$ is positive for all $\lambda_k \geq \lambda_l \geq \lambda_j \geq \lambda_i$, and $\beta \geq -\frac{2}{5}$.

Proof.

Let

$$\hat{s}(a, b, c) = \sum_{\substack{(n_1, n_2, n_3) \\ \in \text{perm}(a, b, c)}} \lambda_i^{n_1} \lambda_j^{n_2} \lambda_l^{n_3}.$$

Differentiating $\xi(\lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta)$ with respect to λ_k we obtain

$$\begin{aligned}
\xi^{(1)}(\lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta) = & (-691200 + 155520\beta^2 + 34560\beta)\lambda_k^3\hat{s}(1, 1, 1) \\
& + (51840\beta^2 + 195840 + 144000\beta)\lambda_k^3\hat{s}(0, 1, 2) \\
& + (76800\beta + 5760\beta^2 + 53760)\lambda_k^3\hat{s}(0, 0, 3) \\
& + (233280\beta^2 - 1658880 + 38880\beta^3 + 207360\beta)\lambda_k^2\hat{s}(1, 1, 2) \\
& + (90720\beta^2 + 466560\beta + 1710720 + 12960\beta^3)\lambda_k^2\hat{s}(0, 2, 2) \\
& + (57600\beta + 4320\beta^2 + 40320)\lambda_k^2\hat{s}(0, 0, 4) \\
& + (400320\beta + 4320\beta^3 + 357120 + 47520\beta^2)\lambda_k^2\hat{s}(0, 1, 3) \\
& + (155520\beta^2 - 1105920 + 138240\beta + 25920\beta^3)\lambda_k\hat{s}(1, 1, 3) \\
& + (155520\beta^2 - 622080\beta + 77760\beta^3)\lambda_k\hat{s}(1, 2, 2) \\
& + (25920\beta^2 + 97920 + 72000\beta)\lambda_k\hat{s}(0, 1, 4) \\
& + (60480\beta^2 + 311040\beta + 1140480 + 8640\beta^3)\lambda_k\hat{s}(0, 2, 3) \\
& + (133440\beta + 1440\beta^3 + 119040 + 15840\beta^2)\hat{s}(0, 3, 3) \\
& + (77760\beta^2 - 311040\beta + 38880\beta^3)\hat{s}(2, 2, 2) \\
& + (-552960 + 69120\beta + 77760\beta^2 + 12960\beta^3)\hat{s}(1, 2, 3) \\
& + (-172800 + 38880\beta^2 + 8640\beta)\hat{s}(1, 1, 4) \\
& + (12960\beta^2 + 48960 + 36000\beta)\hat{s}(0, 2, 4),
\end{aligned}$$

$$\begin{aligned}
\xi^{(2)}(\lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta) = & 3(-691200 + 155520\beta^2 + 34560\beta)\lambda_k^2\hat{s}(1, 1, 1) \\
& + 3(51840\beta^2 + 195840 + 144000\beta)\lambda_k^2\hat{s}(0, 1, 2) \\
& + 3(76800\beta + 5760\beta^2 + 53760)\lambda_k^2\hat{s}(0, 0, 3) \\
& + 2(233280\beta^2 - 1658880 + 38880\beta^3 + 207360\beta)\lambda_k\hat{s}(1, 1, 2) \\
& + 2(90720\beta^2 + 466560\beta + 1710720 + 12960\beta^3)\lambda_k\hat{s}(0, 2, 2) \\
& + 2(57600\beta + 4320\beta^2 + 40320)\lambda_k\hat{s}(0, 0, 4) \\
& + 2(400320\beta + 4320\beta^3 + 357120 + 47520\beta^2)\lambda_k\hat{s}(0, 1, 3) \\
& + (155520\beta^2 - 1105920 + 138240\beta + 25920\beta^3)\hat{s}(1, 1, 3) \\
& + (155520\beta^2 - 622080\beta + 77760\beta^3)\hat{s}(1, 2, 2) \\
& + (25920\beta^2 + 97920 + 72000\beta)\hat{s}(0, 1, 4) \\
& + (60480\beta^2 + 311040\beta + 1140480 + 8640\beta^3)\hat{s}(0, 2, 3),
\end{aligned}$$

$$\begin{aligned} \xi^{(3)}(\lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta) = & 6(-691200 + 155520\beta^2 + 34560\beta)\lambda_k\hat{s}(1, 1, 1) \\ & +6(51840\beta^2 + 195840 + 144000\beta)\lambda_k\hat{s}(0, 1, 2) \\ & +6(76800\beta + 5760\beta^2 + 53760)\lambda_k\hat{s}(0, 0, 3) \\ & +2(233280\beta^2 - 1658880 + 38880\beta^3 + 207360\beta)\hat{s}(1, 1, 2) \\ & +2(90720\beta^2 + 466560\beta + 1710720 + 12960\beta^3)\hat{s}(0, 2, 2) \\ & +2(57600\beta + 4320\beta^2 + 40320)\hat{s}(0, 0, 4) \\ & +2(400320\beta + 4320\beta^3 + 357120 + 47520\beta^2)\hat{s}(0, 1, 3), \end{aligned}$$

$$\begin{aligned} \xi^{(4)}(\lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta) = & 6(-691200 + 155520\beta^2 + 34560\beta)\hat{s}(1, 1, 1) \\ & +6(51840\beta^2 + 195840 + 144000\beta)\hat{s}(0, 1, 2) \\ & +6(76800\beta + 5760\beta^2 + 53760)\hat{s}(0, 0, 3). \end{aligned}$$

Define

$$\begin{aligned} \xi_0(\lambda_l; \lambda_j; \lambda_i; \beta) & \stackrel{\text{def}}{=} \xi(\lambda_l; \lambda_l; \lambda_j; \lambda_i; \beta), \\ \xi_1(\lambda_l; \lambda_j; \lambda_i; \beta) & \stackrel{\text{def}}{=} \xi^{(1)}(\lambda_l; \lambda_l; \lambda_j; \lambda_i; \beta), \\ \xi_2(\lambda_l; \lambda_j; \lambda_i; \beta) & \stackrel{\text{def}}{=} \xi^{(2)}(\lambda_l; \lambda_l; \lambda_j; \lambda_i; \beta), \\ \xi_3(\lambda_l; \lambda_j; \lambda_i; \beta) & \stackrel{\text{def}}{=} \xi^{(3)}(\lambda_l; \lambda_l; \lambda_j; \lambda_i; \beta), \\ \xi_4(\lambda_l; \lambda_j; \lambda_i; \beta) & \stackrel{\text{def}}{=} \xi^{(4)}(\lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta). \end{aligned}$$

We will prove that the above equations are non-negative for all $\lambda_k \geq \lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$ and, hence, deduce that $\xi(\lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta)$ is positive for all $\lambda_k \geq \lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Now

$$\begin{aligned}
\xi_0(\lambda_l; \lambda_j; \lambda_i; \beta) = & (26880 + 38400\beta + 2880\beta^2)\lambda_l^7 \\
& + (205440\beta + 1440\beta^3 + 41760\beta^2 + 216960)(\lambda_j + \lambda_i)\lambda_l^6 \\
& + (383040\beta + 1238400 + 8640\beta^3 + 86400\beta^2)(\lambda_j^2 + \lambda_i^2)\lambda_l^5 \\
& + (155520\beta + 25920\beta^3 + 233280\beta^2 - 1451520)\lambda_i\lambda_j\lambda_l^5 \\
& + (-100800\beta + 259200\beta^2 + 64800\beta^3 - 1008000)(\lambda_i\lambda_j^2 + \lambda_i^2\lambda_j)\lambda_l^4 \\
& + (460800\beta + 64800\beta^2 + 835200 + 7200\beta^3)(\lambda_j^3 + \lambda_i^3)\lambda_l^4 \\
& + (187200\beta^2 - 867840 + 405120\beta + 28800\beta^3)(\lambda_i\lambda_j^3 + \lambda_i^3\lambda_j)\lambda_l^3 \\
& + (124800 + 110400\beta + 28800\beta^2)(\lambda_j^4 + \lambda_i^4)\lambda_l^3 \\
& + (216000\beta^2 + 1140480 - 311040\beta + 86400\beta^3)\lambda_i^2\lambda_j^2\lambda_l^3 \\
& + (380160\beta + 138240\beta^2 + 587520 + 21600\beta^3)(\lambda_i^2\lambda_j^3 + \lambda_i^3\lambda_j^2)\lambda_l^2 \\
& + (80640\beta - 74880 + 64800\beta^2)(\lambda_i\lambda_j^4 + \lambda_i^4\lambda_j)\lambda_l^2 \\
& + (25920\beta^2 + 97920 + 72000\beta)(\lambda_i^2\lambda_j^4 + \lambda_i^4\lambda_j^2)\lambda_l \\
& + (266880\beta + 238080 + 31680\beta^2 + 2880\beta^3)\lambda_i^3\lambda_j^3\lambda_l \\
& + (19200\beta + 1440\beta^2 + 13440)(\lambda_i^3\lambda_j^4 + \lambda_i^4\lambda_j^3),
\end{aligned}$$

and, hence,

$$\begin{aligned}
\xi_0^{(6)}(\lambda_l; \lambda_j; \lambda_i; \beta) = & (135475200 + 193536000\beta + 14515200\beta^2)\lambda_l \\
& + (147916800\beta + 1036800\beta^3 + 30067200\beta^2 + 156211200)(\lambda_i + \lambda_j) \\
\geq & 0 \text{ for all } \beta \geq -0.7412.
\end{aligned}$$

Define $\xi_{05}(\lambda_j; \lambda_i; \beta) = \xi_0^{(5)}(\lambda_l; \lambda_j; \lambda_i; \beta)$. We wish to show that ξ_{05} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{05} with respect to λ_j gives

$$\begin{aligned}
\xi_{05}(\lambda_j; \lambda_i; \beta) = & (372556800 + 290649600\beta + 47692800\beta^2 + 2073600\beta^3)\lambda_j^2 \\
& + (166579200\beta + 4147200\beta^3 + 58060800\beta^2 - 17971200)\lambda_i\lambda_j \\
& + (45964800\beta + 148608000 + 1036800\beta^3 + 10368000\beta^2)\lambda_i^2, \\
\xi_{05}^{(1)}(\lambda_j; \lambda_i; \beta) = & (745113600 + 581299200\beta + 95385600\beta^2 + 4147200\beta^3)\lambda_j \\
& + (166579200\beta + 4147200\beta^3 + 58060800\beta^2 - 17971200)\lambda_i, \\
\xi_{05}^{(2)}(\lambda_j; \lambda_i; \beta) = & 745113600 + 581299200\beta + 95385600\beta^2 + 4147200\beta^3 \\
\geq & 0 \text{ for all } \beta \geq -1.7421.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{05}(\lambda_i; \lambda_i; \beta) = & 2419200\lambda_i^2(208 + 208\beta + 48\beta^2 + 3\beta^3) \geq 0 \text{ for all } \beta \geq -1.4294, \\
\xi_{05}^{(1)}(\lambda_i; \lambda_i; \beta) = & 1382400\lambda_i(526 + 541\beta + 111\beta^2 + 6\beta^3) \geq 0 \text{ for all } \beta \geq -1.2898.
\end{aligned}$$

Hence, $\xi_0^{(5)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{04}(\lambda_j; \lambda_i; \beta) = \xi_0^{(4)}(\lambda_i; \lambda_j; \lambda_i; \beta)$. We wish to show that ξ_{04} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{04} with respect to λ_j gives

$$\begin{aligned} \xi_{04}(\lambda_j; \lambda_i; \beta) &= (269337600 + 163238400\beta + 29376000\beta^2 + 1728000\beta^3)\lambda_j^3 \\ &\quad + (90201600\beta + 5184000\beta^3 + 49248000\beta^2 - 120268800)\lambda_i\lambda_j^2 \\ &\quad + (43545600\beta + 124416000 + 2592000\beta^3 + 16588800\beta^2)\lambda_i^2\lambda_j \\ &\quad + (11059200\beta + 1555200\beta^2 + 20044800 + 172800\beta^3)\lambda_i^3, \\ \xi_{04}^{(1)}(\lambda_j; \lambda_i; \beta) &= (808012800 + 489715200\beta + 88128000\beta^2 + 5184000\beta^3)\lambda_j^2 \\ &\quad + (180403200\beta + 10368000\beta^3 + 98496000\beta^2 - 240537600)\lambda_i\lambda_j \\ &\quad + (43545600\beta + 124416000 + 2592000\beta^3 + 16588800\beta^2)\lambda_i^2, \\ \xi_{04}^{(2)}(\lambda_j; \lambda_i; \beta) &= (979430400\beta + 1616025600 + 176256000\beta^2 + 10368000\beta^3)\lambda_j \\ &\quad + (180403200\beta + 10368000\beta^3 + 98496000\beta^2 - 240537600)\lambda_i, \\ \xi_{04}^{(3)}(\lambda_j; \lambda_i; \beta) &= 979430400\beta + 1616025600 + 176256000\beta^2 + 10368000\beta^3 \\ &\geq 0 \text{ for all } \beta \geq -2.9237. \end{aligned}$$

Now

$$\begin{aligned} \xi_{04}(\lambda_i; \lambda_i; \beta) &= 1612800\lambda_i^3(182 + 191\beta + 60\beta^2 + 6\beta^3) \geq 0 \text{ for all } \beta \geq -1.7321, \\ \xi_{04}^{(1)}(\lambda_i; \lambda_i; \beta) &= 1209600\lambda_i^2(572 + 590\beta + 168\beta^2 + 15\beta^3) \geq 0 \text{ for all } \beta \geq -1.5802, \\ \xi_{04}^{(2)}(\lambda_i; \lambda_i; \beta) &= 345600\lambda_i(3356\beta + 3980 + 795\beta^2 + 60\beta^3) \\ &\geq 0 \text{ for all } \beta \geq -1.9649. \end{aligned}$$

Hence, $\xi_0^{(4)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{03}(\lambda_j; \lambda_i; \beta) = \xi_0^{(3)}(\lambda_j; \lambda_j; \lambda_i; \beta)$. We wish to show that ξ_{03} is non-negative for

all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{03} with respect to λ_j gives

$$\begin{aligned}
\xi_{03}(\lambda_j; \lambda_i; \beta) &= (126777600 + 67420800\beta + 12528000\beta^2 + 864000\beta^3)\lambda_j^4 \\
&\quad + (33995520\beta + 3456000\beta^3 + 26352000\beta^2 - 90455040)\lambda_i\lambda_j^3 \\
&\quad + (18696960\beta + 56954880 + 2592000\beta^3 + 12700800\beta^2)\lambda_i^2\lambda_j^2 \\
&\quad + (2678400\beta^2 + 14837760 + 13489920\beta + 345600\beta^3)\lambda_i^3\lambda_j \\
&\quad + (748800 + 662400\beta + 172800\beta^2)\lambda_i^4, \\
\xi_{03}^{(1)}(\lambda_j; \lambda_i; \beta) &= (507110400 + 269683200\beta + 50112000\beta^2 + 3456000\beta^3)\lambda_j^3 \\
&\quad + (101986560\beta + 10368000\beta^3 + 79056000\beta^2 - 271365120)\lambda_i\lambda_j^2 \\
&\quad + (37393920\beta + 113909760 + 5184000\beta^3 + 25401600\beta^2)\lambda_i^2\lambda_j \\
&\quad + (2678400\beta^2 + 14837760 + 13489920\beta + 345600\beta^3)\lambda_i^3, \\
\xi_{03}^{(2)}(\lambda_j; \lambda_i; \beta) &= (1521331200 + 809049600\beta + 150336000\beta^2 + 10368000\beta^3)\lambda_j^2 \\
&\quad + (203973120\beta + 20736000\beta^3 + 158112000\beta^2 - 542730240)\lambda_i\lambda_j \\
&\quad + (37393920\beta + 113909760 + 5184000\beta^3 + 25401600\beta^2)\lambda_i^2, \\
\xi_{03}^{(3)}(\lambda_j; \lambda_i; \beta) &= (3042662400 + 1618099200\beta + 300672000\beta^2 + 20736000\beta^3)\lambda_j \\
&\quad + (203973120\beta + 20736000\beta^3 + 158112000\beta^2 - 542730240)\lambda_i, \\
\xi_{03}^{(4)}(\lambda_j; \lambda_i; \beta) &= 3042662400 + 1618099200\beta + 300672000\beta^2 + 20736000\beta^3 \\
&\geq 0 \text{ for all } \beta \geq -4.2760.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{03}(\lambda_i; \lambda_i; \beta) &= 3628800\lambda_i^4(2\beta + 5)(\beta + 3)(\beta + 2) \geq 0 \text{ for all } \beta \geq -2, \\
\xi_{03}^{(1)}(\lambda_i; \lambda_i; \beta) &= 806400\lambda_i^3(452 + 524\beta + 195\beta^2 + 24\beta^3) \geq 0 \text{ for all } \beta \geq -1.8116, \\
\xi_{03}^{(2)}(\lambda_i; \lambda_i; \beta) &= 483840\lambda_i^2(2258 + 2171\beta + 690\beta^2 + 75\beta^3) \geq 0 \text{ for all } \beta \geq -2.3123, \\
\xi_{03}^{(3)}(\lambda_i; \lambda_i; \beta) &= 103680\lambda_i(24112 + 17574\beta + 4425\beta^2 + 400\beta^3) \geq 0 \text{ for all } \beta \geq -3.2529.
\end{aligned}$$

Hence, $\xi_0^{(3)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{02}(\lambda_j; \lambda_i; \beta) = \xi_0^{(2)}(\lambda_j; \lambda_j; \lambda_i; \beta)/\lambda_j$. We wish to show that ξ_{02} is non-negative

for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{02} with respect to λ_j gives

$$\begin{aligned} \xi_{02}(\lambda_j; \lambda_i; \beta) &= (43176960 + 21628800\beta + 4052160\beta^2 + 302400\beta^3)\lambda_j^4 \\ &\quad + (10656000\beta + 1512000\beta^3 + 10281600\beta^2 - 39974400)\lambda_i\lambda_j^3 \\ &\quad + (5345280\beta + 20689920 + 1512000\beta^3 + 6410880\beta^2)\lambda_i^2\lambda_j^2 \\ &\quad + (8720640\beta + 2177280\beta^2 + 5990400 + 302400\beta^3)\lambda_i^3\lambda_j \\ &\quad + (599040 + 823680\beta + 302400\beta^2)\lambda_i^4, \\ \xi_{02}^{(1)}(\lambda_j; \lambda_i; \beta) &= (172707840 + 86515200\beta + 16208640\beta^2 + 1209600\beta^3)\lambda_j^3 \\ &\quad + (31968000\beta + 4536000\beta^3 + 30844800\beta^2 - 119923200)\lambda_i\lambda_j^2 \\ &\quad + (10690560\beta + 41379840 + 3024000\beta^3 + 12821760\beta^2)\lambda_i^2\lambda_j \\ &\quad + (8720640\beta + 2177280\beta^2 + 5990400 + 302400\beta^3)\lambda_i^3, \\ \xi_{02}^{(2)}(\lambda_j; \lambda_i; \beta) &= (518123520 + 259545600\beta + 48625920\beta^2 + 3628800\beta^3)\lambda_j^2 \\ &\quad + (63936000\beta + 9072000\beta^3 + 61689600\beta^2 - 239846400)\lambda_i\lambda_j \\ &\quad + (10690560\beta + 41379840 + 3024000\beta^3 + 12821760\beta^2)\lambda_i^2, \\ \xi_{02}^{(3)}(\lambda_j; \lambda_i; \beta) &= (1036247040 + 519091200\beta + 97251840\beta^2 + 7257600\beta^3)\lambda_j \\ &\quad + (63936000\beta + 9072000\beta^3 + 61689600\beta^2 - 239846400)\lambda_i, \\ \xi_{02}^{(4)}(\lambda_j; \lambda_i; \beta) &= 1036247040 + 519091200\beta + 97251840\beta^2 + 7257600\beta^3 \\ &\geq 0 \text{ for all } \beta \geq -4.5982. \end{aligned}$$

Now

$$\begin{aligned} \xi_{02}(\lambda_i; \lambda_i; \beta) &= 725760\lambda_i^4(5\beta + 7)(\beta + 3)(\beta + 2) \geq 0 \text{ for all } \beta \geq -2, \\ \xi_{02}^{(1)}(\lambda_i; \lambda_i; \beta) &= 362880\lambda_i^3(25\beta + 46)(\beta + 3)(\beta + 2) \geq 0 \text{ for all } \beta \geq -\frac{46}{25}, \\ \xi_{02}^{(2)}(\lambda_i; \lambda_i; \beta) &= 80640\lambda_i^2(3964 + 4144\beta + 1527\beta^2 + 195\beta^3) \geq 0 \text{ for all } \beta \geq -3.0390, \\ \xi_{02}^{(3)}(\lambda_i; \lambda_i; \beta) &= 120960\lambda_i(6584 + 4820\beta + 1314\beta^2 + 135\beta^3) \geq 0 \text{ for all } \beta \geq -3.5381. \end{aligned}$$

Hence, $\xi_0^{(2)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{01}(\lambda_j; \lambda_i; \beta) = \xi_0^{(1)}(\lambda_j; \lambda_j; \lambda_i; \beta)/\lambda_j^2$. We wish to show that ξ_{01} is non-negative

for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{01} with respect to λ_j gives

$$\begin{aligned}
\xi_{01}(\lambda_j; \lambda_i; \beta) &= (11397120 + 5591040\beta + 1048320\beta^2 + 80640\beta^3)\lambda_j^4 \\
&\quad + (2983680\beta + 483840\beta^3 + 3144960\beta^2 - 12741120)\lambda_i\lambda_j^3 \\
&\quad + (1411200\beta + 6854400 + 604800\beta^3 + 2419200\beta^2)\lambda_i^2\lambda_j^2 \\
&\quad + (4085760\beta + 1128960\beta^2 + 2150400 + 161280\beta^3)\lambda_i^3\lambda_j \\
&\quad + (322560 + 564480\beta + 241920\beta^2)\lambda_i^4, \\
\xi_{01}^{(1)}(\lambda_j; \lambda_i; \beta) &= (45588480 + 22364160\beta + 4193280\beta^2 + 322560\beta^3)\lambda_j^3 \\
&\quad + (8951040\beta + 1451520\beta^3 + 9434880\beta^2 - 38223360)\lambda_i\lambda_j^2 \\
&\quad + (2822400\beta + 13708800 + 1209600\beta^3 + 4838400\beta^2)\lambda_i^2\lambda_j \\
&\quad + (4085760\beta + 1128960\beta^2 + 2150400 + 161280\beta^3)\lambda_i^3, \\
\xi_{01}^{(2)}(\lambda_j; \lambda_i; \beta) &= (136765440 + 67092480\beta + 12579840\beta^2 + 967680\beta^3)\lambda_j^2 \\
&\quad + (17902080\beta + 2903040\beta^3 + 18869760\beta^2 - 76446720)\lambda_i\lambda_j \\
&\quad + (2822400\beta + 13708800 + 1209600\beta^3 + 4838400\beta^2)\lambda_i^2, \\
\xi_{01}^{(3)}(\lambda_j; \lambda_i; \beta) &= (273530880 + 134184960\beta + 25159680\beta^2 + 1935360\beta^3)\lambda_j \\
&\quad + (17902080\beta + 2903040\beta^3 + 18869760\beta^2 - 76446720)\lambda_i, \\
\xi_{01}^{(4)}(\lambda_j; \lambda_i; \beta) &= 273530880 + 134184960\beta + 25159680\beta^2 + 1935360\beta^3 \\
&\geq 0 \text{ for all } \beta \geq -4.6109.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{01}(\lambda_i; \lambda_i; \beta) &= 1330560\lambda_i^4(\beta + 3)(\beta + 2)(\beta + 1) \geq 0 \text{ for all } \beta \geq -1, \\
\xi_{01}^{(1)}(\lambda_i; \lambda_i; \beta) &= 241920\lambda_i^3(13\beta + 16)(\beta + 3)(\beta + 2) \geq 0 \text{ for all } \beta \geq -\frac{16}{13}, \\
\xi_{01}^{(2)}(\lambda_i; \lambda_i; \beta) &= 725760\lambda_i^2(\beta + 3)(7\beta^2 + 29\beta + 34) \geq 0 \text{ for all } \beta \geq -3, \\
\xi_{01}^{(3)}(\lambda_i; \lambda_i; \beta) &= 161280\lambda_i(1222 + 943\beta + 273\beta^2 + 30\beta^3) \geq 0 \text{ for all } \beta \geq -3.3405.
\end{aligned}$$

Hence, $\xi_0^{(1)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{00}(\lambda_j; \lambda_i; \beta) = \xi_0(\lambda_j; \lambda_j; \lambda_i; \beta)/\lambda_j^3$. We wish to show that ξ_{00} is non-negative for

all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{00} with respect to λ_j gives

$$\begin{aligned} \xi_{00}(\lambda_j; \lambda_i; \beta) &= (2442240 + 1198080\beta + 224640\beta^2 + 17280\beta^3)\lambda_j^4 \\ &\quad + (745920\beta + 120960\beta^3 + 786240\beta^2 - 3185280)\lambda_i\lambda_j^3 \\ &\quad + (423360\beta + 2056320 + 181440\beta^3 + 725760\beta^2)\lambda_i^2\lambda_j^2 \\ &\quad + (1532160\beta + 423360\beta^2 + 806400 + 60480\beta^3)\lambda_i^3\lambda_j \\ &\quad + (161280 + 282240\beta + 120960\beta^2)\lambda_i^4, \\ \xi_{00}^{(1)}(\lambda_j; \lambda_i; \beta) &= (9768960 + 4792320\beta + 898560\beta^2 + 69120\beta^3)\lambda_j^3 \\ &\quad + (2237760\beta + 362880\beta^3 + 2358720\beta^2 - 9555840)\lambda_i\lambda_j^2 \\ &\quad + (846720\beta + 4112640 + 362880\beta^3 + 1451520\beta^2)\lambda_i^2\lambda_j \\ &\quad + (1532160\beta + 423360\beta^2 + 806400 + 60480\beta^3)\lambda_i^3, \\ \xi_{00}^{(2)}(\lambda_j; \lambda_i; \beta) &= (29306880 + 14376960\beta + 2695680\beta^2 + 207360\beta^3)\lambda_j^2 \\ &\quad + (4475520\beta + 725760\beta^3 + 4717440\beta^2 - 19111680)\lambda_i\lambda_j \\ &\quad + (846720\beta + 4112640 + 362880\beta^3 + 1451520\beta^2)\lambda_i^2, \\ \xi_{00}^{(3)}(\lambda_j; \lambda_i; \beta) &= (58613760 + 28753920\beta + 5391360\beta^2 + 414720\beta^3)\lambda_j \\ &\quad + (4475520\beta + 725760\beta^3 + 4717440\beta^2 - 19111680)\lambda_i, \\ \xi_{00}^{(4)}(\lambda_j; \lambda_i; \beta) &= 58613760 + 28753920\beta + 5391360\beta^2 + 414720\beta^3 \\ &\geq 0 \text{ for all } \beta \geq -4.6109. \end{aligned}$$

Now

$$\begin{aligned} \xi_{00}(\lambda_i; \lambda_i; \beta) &= 380160\lambda_i^4(\beta + 3)(\beta + 2)(\beta + 1) > 0 \text{ for all } \beta > -1, \\ \xi_{00}^{(1)}(\lambda_i; \lambda_i; \beta) &= 855360\lambda_i^3(\beta + 3)(\beta + 2)(\beta + 1) > 0 \text{ for all } \beta > -1, \\ \xi_{00}^{(2)}(\lambda_i; \lambda_i; \beta) &= 51840\lambda_i^2(25\beta + 46)(\beta + 3)(\beta + 2) > 0 \text{ for all } \beta > -\frac{46}{25}, \\ \xi_{00}^{(3)}(\lambda_i; \lambda_i; \beta) &= 51840\lambda_i(\beta + 3)(22\beta^2 + 129\beta + 254) > 0 \text{ for all } \beta > -3. \end{aligned}$$

Hence, ξ_0 is positive for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$. This implies that $\xi(\lambda_l; \lambda_l; \lambda_j; \lambda_i; \beta)$ is positive for all $\lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Differentiating $\xi_1(\lambda_i; \lambda_j; \lambda_i; \beta)$ with respect to λ_i we obtain

$$\begin{aligned} \xi_1(\lambda_i; \lambda_j; \lambda_i; \beta) &= (134400\beta + 94080 + 10080\beta^2)\lambda_i^6 \\ &\quad + (4320\beta^3 + 650880 + 125280\beta^2 + 616320\beta)(\lambda_j + \lambda_i)\lambda_i^5 \\ &\quad + (216000\beta^2 + 21600\beta^3 + 957600\beta + 3096000)(\lambda_j^2 + \lambda_i^2)\lambda_i^4 \\ &\quad + (-3628800 + 64800\beta^3 + 388800\beta + 583200\beta^2)\lambda_i\lambda_j\lambda_i^4 \\ &\quad + (-201600\beta + 518400\beta^2 + 129600\beta^3 - 2016000)(\lambda_i\lambda_j^2 + \lambda_i^2\lambda_j)\lambda_i^3 \\ &\quad + (14400\beta^3 + 921600\beta + 1670400 + 129600\beta^2)(\lambda_j^3 + \lambda_i^3)\lambda_i^3 \\ &\quad + (-1301760 + 607680\beta + 43200\beta^3 + 280800\beta^2)(\lambda_i\lambda_j^3 + \lambda_i^3\lambda_j)\lambda_i^2 \\ &\quad + (165600\beta + 43200\beta^2 + 187200)(\lambda_j^4 + \lambda_i^4)\lambda_i^2 \\ &\quad + (1710720 - 466560\beta + 129600\beta^3 + 324000\beta^2)\lambda_i^2\lambda_j^2\lambda_i^2 \\ &\quad + (380160\beta + 138240\beta^2 + 587520 + 21600\beta^3)(\lambda_i^2\lambda_j^3 + \lambda_i^3\lambda_j^2)\lambda_i \\ &\quad + (80640\beta - 74880 + 64800\beta^2)(\lambda_i\lambda_j^4 + \lambda_i^4\lambda_j)\lambda_i \\ &\quad + (133440\beta + 1440\beta^3 + 119040 + 15840\beta^2)\lambda_i^3\lambda_j^3 \\ &\quad + (12960\beta^2 + 48960 + 36000\beta)(\lambda_i^2\lambda_j^4 + \lambda_i^4\lambda_j^2), \end{aligned}$$

and, hence,

$$\begin{aligned} \xi_1^{(5)}(\lambda_i; \lambda_j; \lambda_i; \beta) &= (96768000\beta + 67737600 + 7257600\beta^2)\lambda_i \\ &\quad + (518400\beta^3 + 78105600 + 15033600\beta^2 + 73958400\beta)(\lambda_i + \lambda_j) \\ &\geq 0 \text{ for all } \beta \geq -0.7412. \end{aligned}$$

Define $\xi_{14}(\lambda_j; \lambda_i; \beta) = \xi_1^{(4)}(\lambda_i; \lambda_j; \lambda_i; \beta)$. We wish to show that ξ_{14} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{14} with respect to λ_j gives

$$\begin{aligned} \xi_{14}(\lambda_j; \lambda_i; \beta) &= (145324800\beta + 186278400 + 23846400\beta^2 + 1036800\beta^3)\lambda_j^2 \\ &\quad + (2073600\beta^3 - 8985600 + 29030400\beta^2 + 83289600\beta)\lambda_i\lambda_j \\ &\quad + (5184000\beta^2 + 518400\beta^3 + 22982400\beta + 74304000)\lambda_i^2, \\ \xi_{14}^{(1)}(\lambda_j; \lambda_i; \beta) &= (290649600\beta + 372556800 + 47692800\beta^2 + 2073600\beta^3)\lambda_j \\ &\quad + (2073600\beta^3 - 8985600 + 29030400\beta^2 + 83289600\beta)\lambda_i, \\ \xi_{14}^{(2)}(\lambda_j; \lambda_i; \beta) &= 290649600\beta + 372556800 + 47692800\beta^2 + 2073600\beta^3 \\ &\geq 0 \text{ for all } \beta \geq -1.7421. \end{aligned}$$

Now

$$\begin{aligned} \xi_{14}(\lambda_i; \lambda_i; \beta) &= 1209600\lambda_i^2(208\beta + 208 + 48\beta^2 + 3\beta^3) \geq 0 \text{ for all } \beta \geq -1.4294, \\ \xi_{14}^{(1)}(\lambda_i; \lambda_i; \beta) &= 691200\lambda_i(541\beta + 526 + 111\beta^2 + 6\beta^3) \geq 0 \text{ for all } \beta \geq -1.2898. \end{aligned}$$

Hence, $\xi_1^{(4)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{13}(\lambda_j; \lambda_i; \beta) = \xi_1^{(3)}(\lambda_j; \lambda_j; \lambda_i; \beta)$. We wish to show that ξ_{13} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{13} with respect to λ_j gives

$$\begin{aligned} \xi_{13}(\lambda_j; \lambda_i; \beta) &= (81619200\beta + 134668800 + 14688000\beta^2 + 864000\beta^3)\lambda_j^3 \\ &\quad + (-60134400 + 2592000\beta^3 + 45100800\beta + 24624000\beta^2)\lambda_i\lambda_j^2 \\ &\quad + (8294400\beta^2 + 1296000\beta^3 + 21772800\beta + 62208000)\lambda_i^2\lambda_j \\ &\quad + (86400\beta^3 + 5529600\beta + 10022400 + 777600\beta^2)\lambda_i^3, \\ \xi_{13}^{(1)}(\lambda_j; \lambda_i; \beta) &= (244857600\beta + 404006400 + 44064000\beta^2 + 2592000\beta^3)\lambda_j^2 \\ &\quad + (-120268800 + 5184000\beta^3 + 90201600\beta + 49248000\beta^2)\lambda_i\lambda_j \\ &\quad + (8294400\beta^2 + 1296000\beta^3 + 21772800\beta + 62208000)\lambda_i^2, \\ \xi_{13}^{(2)}(\lambda_j; \lambda_i; \beta) &= (489715200\beta + 808012800 + 88128000\beta^2 + 5184000\beta^3)\lambda_j \\ &\quad + (-120268800 + 5184000\beta^3 + 90201600\beta + 49248000\beta^2)\lambda_i, \\ \xi_{13}^{(3)}(\lambda_j; \lambda_i; \beta) &= 489715200\beta + 808012800 + 88128000\beta^2 + 5184000\beta^3 \\ &\geq 0 \text{ for all } \beta \geq -2.9237. \end{aligned}$$

Now

$$\begin{aligned} \xi_{13}(\lambda_i; \lambda_i; \beta) &= 806400\lambda_i^3(191\beta + 182 + 60\beta^2 + 6\beta^3) \geq 0 \text{ for all } \beta \geq -1.7321, \\ \xi_{13}^{(1)}(\lambda_i; \lambda_i; \beta) &= 604800\lambda_i^2(590\beta + 572 + 168\beta^2 + 15\beta^3) \geq 0 \text{ for all } \beta \geq -1.5802, \\ \xi_{13}^{(2)}(\lambda_i; \lambda_i; \beta) &= 172800\lambda_i(3356\beta + 3980 + 795\beta^2 + 60\beta^3) \geq 0 \text{ for all } \beta \geq -1.9649. \end{aligned}$$

Hence, $\xi_1^{(3)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{12}(\lambda_j; \lambda_i; \beta) = \xi_1^{(2)}(\lambda_j; \lambda_j; \lambda_i; \beta)$. We wish to show that ξ_{12} is non-negative for

all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{12} with respect to λ_j gives

$$\begin{aligned}
\xi_{12}(\lambda_j; \lambda_i; \beta) &= (33710400\beta + 63388800 + 6264000\beta^2 + 432000\beta^3)\lambda_j^4 \\
&\quad + (-45227520 + 1728000\beta^3 + 16997760\beta + 13176000\beta^2)\lambda_i\lambda_j^3 \\
&\quad + (6350400\beta^2 + 1296000\beta^3 + 9348480\beta + 28477440)\lambda_i^2\lambda_j^2 \\
&\quad + (7418880 + 6744960\beta + 172800\beta^3 + 1339200\beta^2)\lambda_i^3\lambda_j \\
&\quad + (331200\beta + 86400\beta^2 + 374400)\lambda_i^4, \\
\xi_{12}^{(1)}(\lambda_j; \lambda_i; \beta) &= (134841600\beta + 253555200 + 25056000\beta^2 + 1728000\beta^3)\lambda_j^3 \\
&\quad + (-135682560 + 5184000\beta^3 + 50993280\beta + 39528000\beta^2)\lambda_i\lambda_j^2 \\
&\quad + (12700800\beta^2 + 2592000\beta^3 + 18696960\beta + 56954880)\lambda_i^2\lambda_j \\
&\quad + (7418880 + 6744960\beta + 172800\beta^3 + 1339200\beta^2)\lambda_i^3, \\
\xi_{12}^{(2)}(\lambda_j; \lambda_i; \beta) &= (404524800\beta + 760665600 + 75168000\beta^2 + 5184000\beta^3)\lambda_j^2 \\
&\quad + (-271365120 + 10368000\beta^3 + 101986560\beta + 79056000\beta^2)\lambda_i\lambda_j \\
&\quad + (12700800\beta^2 + 2592000\beta^3 + 18696960\beta + 56954880)\lambda_i^2, \\
\xi_{12}^{(3)}(\lambda_j; \lambda_i; \beta) &= (809049600\beta + 1521331200 + 150336000\beta^2 + 10368000\beta^3)\lambda_j \\
&\quad + (-271365120 + 10368000\beta^3 + 101986560\beta + 79056000\beta^2)\lambda_i, \\
\xi_{12}^{(4)}(\lambda_j; \lambda_i; \beta) &= 809049600\beta + 1521331200 + 150336000\beta^2 + 10368000\beta^3 \\
&\geq 0 \text{ for all } \beta \geq -4.2760.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{12}(\lambda_i; \lambda_i; \beta) &= 1814400\lambda_i^4(2\beta + 5)(\beta + 3)(\beta + 2) \geq 0 \text{ for all } \beta \geq -2, \\
\xi_{12}^{(1)}(\lambda_i; \lambda_i; \beta) &= 403200\lambda_i^3(524\beta + 452 + 195\beta^2 + 24\beta^3) \geq 0 \text{ for all } \beta \geq -1.8116, \\
\xi_{12}^{(2)}(\lambda_i; \lambda_i; \beta) &= 241920\lambda_i^2(2171\beta + 2258 + 690\beta^2 + 75\beta^3) \geq 0 \text{ for all } \beta \geq -2.3123, \\
\xi_{12}^{(3)}(\lambda_i; \lambda_i; \beta) &= 51840\lambda_i(17574\beta + 24112 + 4425\beta^2 + 400\beta^3) \geq 0 \text{ for all } \beta \geq -3.2529.
\end{aligned}$$

Hence, $\xi_1^{(2)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{11}(\lambda_j; \lambda_i; \beta) = \xi_1^{(1)}(\lambda_j; \lambda_j; \lambda_i; \beta)/\lambda_j$. We wish to show that ξ_{11} is non-negative

for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{11} with respect to λ_j gives

$$\begin{aligned} \xi_{11}(\lambda_j; \lambda_i; \beta) &= (10814400\beta + 21588480 + 2026080\beta^2 + 151200\beta^3)\lambda_j^4 \\ &\quad + (-19987200 + 756000\beta^3 + 5328000\beta + 5140800\beta^2)\lambda_i\lambda_j^3 \\ &\quad + (3205440\beta^2 + 756000\beta^3 + 2672640\beta + 10344960)\lambda_i^2\lambda_j^2 \\ &\quad + (151200\beta^3 + 4360320\beta + 2995200 + 1088640\beta^2)\lambda_i^3\lambda_j \\ &\quad + (411840\beta + 151200\beta^2 + 299520)\lambda_i^4, \\ \xi_{11}^{(1)}(\lambda_j; \lambda_i; \beta) &= (43257600\beta + 86353920 + 8104320\beta^2 + 604800\beta^3)\lambda_j^3 \\ &\quad + (-59961600 + 2268000\beta^3 + 15984000\beta + 15422400\beta^2)\lambda_i\lambda_j^2 \\ &\quad + (6410880\beta^2 + 1512000\beta^3 + 5345280\beta + 20689920)\lambda_i^2\lambda_j \\ &\quad + (151200\beta^3 + 4360320\beta + 2995200 + 1088640\beta^2)\lambda_i^3, \\ \xi_{11}^{(2)}(\lambda_j; \lambda_i; \beta) &= (129772800\beta + 259061760 + 24312960\beta^2 + 1814400\beta^3)\lambda_j^2 \\ &\quad + (-119923200 + 4536000\beta^3 + 31968000\beta + 30844800\beta^2)\lambda_i\lambda_j \\ &\quad + (6410880\beta^2 + 1512000\beta^3 + 5345280\beta + 20689920)\lambda_i^2, \\ \xi_{11}^{(3)}(\lambda_j; \lambda_i; \beta) &= (259545600\beta + 518123520 + 48625920\beta^2 + 3628800\beta^3)\lambda_j \\ &\quad + (-119923200 + 4536000\beta^3 + 31968000\beta + 30844800\beta^2)\lambda_i, \\ \xi_{11}^{(4)}(\lambda_j; \lambda_i; \beta) &= 259545600\beta + 518123520 + 48625920\beta^2 + 3628800\beta^3 \\ &\geq 0 \text{ for all } \beta \geq -4.5982. \end{aligned}$$

Now

$$\begin{aligned} \xi_{11}(\lambda_i; \lambda_i; \beta) &= 362880\lambda_i^4(5\beta + 7)(\beta + 3)(\beta + 2) \geq 0 \text{ for all } \beta \geq -\frac{7}{5}, \\ \xi_{11}^{(1)}(\lambda_i; \lambda_i; \beta) &= 181440\lambda_i^3(25\beta + 46)(\beta + 3)(\beta + 2) \geq 0 \text{ for all } \beta \geq -\frac{46}{25}, \\ \xi_{11}^{(2)}(\lambda_i; \lambda_i; \beta) &= 40320\lambda_i^2(4144\beta + 3964 + 1527\beta^2 + 195\beta^3) \geq 0 \text{ for all } \beta \geq -3.0390, \\ \xi_{11}^{(3)}(\lambda_i; \lambda_i; \beta) &= 60480\lambda_i(4820\beta + 6584 + 1314\beta^2 + 135\beta^3) \geq 0 \text{ for all } \beta \geq -3.5381. \end{aligned}$$

Hence, $\xi_1^{(1)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{10}(\lambda_j; \lambda_i; \beta) = \xi_1(\lambda_j; \lambda_j; \lambda_i; \beta)/\lambda_j^2$. We wish to show that ξ_{10} is non-negative for

all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{10} with respect to λ_j gives

$$\begin{aligned} \xi_{10}(\lambda_j; \lambda_i; \beta) &= (2795520\beta + 5698560 + 524160\beta^2 + 40320\beta^3)\lambda_j^4 \\ &\quad + (-6370560 + 241920\beta^3 + 1491840\beta + 1572480\beta^2)\lambda_i\lambda_j^3 \\ &\quad + (1209600\beta^2 + 302400\beta^3 + 705600\beta + 3427200)\lambda_i^2\lambda_j^2 \\ &\quad + (80640\beta^3 + 2042880\beta + 1075200 + 564480\beta^2)\lambda_i^3\lambda_j \\ &\quad + (282240\beta + 120960\beta^2 + 161280)\lambda_i^4, \\ \xi_{10}^{(1)}(\lambda_j; \lambda_i; \beta) &= (11182080\beta + 22794240 + 2096640\beta^2 + 161280\beta^3)\lambda_j^3 \\ &\quad + (-19111680 + 725760\beta^3 + 4475520\beta + 4717440\beta^2)\lambda_i\lambda_j^2 \\ &\quad + (2419200\beta^2 + 604800\beta^3 + 1411200\beta + 6854400)\lambda_i^2\lambda_j \\ &\quad + (80640\beta^3 + 2042880\beta + 1075200 + 564480\beta^2)\lambda_i^3, \\ \xi_{10}^{(2)}(\lambda_j; \lambda_i; \beta) &= (33546240\beta + 68382720 + 6289920\beta^2 + 483840\beta^3)\lambda_j^2 \\ &\quad + (-38223360 + 1451520\beta^3 + 8951040\beta + 9434880\beta^2)\lambda_i\lambda_j \\ &\quad + (2419200\beta^2 + 604800\beta^3 + 1411200\beta + 6854400)\lambda_i^2, \\ \xi_{10}^{(3)}(\lambda_j; \lambda_i; \beta) &= (67092480\beta + 136765440 + 12579840\beta^2 + 967680\beta^3)\lambda_j \\ &\quad + (-38223360 + 1451520\beta^3 + 8951040\beta + 9434880\beta^2)\lambda_i, \\ \xi_{10}^{(4)}(\lambda_j; \lambda_i; \beta) &= 67092480\beta + 136765440 + 12579840\beta^2 + 967680\beta^3 \\ &\geq 0 \text{ for all } \beta \geq -4.6109. \end{aligned}$$

Now

$$\begin{aligned} \xi_{10}(\lambda_i; \lambda_i; \beta) &= 665280\lambda_i^4(\beta + 3)(\beta + 2)(\beta + 1) > 0 \text{ for all } \beta > -1, \\ \xi_{10}^{(1)}(\lambda_i; \lambda_i; \beta) &= 120960\lambda_i^3(13\beta + 16)(\beta + 3)(\beta + 2) > 0 \text{ for all } \beta > -\frac{16}{13}, \\ \xi_{10}^{(2)}(\lambda_i; \lambda_i; \beta) &= 362880\lambda_i^2(\beta + 3)(7\beta^2 + 29\beta + 34) > 0 \text{ for all } \beta > -3, \\ \xi_{10}^{(3)}(\lambda_i; \lambda_i; \beta) &= 80640\lambda_i(943\beta + 1222 + 273\beta^2 + 30\beta^3) > 0 \text{ for all } \beta > -3.3405. \end{aligned}$$

Hence, ξ_1 is positive for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$. This implies that $\xi(\lambda_l; \lambda_l; \lambda_j; \lambda_i; \beta)$ is positive for all $\lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Now

$$\begin{aligned} \xi_2(\lambda_i; \lambda_j; \lambda_i; \beta) = & (25920\beta^2 + 345600\beta + 241920)\lambda_i^5 \\ & +(8640\beta^3 + 276480\beta^2 + 1399680 + 1304640\beta)(\lambda_j + \lambda_i)\lambda_i^4 \\ & +(5149440 + 1676160\beta + 397440\beta^2 + 34560\beta^3)(\lambda_j^2 + \lambda_i^2)\lambda_i^3 \\ & +(-6497280 + 1088640\beta^2 + 103680\beta^3 + 656640\beta)\lambda_i\lambda_j\lambda_i^3 \\ & +(155520\beta^3 - 2730240 + 224640\beta + 777600\beta^2)(\lambda_i\lambda_j^2 + \lambda_i^2\lambda_j)\lambda_i^2 \\ & +(2016000 + 17280\beta^3 + 172800\beta^2 + 1342080\beta)(\lambda_j^3 + \lambda_i^3)\lambda_i^2 \\ & +(34560\beta^2 + 178560 + 187200\beta)(\lambda_j^4 + \lambda_i^4)\lambda_i \\ & +(250560\beta^2 + 34560\beta^3 - 391680 + 938880\beta)(\lambda_i\lambda_j^3 + \lambda_i^3\lambda_j)\lambda_i \\ & +(103680\beta^3 + 3421440 + 311040\beta + 336960\beta^2)\lambda_i^2\lambda_j^2\lambda_i \\ & +(60480\beta^2 + 311040\beta + 1140480 + 8640\beta^3)(\lambda_i^3\lambda_j^2 + \lambda_i^2\lambda_j^3) \\ & +(25920\beta^2 + 97920 + 72000\beta)(\lambda_i\lambda_j^4 + \lambda_i^4\lambda_j), \end{aligned}$$

and, hence,

$$\begin{aligned} \xi_2^{(4)}(\lambda_i; \lambda_j; \lambda_i; \beta) = & (3110400\beta^2 + 41472000\beta + 29030400)\lambda_i \\ & +(207360\beta^3 + 6635520\beta^2 + 33592320 + 31311360\beta)(\lambda_i + \lambda_j) \\ \geq & 0 \text{ for all } \beta \geq -0.7412. \end{aligned}$$

Define $\xi_{23}(\lambda_j; \lambda_i; \beta) = \xi_2^{(3)}(\lambda_j; \lambda_j; \lambda_i; \beta)$. We wish to show that ξ_{23} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{23} with respect to λ_j gives

$$\begin{aligned} \xi_{23}(\lambda_j; \lambda_i; \beta) = & (10575360\beta^2 + 62104320\beta + 79004160 + 414720\beta^3)\lambda_j^2 \\ & +(829440\beta^3 + 13167360\beta^2 - 5391360 + 35251200\beta)\lambda_i\lambda_j \\ & +(30896640 + 10056960\beta + 2384640\beta^2 + 207360\beta^3)\lambda_i^2, \\ \xi_{23}^{(1)}(\lambda_j; \lambda_i; \beta) = & (21150720\beta^2 + 124208640\beta + 158008320 + 829440\beta^3)\lambda_j \\ & +(829440\beta^3 + 13167360\beta^2 - 5391360 + 35251200\beta)\lambda_i, \\ \xi_{23}^{(2)}(\lambda_j; \lambda_i; \beta) = & 21150720\beta^2 + 124208640\beta + 158008320 + 829440\beta^3 \geq 0 \text{ for all } \beta \geq -1.7669. \end{aligned}$$

Now

$$\begin{aligned} \xi_{23}(\lambda_i; \lambda_i; \beta) = & 1451520\lambda_i^2(\beta + 4)(\beta^2 + 14\beta + 18) \geq 0 \text{ for all } \beta \geq -1.4322, \\ \xi_{23}^{(1)}(\lambda_i; \lambda_i; \beta) = & 103680\lambda_i(331\beta^2 + 1538\beta + 1472 + 16\beta^3) \geq 0 \text{ for all } \beta \geq -1.2958. \end{aligned}$$

Hence, $\xi_2^{(3)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{22}(\lambda_j; \lambda_i; \beta) = \xi_2^{(2)}(\lambda_j; \lambda_j; \lambda_i; \beta)$. We wish to show that ξ_{22} is non-negative for

all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{22} with respect to λ_j gives

$$\begin{aligned}\xi_{22}(\lambda_j; \lambda_i; \beta) &= (6566400\beta^2 + 35308800\beta + 56563200 + 345600\beta^3)\lambda_j^3 \\ &\quad + (-27648000 + 11404800\beta^2 + 1036800\beta^3 + 20044800\beta)\lambda_i\lambda_j^2 \\ &\quad + (25436160 + 10506240\beta + 3939840\beta^2 + 518400\beta^3)\lambda_i^2\lambda_j \\ &\quad + (4032000 + 34560\beta^3 + 345600\beta^2 + 2684160\beta)\lambda_i^3, \\ \xi_{22}^{(1)}(\lambda_j; \lambda_i; \beta) &= (19699200\beta^2 + 105926400\beta + 169689600 + 1036800\beta^3)\lambda_j^2 \\ &\quad + (-55296000 + 22809600\beta^2 + 2073600\beta^3 + 40089600\beta)\lambda_i\lambda_j \\ &\quad + (25436160 + 10506240\beta + 3939840\beta^2 + 518400\beta^3)\lambda_i^2, \\ \xi_{22}^{(2)}(\lambda_j; \lambda_i; \beta) &= (39398400\beta^2 + 211852800\beta + 339379200 + 2073600\beta^3)\lambda_j \\ &\quad + (-55296000 + 22809600\beta^2 + 2073600\beta^3 + 40089600\beta)\lambda_i, \\ \xi_{22}^{(3)}(\lambda_j; \lambda_i; \beta) &= 39398400\beta^2 + 211852800\beta + 339379200 + 2073600\beta^3 \geq 0 \text{ for all } \beta \geq -3.0812.\end{aligned}$$

Now

$$\begin{aligned}\xi_{22}(\lambda_i; \lambda_i; \beta) &= 161280\lambda_i^3(138\beta^2 + 425\beta + 362 + 12\beta^3) \geq 0 \text{ for all } \beta \geq -1.4432, \\ \xi_{22}^{(1)}(\lambda_i; \lambda_i; \beta) &= 241920\lambda_i^2(192\beta^2 + 647\beta + 578 + 15\beta^3) \geq 0 \text{ for all } \beta \geq -1.4382, \\ \xi_{22}^{(2)}(\lambda_i; \lambda_i; \beta) &= 1036800\lambda_i(60\beta^2 + 243\beta + 274 + 4\beta^3) \geq 0 \text{ for all } \beta \geq -1.9255.\end{aligned}$$

Hence, $\xi_2^{(2)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{21}(\lambda_j; \lambda_i; \beta) = \xi_2^{(1)}(\lambda_j; \lambda_j; \lambda_i; \beta)$. We wish to show that ξ_{21} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{21} with respect to λ_j gives

$$\begin{aligned}\xi_{21}(\lambda_j; \lambda_i; \beta) &= (2808000\beta^2 + 14846400\beta + 26467200 + 172800\beta^3)\lambda_j^4 \\ &\quad + (691200\beta^3 + 6177600\beta^2 - 19745280 + 8576640\beta)\lambda_i\lambda_j^3 \\ &\quad + (13409280 + 5788800\beta + 3084480\beta^2 + 518400\beta^3)\lambda_i^2\lambda_j^2 \\ &\quad + (3640320 + 69120\beta^3 + 596160\beta^2 + 3623040\beta)\lambda_i^3\lambda_j \\ &\quad + (34560\beta^2 + 178560 + 187200\beta)\lambda_i^4, \\ \xi_{21}^{(1)}(\lambda_j; \lambda_i; \beta) &= (11232000\beta^2 + 59385600\beta + 105868800 + 691200\beta^3)\lambda_j^3 \\ &\quad + (2073600\beta^3 + 18532800\beta^2 - 59235840 + 25729920\beta)\lambda_i\lambda_j^2 \\ &\quad + (26818560 + 11577600\beta + 6168960\beta^2 + 1036800\beta^3)\lambda_i^2\lambda_j \\ &\quad + (3640320 + 69120\beta^3 + 596160\beta^2 + 3623040\beta)\lambda_i^3, \\ \xi_{21}^{(2)}(\lambda_j; \lambda_i; \beta) &= (33696000\beta^2 + 178156800\beta + 317606400 + 2073600\beta^3)\lambda_j^2 \\ &\quad + (4147200\beta^3 + 37065600\beta^2 - 118471680 + 51459840\beta)\lambda_i\lambda_j \\ &\quad + (26818560 + 11577600\beta + 6168960\beta^2 + 1036800\beta^3)\lambda_i^2, \\ \xi_{21}^{(3)}(\lambda_j; \lambda_i; \beta) &= (67392000\beta^2 + 356313600\beta + 635212800 + 4147200\beta^3)\lambda_j \\ &\quad + (4147200\beta^3 + 37065600\beta^2 - 118471680 + 51459840\beta)\lambda_i, \\ \xi_{21}^{(4)}(\lambda_j; \lambda_i; \beta) &= 67392000\beta^2 + 356313600\beta + 635212800 + 4147200\beta^3 \\ &\geq 0 \text{ for all } \beta \geq -7.5855.\end{aligned}$$

Now

$$\begin{aligned} \xi_{21}(\lambda_i; \lambda_i; \beta) &= 362880\lambda_i^4(\beta + 3)(4\beta^2 + 23\beta + 22) \geq 0 \text{ for all } \beta \geq -1.2120, \\ \xi_{21}^{(1)}(\lambda_i; \lambda_i; \beta) &= 80640\lambda_i^3(453\beta^2 + 1244\beta + 956 + 48\beta^3) \geq 0 \text{ for all } \beta \geq -1.2965, \\ \xi_{21}^{(2)}(\lambda_i; \lambda_i; \beta) &= 241920\lambda_i^2(318\beta^2 + 997\beta + 934 + 30\beta^3) \geq 0 \text{ for all } \beta \geq -1.7649, \\ \xi_{21}^{(3)}(\lambda_i; \lambda_i; \beta) &= 67392000\beta^2 + 356313600\beta + 635212800 + 4147200\beta^3 \geq 0 \text{ for all } \beta \geq -7.5855. \end{aligned}$$

Hence, $\xi_2^{(1)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{20}(\lambda_j; \lambda_i; \beta) = \xi_2(\lambda_j; \lambda_j; \lambda_i; \beta)/\lambda_j$. We wish to show that ξ_{20} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{20} with respect to λ_j gives

$$\begin{aligned} \xi_{20}(\lambda_j; \lambda_i; \beta) &= (907200\beta^2 + 4855680\beta + 8985600 + 60480\beta^3)\lambda_j^4 \\ &\quad + (-8121600 + 2419200\beta^2 + 302400\beta^3 + 3196800\beta)\lambda_i\lambda_j^3 \\ &\quad + (6981120 + 2522880\beta + 1572480\beta^2 + 302400\beta^3)\lambda_i^2\lambda_j^2 \\ &\quad + (2764800 + 60480\beta^3 + 483840\beta^2 + 2592000\beta)\lambda_i^3\lambda_j \\ &\quad + (60480\beta^2 + 276480 + 259200\beta)\lambda_i^4, \\ \xi_{20}^{(1)}(\lambda_j; \lambda_i; \beta) &= (3628800\beta^2 + 19422720\beta + 35942400 + 241920\beta^3)\lambda_j^3 \\ &\quad + (-24364800 + 7257600\beta^2 + 907200\beta^3 + 9590400\beta)\lambda_i\lambda_j^2 \\ &\quad + (13962240 + 5045760\beta + 3144960\beta^2 + 604800\beta^3)\lambda_i^2\lambda_j \\ &\quad + (2764800 + 60480\beta^3 + 483840\beta^2 + 2592000\beta)\lambda_i^3, \\ \xi_{20}^{(2)}(\lambda_j; \lambda_i; \beta) &= (10886400\beta^2 + 58268160\beta + 107827200 + 725760\beta^3)\lambda_j^2 \\ &\quad + (-48729600 + 14515200\beta^2 + 1814400\beta^3 + 19180800\beta)\lambda_i\lambda_j \\ &\quad + (13962240 + 5045760\beta + 3144960\beta^2 + 604800\beta^3)\lambda_i^2, \\ \xi_{20}^{(3)}(\lambda_j; \lambda_i; \beta) &= (21772800\beta^2 + 116536320\beta + 215654400 + 1451520\beta^3)\lambda_j \\ &\quad + (-48729600 + 14515200\beta^2 + 1814400\beta^3 + 19180800\beta)\lambda_i, \\ \xi_{20}^{(4)}(\lambda_j; \lambda_i; \beta) &= 21772800\beta^2 + 116536320\beta + 215654400 + 1451520\beta^3 \\ &\geq 0 \text{ for all } \beta \geq -4.4853. \end{aligned}$$

Now

$$\begin{aligned} \xi_{20}(\lambda_i; \lambda_i; \beta) &= 362880\lambda_i^4(2\beta + 5)(\beta + 3)(\beta + 2) > 0 \text{ for all } \beta > -2, \\ \xi_{20}^{(1)}(\lambda_i; \lambda_i; \beta) &= 362880\lambda_i^3(\beta + 3)(5\beta^2 + 25\beta + 26) > 0 \text{ for all } \beta > -1.4753, \\ \xi_{20}^{(2)}(\lambda_i; \lambda_i; \beta) &= 241920\lambda_i^2(118\beta^2 + 341\beta + 302 + 13\beta^3) > 0 \text{ for all } \beta > -1.6897, \\ \xi_{20}^{(3)}(\lambda_i; \lambda_i; \beta) &= 362880\lambda_i(100\beta^2 + 374\beta + 460 + 9\beta^3) > 0 \text{ for all } \beta > -2.7709. \end{aligned}$$

Hence, ξ_2 is positive for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$. This implies that $\xi(\lambda_l; \lambda_l; \lambda_j; \lambda_i; \beta)$ is positive for all $\lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Now

$$\begin{aligned}
\xi_3(\lambda_l; \lambda_j; \lambda_i; \beta) = & (403200 + 576000\beta + 43200\beta^2)\lambda_l^4 \\
& + (1889280 + 8640\beta^3 + 1664640\beta + 406080\beta^2)(\lambda_i + \lambda_j)\lambda_l^3 \\
& + (4596480 + 1797120\beta + 492480\beta^2 + 25920\beta^3)(\lambda_j^2 + \lambda_i^2)\lambda_l^2 \\
& + (77760\beta^3 + 622080\beta - 7464960 + 1399680\beta^2)\lambda_i\lambda_j\lambda_l^2 \\
& + (1036800 + 1261440\beta + 8640\beta^3 + 129600\beta^2)(\lambda_j^3 + \lambda_i^3)\lambda_l \\
& + (777600\beta^2 - 2142720 + 1278720\beta + 77760\beta^3)(\lambda_i\lambda_j^2 + \lambda_i^2\lambda_j)\lambda_l \\
& + (714240 + 95040\beta^2 + 800640\beta + 8640\beta^3)(\lambda_i\lambda_j^3 + \lambda_i^3\lambda_j) \\
& + (25920\beta^3 + 181440\beta^2 + 933120\beta + 3421440)\lambda_i^2\lambda_j^2 \\
& + (8640\beta^2 + 115200\beta + 80640)(\lambda_j^4 + \lambda_i^4),
\end{aligned}$$

and, hence,

$$\begin{aligned}
\xi_3^{(3)}(\lambda_l; \lambda_j; \lambda_i; \beta) = & (9676800 + 13824000\beta + 1036800\beta^2)\lambda_l \\
& + (11335680 + 51840\beta^3 + 9987840\beta + 2436480\beta^2)(\lambda_j + \lambda_i) \\
\geq & 0 \text{ for all } \beta \geq -0.7412.
\end{aligned}$$

Define $\xi_{32}(\lambda_j; \lambda_i; \beta) = \xi_3^{(2)}(\lambda_j; \lambda_j; \lambda_i; \beta)$. We wish to show that ξ_{32} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{32} with respect to λ_j gives

$$\begin{aligned}
\xi_{32}(\lambda_j; \lambda_i; \beta) = & (25367040 + 20494080\beta + 3939840\beta^2 + 103680\beta^3)\lambda_j^2 \\
& + (-3594240 + 207360\beta^3 + 11232000\beta + 5235840\beta^2)\lambda_i\lambda_j \\
& + (9192960 + 3594240\beta + 984960\beta^2 + 51840\beta^3)\lambda_i^2, \\
\xi_{32}^{(1)}(\lambda_j; \lambda_i; \beta) = & (50734080 + 40988160\beta + 7879680\beta^2 + 207360\beta^3)\lambda_j \\
& + (-3594240 + 207360\beta^3 + 11232000\beta + 5235840\beta^2)\lambda_i, \\
\xi_{32}^{(2)}(\lambda_j; \lambda_i; \beta) = & 50734080 + 40988160\beta + 7879680\beta^2 + 207360\beta^3 \\
\geq & 0 \text{ for all } \beta \geq -1.8910.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{32}(\lambda_i; \lambda_i; \beta) = & 120960\lambda_i^2(256 + 292\beta + 84\beta^2 + 3\beta^3) \geq 0 \text{ for all } \beta \geq -1.4510, \\
\xi_{32}^{(1)}(\lambda_i; \lambda_i; \beta) = & 17280\lambda_i(2728 + 3022\beta + 759\beta^2 + 24\beta^3) \geq 0 \text{ for all } \beta \geq -1.3255.
\end{aligned}$$

Hence, $\xi_3^{(2)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{31}(\lambda_j; \lambda_i; \beta) = \xi_3^{(1)}(\lambda_j; \lambda_j; \lambda_i; \beta)$. We wish to show that ξ_{31} is non-negative for

all $\lambda_j \geq \lambda_i \geq 0$ for all $\beta \geq .$ Differentiating ξ_{31} with respect to λ_j gives

$$\begin{aligned} \xi_{31}(\lambda_j; \lambda_i; \beta) &= (17510400 + 12153600\beta + 2505600\beta^2 + 86400\beta^3)\lambda_j^3 \\ &\quad + (-11404800 + 259200\beta^3 + 7516800\beta + 4795200\beta^2)\lambda_i\lambda_j^2 \\ &\quad + (7050240 + 4872960\beta + 1762560\beta^2 + 129600\beta^3)\lambda_i^2\lambda_j \\ &\quad + (1036800 + 1261440\beta + 8640\beta^3 + 129600\beta^2)\lambda_i^3, \\ \xi_{31}^{(1)}(\lambda_j; \lambda_i; \beta) &= (52531200 + 36460800\beta + 7516800\beta^2 + 259200\beta^3)\lambda_j^2 \\ &\quad + (-22809600 + 518400\beta^3 + 15033600\beta + 9590400\beta^2)\lambda_i\lambda_j \\ &\quad + (7050240 + 4872960\beta + 1762560\beta^2 + 129600\beta^3)\lambda_i^2, \\ \xi_{31}^{(2)}(\lambda_j; \lambda_i; \beta) &= (105062400 + 72921600\beta + 15033600\beta^2 + 518400\beta^3)\lambda_j \\ &\quad + (-22809600 + 518400\beta^3 + 15033600\beta + 9590400\beta^2)\lambda_i, \\ \xi_{31}^{(3)}(\lambda_j; \lambda_i; \beta) &= 105062400 + 72921600\beta + 15033600\beta^2 + 518400\beta^3 \\ &\geq 0 \text{ for all } \beta \geq -23.3467. \end{aligned}$$

Now

$$\begin{aligned} \xi_{31}(\lambda_i; \lambda_i; \beta) &= 161280\lambda_i^3(88 + 160\beta + 57\beta^2 + 3\beta^3) \geq 0 \text{ for all } \beta \geq -0.7350, \\ \xi_{31}^{(1)}(\lambda_i; \lambda_i; \beta) &= 60480\lambda_i^2(608 + 932\beta + 312\beta^2 + 15\beta^3) \geq 0 \text{ for all } \beta \geq -0.9275, \\ \xi_{31}^{(2)}(\lambda_i; \lambda_i; \beta) &= 86400\lambda_i(952 + 1018\beta + 285\beta^2 + 12\beta^3) \geq 0 \text{ for all } \beta \geq -1.6188. \end{aligned}$$

Hence, $\xi_3^{(1)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{30}(\lambda_j; \lambda_i; \beta) = \xi_3(\lambda_j; \lambda_j; \lambda_i; \beta)$. We wish to show that ξ_{30} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{30} with respect to λ_j gives

$$\begin{aligned} \xi_{30}(\lambda_j; \lambda_i; \beta) &= (8006400 + 5414400\beta + 1080000\beta^2 + 43200\beta^3)\lambda_j^4 \\ &\quad + (172800\beta^3 + 4366080\beta - 7004160 + 2678400\beta^2)\lambda_i\lambda_j^3 \\ &\quad + (5875200 + 4008960\beta + 1451520\beta^2 + 129600\beta^3)\lambda_i^2\lambda_j^2 \\ &\quad + (1751040 + 224640\beta^2 + 2062080\beta + 17280\beta^3)\lambda_i^3\lambda_j \\ &\quad + (8640\beta^2 + 115200\beta + 80640)\lambda_i^4, \\ \xi_{30}^{(1)}(\lambda_j; \lambda_i; \beta) &= (32025600 + 21657600\beta + 4320000\beta^2 + 172800\beta^3)\lambda_j^3 \\ &\quad + (518400\beta^3 + 13098240\beta - 21012480 + 8035200\beta^2)\lambda_i\lambda_j^2 \\ &\quad + (11750400 + 8017920\beta + 2903040\beta^2 + 259200\beta^3)\lambda_i^2\lambda_j \\ &\quad + (1751040 + 224640\beta^2 + 2062080\beta + 17280\beta^3)\lambda_i^3, \\ \xi_{30}^{(2)}(\lambda_j; \lambda_i; \beta) &= (96076800 + 64972800\beta + 12960000\beta^2 + 518400\beta^3)\lambda_j^2 \\ &\quad + (1036800\beta^3 + 26196480\beta - 42024960 + 16070400\beta^2)\lambda_i\lambda_j \\ &\quad + (11750400 + 8017920\beta + 2903040\beta^2 + 259200\beta^3)\lambda_i^2, \\ \xi_{30}^{(3)}(\lambda_j; \lambda_i; \beta) &= (192153600 + 129945600\beta + 25920000\beta^2 + 1036800\beta^3)\lambda_j \\ &\quad + (1036800\beta^3 + 26196480\beta - 42024960 + 16070400\beta^2)\lambda_i, \\ \xi_{30}^{(4)}(\lambda_j; \lambda_i; \beta) &= 192153600 + 129945600\beta + 25920000\beta^2 + 1036800\beta^3 \\ &\geq 0 \text{ for all } \beta \geq -18.8822. \end{aligned}$$

Now

$$\begin{aligned}\xi_{30}(\lambda_i; \lambda_i; \beta) &= 362880\lambda_i^4(\beta + 3)(\beta^2 + 12\beta + 8) > 0 \text{ for all } \beta > -0.7085, \\ \xi_{30}^{(1)}(\lambda_i; \lambda_i; \beta) &= 322560\lambda_i^3(76 + 139\beta + 48\beta^2 + 3\beta^3) > 0 \text{ for all } \beta > -0.7158, \\ \xi_{30}^{(2)}(\lambda_i; \lambda_i; \beta) &= 120960\lambda_i^2(544 + 820\beta + 264\beta^2 + 15\beta^3) > 0 \text{ for all } \beta > -0.9237, \\ \xi_{30}^{(3)}(\lambda_i; \lambda_i; \beta) &= 103680\lambda_i(1448 + 1506\beta + 405\beta^2 + 20\beta^3) > 0 \text{ for all } \beta > -1.5818.\end{aligned}$$

Hence, ξ_3 is positive for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$. This implies that $\xi(\lambda_l; \lambda_l; \lambda_j; \lambda_i; \beta)$ is positive for all $\lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Differentiating $\xi_4(\lambda_l; \lambda_j; \lambda_i; \beta)$ with respect to λ_l we obtain

$$\begin{aligned}\xi_4(\lambda_l; \lambda_j; \lambda_i; \beta) &= (207360\beta - 4147200 + 933120\beta^2)\lambda_i\lambda_j\lambda_l \\ &\quad + (864000\beta + 1175040 + 311040\beta^2)(\lambda_j^2 + \lambda_i^2)\lambda_l \\ &\quad + (864000\beta + 1175040 + 311040\beta^2)(\lambda_i + \lambda_j)\lambda_l^2 \\ &\quad + (460800\beta + 322560 + 34560\beta^2)(\lambda_i^3 + \lambda_j^3) \\ &\quad + (864000\beta + 1175040 + 311040\beta^2)(\lambda_i\lambda_j^2 + \lambda_i^2\lambda_j) \\ &\quad + (460800\beta + 322560 + 34560\beta^2)\lambda_l^3,\end{aligned}$$

and, hence,

$$\begin{aligned}\xi_4^{(2)}(\lambda_l; \lambda_j; \lambda_i; \beta) &= (2764800\beta + 1935360 + 207360\beta^2)\lambda_l \\ &\quad + (1728000\beta + 2350080 + 622080\beta^2)(\lambda_j + \lambda_i) \\ &\geq 0 \text{ for all } \beta \geq -0.7412.\end{aligned}$$

Define $\xi_{41}(\lambda_j; \lambda_i; \beta) = \xi_4^{(1)}(\lambda_j; \lambda_j; \lambda_i; \beta)$. We wish to show that ξ_{41} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{41} with respect to λ_j gives

$$\begin{aligned}\xi_{41}(\lambda_j; \lambda_i; \beta) &= (3974400\beta + 4492800 + 1036800\beta^2)\lambda_j^2 \\ &\quad + (1935360\beta - 1797120 + 1555200\beta^2)\lambda_i\lambda_j \\ &\quad + (864000\beta + 1175040 + 311040\beta^2)\lambda_i^2, \\ \xi_{41}^{(1)}(\lambda_j; \lambda_i; \beta) &= (7948800\beta + 8985600 + 2073600\beta^2)\lambda_j \\ &\quad + (1935360\beta - 1797120 + 1555200\beta^2)\lambda_i, \\ \xi_{41}^{(2)}(\lambda_j; \lambda_i; \beta) &= 7948800\beta + 8985600 + 2073600\beta^2 \geq 0 \text{ for all } \beta.\end{aligned}$$

Now

$$\begin{aligned}\xi_{41}(\lambda_i; \lambda_i; \beta) &= 967680\lambda_i^2(3\beta + 4)(\beta + 1) \geq 0 \text{ for all } \beta \geq -1, \\ \xi_{41}^{(1)}(\lambda_i; \lambda_i; \beta) &= 34560\lambda_i(105\beta^2 + 208 + 286\beta) \geq 0 \text{ for all } \beta.\end{aligned}$$

Hence, $\xi_{41}^{(1)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{40}(\lambda_j; \lambda_i; \beta) = \xi_4(\lambda_j; \lambda_j; \lambda_i; \beta)$. We wish to show that ξ_{40} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{40} with respect to λ_j gives

$$\begin{aligned} \xi_{40}(\lambda_j; \lambda_i; \beta) &= (2649600\beta + 2995200 + 691200\beta^2)\lambda_j^3 \\ &\quad + (1935360\beta - 1797120 + 1555200\beta^2)\lambda_i\lambda_j^2 \\ &\quad + (1728000\beta + 2350080 + 622080\beta^2)\lambda_i^2\lambda_j \\ &\quad + (460800\beta + 322560 + 34560\beta^2)\lambda_i^3, \\ \xi_{40}^{(1)}(\lambda_j; \lambda_i; \beta) &= (7948800\beta + 8985600 + 2073600\beta^2)\lambda_j^2 \\ &\quad + (3870720\beta - 3594240 + 3110400\beta^2)\lambda_i\lambda_j \\ &\quad + (1728000\beta + 2350080 + 622080\beta^2)\lambda_i^2, \\ \xi_{40}^{(2)}(\lambda_j; \lambda_i; \beta) &= (15897600\beta + 17971200 + 4147200\beta^2)\lambda_j \\ &\quad + (3870720\beta - 3594240 + 3110400\beta^2)\lambda_i, \\ \xi_{40}^{(3)}(\lambda_j; \lambda_i; \beta) &= 15897600\beta + 17971200 + 4147200\beta^2 \geq 0 \text{ for all } \beta. \end{aligned}$$

Now

$$\begin{aligned} \xi_{40}(\lambda_i; \lambda_i; \beta) &= 967680\lambda_i^3(3\beta + 4)(\beta + 1) > 0 \text{ for all } \beta > -1, \\ \xi_{40}^{(1)}(\lambda_i; \lambda_i; \beta) &= 1935360\lambda_i^2(3\beta + 4)(\beta + 1) > 0 \text{ for all } \beta > -1, \\ \xi_{40}^{(2)}(\lambda_i; \lambda_i; \beta) &= 69120\lambda_i(105\beta^2 + 208 + 286\beta) > 0 \text{ for all } \beta. \end{aligned}$$

Hence, ξ_4 is positive for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$. This implies that $\xi(\lambda_l; \lambda_l; \lambda_j; \lambda_i; \beta)$ is positive for all $\lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

□

Lemma B.17 Let

$$\begin{aligned} \xi(\lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta) &= (345600 + 420480\beta + 43200\beta^2 + 2880\beta^3)t(0, 0, 3, 3) \\ &\quad + (1140480 + 924480\beta + 237600\beta^2 + 25920\beta^3)t(0, 1, 2, 3) \\ &\quad + (152640\beta + 23040 + 90720\beta^2)t(0, 1, 1, 4) \\ &\quad + (77760\beta^3 - 4008960 + 449280\beta + 622080\beta^2)t(1, 1, 1, 3) \\ &\quad + (777600\beta^2 - 829440\beta + 233280\beta^3 - 3317760)t(1, 1, 2, 2) \\ &\quad + (129600\beta + 30240\beta^2 + 138240)t(0, 0, 2, 4) \\ &\quad + (349920\beta^2 + 77760\beta^3 + 5132160 + 1088640\beta)t(0, 2, 2, 2), \end{aligned}$$

where

$$t(a, b, c, d) = \sum_{\substack{(n_1, n_2, n_3, n_4) \\ \in \text{perm}(a, b, c, d)}} \lambda_i^{n_1} \lambda_j^{n_2} \lambda_l^{n_3} \lambda_k^{n_4}.$$

The expression $\xi(\lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta)$ is positive for all $\lambda_k \geq \lambda_l \geq \lambda_j \geq \lambda_i$, and $\beta \geq -\frac{2}{5}$.

Proof.

Let

$$\hat{s}(a, b, c) = \sum_{\substack{(n_1, n_2, n_3) \\ \in \text{perm}(a, b, c)}} \lambda_i^{n_1} \lambda_j^{n_2} \lambda_l^{n_3}.$$

Differentiating $\xi(\lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta)$ with respect to λ_k we obtain

$$\begin{aligned} \xi^{(1)}(\lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta) &= 4(30240\beta^2 + 129600\beta + 138240)\lambda_k^3 \hat{s}(0, 0, 2) \\ &+ 4(152640\beta + 23040 + 90720\beta^2)\lambda_k^3 \hat{s}(0, 1, 1) \\ &+ 3(77760\beta^3 - 4008960 + 449280\beta + 622080\beta^2)\lambda_k^2 \hat{s}(1, 1, 1) \\ &+ 3(2880\beta^3 + 43200\beta^2 + 345600 + 420480\beta)\lambda_k^2 \hat{s}(0, 0, 3) \\ &+ 3(924480\beta + 1140480 + 237600\beta^2 + 25920\beta^3)\lambda_k^2 \hat{s}(0, 1, 2) \\ &+ 2(-3317760 + 233280\beta^3 + 777600\beta^2 - 829440\beta)\lambda_k \hat{s}(1, 1, 2) \\ &+ 2(5132160 + 349920\beta^2 + 77760\beta^3 + 1088640\beta)\lambda_k \hat{s}(0, 2, 2) \\ &+ (60480\beta^2 + 259200\beta + 276480)\lambda_k \hat{s}(0, 0, 4) \\ &+ 2(924480\beta + 1140480 + 237600\beta^2 + 25920\beta^3)\lambda_k \hat{s}(0, 1, 3) \\ &+ (-3317760 + 233280\beta^3 + 777600\beta^2 - 829440\beta)\hat{s}(1, 2, 2) \\ &+ (77760\beta^3 - 4008960 + 449280\beta + 622080\beta^2)\hat{s}(1, 1, 3) \\ &+ (924480\beta + 1140480 + 237600\beta^2 + 25920\beta^3)\hat{s}(0, 2, 3) \\ &+ (152640\beta + 23040 + 90720\beta^2)\hat{s}(0, 1, 4), \\ \xi^{(2)}(\lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta) &= 12(30240\beta^2 + 129600\beta + 138240)\lambda_k^2 \hat{s}(0, 0, 2) \\ &+ 12(152640\beta + 23040 + 90720\beta^2)\lambda_k^2 \hat{s}(0, 1, 1) \\ &+ 6(77760\beta^3 - 4008960 + 449280\beta + 622080\beta^2)\lambda_k \hat{s}(1, 1, 1) \\ &+ 6(2880\beta^3 + 43200\beta^2 + 345600 + 420480\beta)\lambda_k \hat{s}(0, 0, 3) \\ &+ 6(924480\beta + 1140480 + 237600\beta^2 + 25920\beta^3)\lambda_k \hat{s}(0, 1, 2) \\ &+ 2(-3317760 + 233280\beta^3 + 777600\beta^2 - 829440\beta)\hat{s}(1, 1, 2) \\ &+ 2(5132160 + 349920\beta^2 + 77760\beta^3 + 1088640\beta)\hat{s}(0, 2, 2) \\ &+ (60480\beta^2 + 259200\beta + 276480)\hat{s}(0, 0, 4) \\ &+ 2(924480\beta + 1140480 + 237600\beta^2 + 25920\beta^3)\hat{s}(0, 1, 3), \\ \xi^{(3)}(\lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta) &= 24(30240\beta^2 + 129600\beta + 138240)\lambda_k \hat{s}(0, 0, 2) \\ &+ 24(152640\beta + 23040 + 90720\beta^2)\lambda_k \hat{s}(0, 1, 1) \\ &+ 6(77760\beta^3 - 4008960 + 449280\beta + 622080\beta^2)\hat{s}(1, 1, 1) \\ &+ 6(2880\beta^3 + 43200\beta^2 + 345600 + 420480\beta)\hat{s}(0, 0, 3) \\ &+ 6(924480\beta + 1140480 + 237600\beta^2 + 25920\beta^3)\hat{s}(0, 1, 2), \\ \xi^{(4)}(\lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta) &= 24(30240\beta^2 + 129600\beta + 138240)\hat{s}(0, 0, 2) \\ &+ 24(152640\beta + 23040 + 90720\beta^2)\hat{s}(0, 1, 1). \end{aligned}$$

Define

$$\begin{aligned}\xi_0(\lambda_l; \lambda_j; \lambda_i; \beta) &\stackrel{\text{def}}{=} \xi(\lambda_l; \lambda_l; \lambda_j; \lambda_i; \beta), \\ \xi_1(\lambda_l; \lambda_j; \lambda_i; \beta) &\stackrel{\text{def}}{=} \xi^{(1)}(\lambda_l; \lambda_l; \lambda_j; \lambda_i; \beta), \\ \xi_2(\lambda_l; \lambda_j; \lambda_i; \beta) &\stackrel{\text{def}}{=} \xi^{(2)}(\lambda_l; \lambda_l; \lambda_j; \lambda_i; \beta), \\ \xi_3(\lambda_l; \lambda_j; \lambda_i; \beta) &\stackrel{\text{def}}{=} \xi^{(3)}(\lambda_l; \lambda_l; \lambda_j; \lambda_i; \beta), \\ \xi_4(\lambda_l; \lambda_j; \lambda_i; \beta) &\stackrel{\text{def}}{=} \xi^{(4)}(\lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta).\end{aligned}$$

We will prove that the above equations are non-negative for all $\lambda_k \geq \lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$ and, hence, deduce that $\xi(\lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta)$ is positive for all $\lambda_k \geq \lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Now

$$\begin{aligned}\xi_0(\lambda_l; \lambda_j; \lambda_i; \beta) &= (2880\beta^3 + 103680\beta^2 + 622080 + 679680\beta)\lambda_l^6 \\ &\quad + (2154240\beta + 2327040 + 656640\beta^2 + 51840\beta^3)(\lambda_i + \lambda_j)\lambda_l^5 \\ &\quad + (3196800\beta + 7689600 + 885600\beta^2 + 129600\beta^3)(\lambda_j^2 + \lambda_i^2)\lambda_l^4 \\ &\quad + (388800\beta^3 - 11289600 + 374400\beta + 2203200\beta^2)\lambda_i\lambda_j\lambda_l^4 \\ &\quad + (190080\beta - 4354560 + 2030400\beta^2 + 518400\beta^3)(\lambda_i\lambda_j^2 + \lambda_i^2\lambda_j)\lambda_l^3 \\ &\quad + (57600\beta^3 + 561600\beta^2 + 2972160 + 2689920\beta)(\lambda_j^3 + \lambda_i^3)\lambda_l^3 \\ &\quad + (129600\beta^3 - 1728000 + 2298240\beta + 1097280\beta^2)(\lambda_i\lambda_j^3 + \lambda_i^3\lambda_j)\lambda_l^2 \\ &\quad + (151200\beta^2 + 411840\beta + 299520)(\lambda_j^4 + \lambda_i^4)\lambda_l^2 \\ &\quad + (6946560 + 1477440\beta^2 + 388800\beta^3 + 1347840\beta)\lambda_i^2\lambda_j^2\lambda_l^2 \\ &\quad + 2(924480\beta + 1140480 + 237600\beta^2 + 25920\beta^3)(\lambda_i^2\lambda_j^3 + \lambda_i^3\lambda_j^2)\lambda_l \\ &\quad + 2(152640\beta + 23040 + 90720\beta^2)(\lambda_i\lambda_j^4 + \lambda_i^4\lambda_j)\lambda_l \\ &\quad + (2880\beta^3 + 43200\beta^2 + 345600 + 420480\beta)\lambda_i^3\lambda_j^3 \\ &\quad + (30240\beta^2 + 129600\beta + 138240)(\lambda_i^2\lambda_j^4 + \lambda_i^4\lambda_j^2),\end{aligned}$$

and, hence,

$$\begin{aligned}\xi_0^{(5)}(\lambda_l; \lambda_j; \lambda_i; \beta) &= (2073600\beta^3 + 74649600\beta^2 + 447897600 + 489369600\beta)\lambda_l \\ &\quad + (258508800\beta + 279244800 + 78796800\beta^2 + 6220800\beta^3)(\lambda_j + \lambda_i) \\ &\geq 0 \text{ for all } \beta \geq -1.0915.\end{aligned}$$

Define $\xi_{04}(\lambda_j; \lambda_i; \beta) = \xi_0^{(4)}(\lambda_l; \lambda_j; \lambda_i; \beta)$. We wish to show that ξ_{04} is non-negative for all

$\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{04} with respect to λ_j gives

$$\begin{aligned}\xi_{04}(\lambda_j; \lambda_i; \beta) &= (10368000\beta^3 + 137376000\beta^2 + 687744000 + 579916800\beta)\lambda_j^2 \\ &\quad + (267494400\beta + 8294400 + 131673600\beta^2 + 15552000\beta^3)\lambda_i\lambda_j \\ &\quad + (76723200\beta + 184550400 + 21254400\beta^2 + 3110400\beta^3)\lambda_i^2, \\ \xi_{04}^{(1)}(\lambda_j; \lambda_i; \beta) &= (20736000\beta^3 + 274752000\beta^2 + 1375488000 + 1159833600\beta)\lambda_j \\ &\quad + (267494400\beta + 8294400 + 131673600\beta^2 + 15552000\beta^3)\lambda_i, \\ \xi_{04}^{(2)}(\lambda_j; \lambda_i; \beta) &= 20736000\beta^3 + 274752000\beta^2 + 1375488000 + 1159833600\beta \\ &\geq 0 \text{ for all } \beta \geq -1.9649.\end{aligned}$$

Now

$$\begin{aligned}\xi_{04}(\lambda_i; \lambda_i; \beta) &= 4838400\lambda_i^2(6\beta^3 + 60\beta^2 + 182 + 191\beta) \geq 0 \text{ for all } \beta \geq -1.7321, \\ \xi_{04}^{(1)}(\lambda_i; \lambda_i; \beta) &= 2419200\lambda_i(15\beta^3 + 168\beta^2 + 572 + 590\beta) \geq 0 \text{ for all } \beta \geq -1.5802.\end{aligned}$$

Hence, $\xi_0^{(4)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{03}(\lambda_j; \lambda_i; \beta) = \xi_0^{(3)}(\lambda_j; \lambda_j; \lambda_i; \beta)$. We wish to show that ξ_{03} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{03} with respect to λ_j gives

$$\begin{aligned}\xi_{03}(\lambda_j; \lambda_i; \beta) &= (6912000\beta^3 + 76464000\beta^2 + 416655360 + 303678720\beta)\lambda_j^3 \\ &\quad + (15552000\beta^3 - 157455360 + 139380480\beta + 104457600\beta^2)\lambda_i\lambda_j^2 \\ &\quad + (77863680\beta + 158423040 + 33436800\beta^2 + 6220800\beta^3)\lambda_i^2\lambda_j \\ &\quad + (345600\beta^3 + 3369600\beta^2 + 17832960 + 16139520\beta)\lambda_i^3, \\ \xi_{03}^{(1)}(\lambda_j; \lambda_i; \beta) &= (20736000\beta^3 + 229392000\beta^2 + 1249966080 + 911036160\beta)\lambda_j^2 \\ &\quad + (31104000\beta^3 - 314910720 + 278760960\beta + 208915200\beta^2)\lambda_i\lambda_j \\ &\quad + (77863680\beta + 158423040 + 33436800\beta^2 + 6220800\beta^3)\lambda_i^2, \\ \xi_{03}^{(2)}(\lambda_j; \lambda_i; \beta) &= (41472000\beta^3 + 458784000\beta^2 + 2499932160 + 1822072320\beta)\lambda_j \\ &\quad + (31104000\beta^3 - 314910720 + 278760960\beta + 208915200\beta^2)\lambda_i, \\ \xi_{03}^{(3)}(\lambda_j; \lambda_i; \beta) &= 41472000\beta^3 + 458784000\beta^2 + 2499932160 + 1822072320\beta \\ &\geq 0 \text{ for all } \beta \geq -3.2529.\end{aligned}$$

Now

$$\begin{aligned}\xi_{03}(\lambda_i; \lambda_i; \beta) &= 14515200\lambda_i^3(2\beta + 5)(\beta + 3)(\beta + 2) \geq 0 \text{ for all } \beta \geq -2, \\ \xi_{03}^{(1)}(\lambda_i; \lambda_i; \beta) &= 2419200\lambda_i^2(24\beta^3 + 195\beta^2 + 452 + 524\beta) \geq 0 \text{ for all } \beta \geq -1.8116, \\ \xi_{03}^{(2)}(\lambda_i; \lambda_i; \beta) &= 967680\lambda_i(75\beta^3 + 690\beta^2 + 2258 + 2171\beta) \geq 0 \text{ for all } \beta \geq -2.3123.\end{aligned}$$

Hence, $\xi_0^{(3)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{02}(\lambda_j; \lambda_i; \beta) = \xi_0^{(2)}(\lambda_j; \lambda_j; \lambda_i; \beta)$. We wish to show that ξ_{02} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{02} with respect to λ_j gives

$$\begin{aligned} \xi_{02}(\lambda_j; \lambda_i; \beta) &= (3024000\beta^3 + 30542400\beta^2 + 175910400 + 118800000\beta)\lambda_j^4 \\ &\quad + (9072000\beta^3 - 118517760 + 53314560\beta + 53948160\beta^2)\lambda_i\lambda_j^3 \\ &\quad + (42197760\beta + 80040960 + 25764480\beta^2 + 5443200\beta^3)\lambda_i^2\lambda_j^2 \\ &\quad + (604800\beta^3 + 14376960 + 20736000\beta + 5564160\beta^2)\lambda_i^3\lambda_j \\ &\quad + (302400\beta^2 + 823680\beta + 599040)\lambda_i^4, \\ \xi_{02}^{(1)}(\lambda_j; \lambda_i; \beta) &= (12096000\beta^3 + 122169600\beta^2 + 703641600 + 475200000\beta)\lambda_j^3 \\ &\quad + (27216000\beta^3 - 355553280 + 159943680\beta + 161844480\beta^2)\lambda_i\lambda_j^2 \\ &\quad + (84395520\beta + 160081920 + 51528960\beta^2 + 10886400\beta^3)\lambda_i^2\lambda_j \\ &\quad + (604800\beta^3 + 14376960 + 20736000\beta + 5564160\beta^2)\lambda_i^3, \\ \xi_{02}^{(2)}(\lambda_j; \lambda_i; \beta) &= (36288000\beta^3 + 366508800\beta^2 + 2110924800 + 1425600000\beta)\lambda_j^2 \\ &\quad + (54432000\beta^3 - 711106560 + 319887360\beta + 323688960\beta^2)\lambda_i\lambda_j \\ &\quad + (84395520\beta + 160081920 + 51528960\beta^2 + 10886400\beta^3)\lambda_i^2, \\ \xi_{02}^{(3)}(\lambda_j; \lambda_i; \beta) &= (72576000\beta^3 + 733017600\beta^2 + 4221849600 + 2851200000\beta)\lambda_j \\ &\quad + (54432000\beta^3 - 711106560 + 319887360\beta + 323688960\beta^2)\lambda_i, \\ \xi_{02}^{(4)}(\lambda_j; \lambda_i; \beta) &= 72576000\beta^3 + 733017600\beta^2 + 4221849600 + 2851200000\beta \\ &\geq 0 \text{ for all } \beta \geq -3.7755. \end{aligned}$$

Now

$$\begin{aligned} \xi_{02}(\lambda_i; \lambda_i; \beta) &= 3628800\lambda_i^4(5\beta + 7)(\beta + 3)(\beta + 2) \geq 0 \text{ for all } \beta \geq -\frac{7}{5}, \\ \xi_{02}^{(1)}(\lambda_i; \lambda_i; \beta) &= 7257600\lambda_i^3(\beta + 3)(\beta + 2)(7\beta + 12) \geq 0 \text{ for all } \beta \geq -\frac{12}{7}, \\ \xi_{02}^{(2)}(\lambda_i; \lambda_i; \beta) &= 483840\lambda_i^2(210\beta^3 + 1533\beta^2 + 3224 + 3782\beta) \geq 0 \text{ for all } \beta \geq -3.0192, \\ \xi_{02}^{(3)}(\lambda_i; \lambda_i; \beta) &= 241920\lambda_i(525\beta^3 + 4368\beta^2 + 14512 + 13108\beta) \geq 0 \text{ for all } \beta \geq -3.2688. \end{aligned}$$

Hence, $\xi_0^{(2)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{01}(\lambda_j; \lambda_i; \beta) = \xi_0^{(1)}(\lambda_j; \lambda_j; \lambda_i; \beta)/\lambda_j$. We wish to show that ξ_{01} is non-negative

for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{01} with respect to λ_j gives

$$\begin{aligned}
\xi_{01}(\lambda_j; \lambda_i; \beta) &= (967680\beta^3 + 9434880\beta^2 + 55641600 + 36529920\beta)\lambda_j^4 \\
&\quad + (3628800\beta^3 - 49996800 + 17740800\beta + 20563200\beta^2)\lambda_i\lambda_j^3 \\
&\quad + (17902080\beta + 33868800 + 13063680\beta^2 + 2903040\beta^3)\lambda_i^2\lambda_j^2 \\
&\quad + (483840\beta^3 + 4354560\beta^2 + 7741440 + 14515200\beta)\lambda_i^3\lambda_j \\
&\quad + (483840\beta^2 + 1128960\beta + 645120)\lambda_i^4, \\
\xi_{01}^{(1)}(\lambda_j; \lambda_i; \beta) &= (3870720\beta^3 + 37739520\beta^2 + 222566400 + 146119680\beta)\lambda_j^3 \\
&\quad + (10886400\beta^3 - 149990400 + 53222400\beta + 61689600\beta^2)\lambda_i\lambda_j^2 \\
&\quad + (35804160\beta + 67737600 + 26127360\beta^2 + 5806080\beta^3)\lambda_i^2\lambda_j \\
&\quad + (483840\beta^3 + 4354560\beta^2 + 7741440 + 14515200\beta)\lambda_i^3, \\
\xi_{01}^{(2)}(\lambda_j; \lambda_i; \beta) &= (11612160\beta^3 + 113218560\beta^2 + 667699200 + 438359040\beta)\lambda_j^2 \\
&\quad + (21772800\beta^3 - 299980800 + 106444800\beta + 123379200\beta^2)\lambda_i\lambda_j \\
&\quad + (35804160\beta + 67737600 + 26127360\beta^2 + 5806080\beta^3)\lambda_i^2, \\
\xi_{01}^{(3)}(\lambda_j; \lambda_i; \beta) &= (23224320\beta^3 + 226437120\beta^2 + 1335398400 + 876718080\beta)\lambda_j \\
&\quad + (21772800\beta^3 - 299980800 + 106444800\beta + 123379200\beta^2)\lambda_i, \\
\xi_{01}^{(4)}(\lambda_j; \lambda_i; \beta) &= 23224320\beta^3 + 226437120\beta^2 + 1335398400 + 876718080\beta \\
&\geq 0 \text{ for all } \beta \geq -3.7954.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{01}(\lambda_i; \lambda_i; \beta) &= 7983360\lambda_i^4(\beta + 3)(\beta + 2)(\beta + 1) \geq 0 \text{ for all } \beta \geq -1, \\
\xi_{01}^{(1)}(\lambda_i; \lambda_i; \beta) &= 725760\lambda_i^3(\beta + 3)(\beta + 2)(29\beta + 34) \geq 0 \text{ for all } \beta \geq -\frac{34}{29}, \\
\xi_{01}^{(2)}(\lambda_i; \lambda_i; \beta) &= 1451520\lambda_i^2(\beta + 3)(27\beta^2 + 100\beta + 100) \geq 0 \text{ for all } \beta \geq -3, \\
\xi_{01}^{(3)}(\lambda_i; \lambda_i; \beta) &= 483840\lambda_i(93\beta^3 + 723\beta^2 + 2140 + 2032\beta) \geq 0 \text{ for all } \beta \geq -3.1755.
\end{aligned}$$

Hence, $\xi_0^{(1)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{00}(\lambda_j; \lambda_i; \beta) = \xi_0(\lambda_j; \lambda_j; \lambda_i; \beta)/\lambda_j^2$. We wish to show that ξ_{00} is non-negative for

all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{00} with respect to λ_j gives

$$\begin{aligned} \xi_{00}(\lambda_j; \lambda_i; \beta) &= (241920\beta^3 + 2358720\beta^2 + 13910400 + 9132480\beta)\lambda_j^4 \\ &\quad + (1088640\beta^3 - 14999040 + 5322240\beta + 6168960\beta^2)\lambda_i\lambda_j^3 \\ &\quad + (6713280\beta + 12700800 + 4898880\beta^2 + 1088640\beta^3)\lambda_i^2\lambda_j^2 \\ &\quad + (241920\beta^3 + 2177280\beta^2 + 3870720 + 7257600\beta)\lambda_i^3\lambda_j \\ &\quad + (362880\beta^2 + 846720\beta + 483840)\lambda_i^4, \\ \xi_{00}^{(1)}(\lambda_j; \lambda_i; \beta) &= (967680\beta^3 + 9434880\beta^2 + 55641600 + 36529920\beta)\lambda_j^3 \\ &\quad + (3265920\beta^3 - 44997120 + 15966720\beta + 18506880\beta^2)\lambda_i\lambda_j^2 \\ &\quad + (13426560\beta + 25401600 + 9797760\beta^2 + 2177280\beta^3)\lambda_i^2\lambda_j \\ &\quad + (241920\beta^3 + 2177280\beta^2 + 3870720 + 7257600\beta)\lambda_i^3, \\ \xi_{00}^{(2)}(\lambda_j; \lambda_i; \beta) &= (2903040\beta^3 + 28304640\beta^2 + 166924800 + 109589760\beta)\lambda_j^2 \\ &\quad + (6531840\beta^3 - 89994240 + 31933440\beta + 37013760\beta^2)\lambda_i\lambda_j \\ &\quad + (13426560\beta + 25401600 + 9797760\beta^2 + 2177280\beta^3)\lambda_i^2, \\ \xi_{00}^{(3)}(\lambda_j; \lambda_i; \beta) &= (5806080\beta^3 + 56609280\beta^2 + 333849600 + 219179520\beta)\lambda_j \\ &\quad + (6531840\beta^3 - 89994240 + 31933440\beta + 37013760\beta^2)\lambda_i, \\ \xi_{00}^{(4)}(\lambda_j; \lambda_i; \beta) &= 5806080\beta^3 + 56609280\beta^2 + 333849600 + 219179520\beta \\ &\geq 0 \text{ for all } \beta \geq -3.7954. \end{aligned}$$

Now

$$\begin{aligned} \xi_{00}(\lambda_i; \lambda_i; \beta) &= 2661120\lambda_i^4(\beta + 3)(\beta + 2)(\beta + 1) > 0 \text{ for all } \beta > -1, \\ \xi_{00}^{(1)}(\lambda_i; \lambda_i; \beta) &= 6652800\lambda_i^3(\beta + 3)(\beta + 2)(\beta + 1) > 0 \text{ for all } \beta > -1, \\ \xi_{00}^{(2)}(\lambda_i; \lambda_i; \beta) &= 362880\lambda_i^2(\beta + 3)(\beta + 2)(32\beta + 47) > 0 \text{ for all } \beta > -\frac{47}{32}, \\ \xi_{00}^{(3)}(\lambda_i; \lambda_i; \beta) &= 725760\lambda_i(\beta + 3)(17\beta^2 + 78\beta + 112) > 0 \text{ for all } \beta > -3. \end{aligned}$$

Hence, ξ_0 is positive for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$. This implies that $\xi(\lambda_l; \lambda_l; \lambda_j; \lambda_i; \beta)$ is positive for all $\lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Now

$$\begin{aligned}
\xi_1(\lambda_l; \lambda_j; \lambda_i; \beta) &= (311040\beta^2 + 2039040\beta + 1866240 + 8640\beta^3)\lambda_l^5 \\
&\quad + (5385600\beta + 5817600 + 1641600\beta^2 + 129600\beta^3)(\lambda_j + \lambda_i)\lambda_l^4 \\
&\quad + (6393600\beta + 15379200 + 1771200\beta^2 + 259200\beta^3)(\lambda_j^2 + \lambda_i^2)\lambda_l^3 \\
&\quad + (777600\beta^3 - 22579200 + 748800\beta + 4406400\beta^2)\lambda_i\lambda_j\lambda_l^3 \\
&\quad + (285120\beta - 6531840 + 3045600\beta^2 + 777600\beta^3)(\lambda_i\lambda_j^2 + \lambda_i^2\lambda_j)\lambda_l^2 \\
&\quad + (4034880\beta + 4458240 + 842400\beta^2 + 86400\beta^3)(\lambda_j^3 + \lambda_i^3)\lambda_l^2 \\
&\quad + (129600\beta^3 - 1728000 + 2298240\beta + 1097280\beta^2)(\lambda_i\lambda_j^3 + \lambda_i^3\lambda_j)\lambda_l \\
&\quad + (151200\beta^2 + 411840\beta + 299520)\lambda_j^4 + \lambda_i^4\lambda_l \\
&\quad + (6946560 + 1477440\beta^2 + 388800\beta^3 + 1347840\beta)\lambda_i^2\lambda_j^2\lambda_l \\
&\quad + (152640\beta + 23040 + 90720\beta^2)(\lambda_i\lambda_j^4 + \lambda_i^4\lambda_j) \\
&\quad + (924480\beta + 1140480 + 237600\beta^2 + 25920\beta^3)(\lambda_i^3\lambda_j^2 + \lambda_i^2\lambda_j^3),
\end{aligned}$$

and, hence,

$$\begin{aligned}
\xi_1^{(4)}(\lambda_l; \lambda_j; \lambda_i; \beta) &= (37324800\beta^2 + 244684800\beta + 223948800 + 1036800\beta^3)\lambda_l \\
&\quad + (129254400\beta + 139622400 + 39398400\beta^2 + 3110400\beta^3)(\lambda_i + \lambda_j) \\
&\geq 0 \text{ for all } \beta \geq -1.0915.
\end{aligned}$$

Define $\xi_{13}(\lambda_j; \lambda_i; \beta) = \xi_1^{(3)}(\lambda_j; \lambda_j; \lambda_i; \beta)$. We wish to show that ξ_{13} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{13} with respect to λ_j gives

$$\begin{aligned}
\xi_{13}(\lambda_j; \lambda_i; \beta) &= (68688000\beta^2 + 289958400\beta + 343872000 + 5184000\beta^3)\lambda_j^2 \\
&\quad + (133747200\beta + 4147200 + 65836800\beta^2 + 7776000\beta^3)\lambda_i\lambda_j \\
&\quad + (38361600\beta + 92275200 + 10627200\beta^2 + 1555200\beta^3)\lambda_i^2, \\
\xi_{13}^{(1)}(\lambda_j; \lambda_i; \beta) &= (137376000\beta^2 + 579916800\beta + 687744000 + 10368000\beta^3)\lambda_j \\
&\quad + (133747200\beta + 4147200 + 65836800\beta^2 + 7776000\beta^3)\lambda_i, \\
\xi_{13}^{(2)}(\lambda_j; \lambda_i; \beta) &= 137376000\beta^2 + 579916800\beta + 687744000 + 10368000\beta^3 \\
&\geq 0 \text{ for all } \beta \geq -1.9649.
\end{aligned}$$

Now

$$\begin{aligned}
\xi_{13}(\lambda_i; \lambda_i; \beta) &= 2419200\lambda_i^2(60\beta^2 + 191\beta + 182 + 6\beta^3) \geq 0 \text{ for all } \beta \geq -1.7321, \\
\xi_{13}^{(1)}(\lambda_i; \lambda_i; \beta) &= 1209600\lambda_i(168\beta^2 + 590\beta + 572 + 15\beta^3) \geq 0 \text{ for all } \beta \geq -1.5802.
\end{aligned}$$

Hence, $\xi_1^{(3)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{12}(\lambda_j; \lambda_i; \beta) = \xi_1^{(2)}(\lambda_j; \lambda_j; \lambda_i; \beta)$. We wish to show that ξ_{12} is non-negative for

all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{12} with respect to λ_j gives

$$\begin{aligned} \xi_{12}(\lambda_j; \lambda_i; \beta) &= (38232000\beta^2 + 151839360\beta + 208327680 + 3456000\beta^3)\lambda_j^3 \\ &\quad + (7776000\beta^3 - 78727680 + 69690240\beta + 52228800\beta^2)\lambda_i\lambda_j^2 \\ &\quad + (38931840\beta + 79211520 + 16718400\beta^2 + 3110400\beta^3)\lambda_i^2\lambda_j \\ &\quad + (8069760\beta + 8916480 + 1684800\beta^2 + 172800\beta^3)\lambda_i^3, \\ \xi_{12}^{(1)}(\lambda_j; \lambda_i; \beta) &= (114696000\beta^2 + 455518080\beta + 624983040 + 10368000\beta^3)\lambda_j^2 \\ &\quad + (15552000\beta^3 - 157455360 + 139380480\beta + 104457600\beta^2)\lambda_i\lambda_j \\ &\quad + (38931840\beta + 79211520 + 16718400\beta^2 + 3110400\beta^3)\lambda_i^2, \\ \xi_{12}^{(2)}(\lambda_j; \lambda_i; \beta) &= (229392000\beta^2 + 911036160\beta + 1249966080 + 20736000\beta^3)\lambda_j \\ &\quad + (15552000\beta^3 - 157455360 + 139380480\beta + 104457600\beta^2)\lambda_i, \\ \xi_{12}^{(3)}(\lambda_j; \lambda_i; \beta) &= 229392000\beta^2 + 911036160\beta + 1249966080 + 20736000\beta^3 \\ &\geq 0 \text{ for all } \beta \geq -3.2529. \end{aligned}$$

Now

$$\begin{aligned} \xi_{12}(\lambda_i; \lambda_i; \beta) &= 7257600\lambda_i^3(2\beta + 5)(\beta + 3)(\beta + 2) \geq 0 \text{ for all } \beta \geq -2, \\ \xi_{12}^{(1)}(\lambda_i; \lambda_i; \beta) &= 1209600\lambda_i^2(195\beta^2 + 524\beta + 452 + 24\beta^3) \geq 0 \text{ for all } \beta \geq -1.8116, \\ \xi_{12}^{(2)}(\lambda_i; \lambda_i; \beta) &= 483840\lambda_i(690\beta^2 + 2171\beta + 2258 + 75\beta^3) \geq 0 \text{ for all } \beta \geq -2.3123. \end{aligned}$$

Hence, $\xi_1^{(2)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{11}(\lambda_j; \lambda_i; \beta) = \xi_1^{(1)}(\lambda_j; \lambda_j; \lambda_i; \beta)$. We wish to show that ξ_{11} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{11} with respect to λ_j gives

$$\begin{aligned} \xi_{11}(\lambda_j; \lambda_i; \beta) &= (15271200\beta^2 + 59400000\beta + 87955200 + 1512000\beta^3)\lambda_j^4 \\ &\quad + (4536000\beta^3 - 59258880 + 26657280\beta + 26974080\beta^2)\lambda_i\lambda_j^3 \\ &\quad + (21098880\beta + 40020480 + 12882240\beta^2 + 2721600\beta^3)\lambda_i^2\lambda_j^2 \\ &\quad + (10368000\beta + 7188480 + 2782080\beta^2 + 302400\beta^3)\lambda_i^3\lambda_j \\ &\quad + (151200\beta^2 + 411840\beta + 299520)\lambda_i^4, \\ \xi_{11}^{(1)}(\lambda_j; \lambda_i; \beta) &= (61084800\beta^2 + 237600000\beta + 351820800 + 6048000\beta^3)\lambda_j^3 \\ &\quad + (13608000\beta^3 - 177776640 + 79971840\beta + 80922240\beta^2)\lambda_i\lambda_j^2 \\ &\quad + (42197760\beta + 80040960 + 25764480\beta^2 + 5443200\beta^3)\lambda_i^2\lambda_j \\ &\quad + (10368000\beta + 7188480 + 2782080\beta^2 + 302400\beta^3)\lambda_i^3, \\ \xi_{11}^{(2)}(\lambda_j; \lambda_i; \beta) &= (183254400\beta^2 + 712800000\beta + 1055462400 + 18144000\beta^3)\lambda_j^2 \\ &\quad + (27216000\beta^3 - 355553280 + 159943680\beta + 161844480\beta^2)\lambda_i\lambda_j \\ &\quad + (42197760\beta + 80040960 + 25764480\beta^2 + 5443200\beta^3)\lambda_i^2, \\ \xi_{11}^{(3)}(\lambda_j; \lambda_i; \beta) &= (366508800\beta^2 + 1425600000\beta + 2110924800 + 36288000\beta^3)\lambda_j \\ &\quad + (27216000\beta^3 - 355553280 + 159943680\beta + 161844480\beta^2)\lambda_i, \\ \xi_{11}^{(4)}(\lambda_j; \lambda_i; \beta) &= 366508800\beta^2 + 1425600000\beta + 2110924800 + 36288000\beta^3 \\ &\geq 0 \text{ for all } \beta \geq -3.7755. \end{aligned}$$

Now

$$\begin{aligned}\xi_{11}(\lambda_i; \lambda_i; \beta) &= 1814400\lambda_i^4(5\beta + 7)(\beta + 3)(\beta + 2) \geq 0 \text{ for all } \beta \geq -\frac{7}{5}, \\ \xi_{11}^{(1)}(\lambda_i; \lambda_i; \beta) &= 3628800\lambda_i^3(\beta + 3)(\beta + 2)(7\beta + 12) \geq 0 \text{ for all } \beta \geq -\frac{12}{7}, \\ \xi_{11}^{(2)}(\lambda_i; \lambda_i; \beta) &= 241920\lambda_i^2(1533\beta^2 + 3782\beta + 3224 + 210\beta^3) \geq 0 \text{ for all } \beta \geq -3.0192, \\ \xi_{11}^{(3)}(\lambda_i; \lambda_i; \beta) &= 120960\lambda_i(4368\beta^2 + 13108\beta + 14512 + 525\beta^3) \geq 0 \text{ for all } \beta \geq -3.2688.\end{aligned}$$

Hence, $\xi_1^{(1)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{10}(\lambda_j; \lambda_i; \beta) = \xi_1(\lambda_j; \lambda_j; \lambda_i; \beta)/\lambda_j$. We wish to show that ξ_{10} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{10} with respect to λ_j gives

$$\begin{aligned}\xi_{10}(\lambda_j; \lambda_i; \beta) &= (4717440\beta^2 + 18264960\beta + 27820800 + 483840\beta^3)\lambda_j^4 \\ &\quad + (1814400\beta^3 - 24998400 + 8870400\beta + 10281600\beta^2)\lambda_i\lambda_j^3 \\ &\quad + (8951040\beta + 16934400 + 6531840\beta^2 + 1451520\beta^3)\lambda_i^2\lambda_j^2 \\ &\quad + (7257600\beta + 3870720 + 2177280\beta^2 + 241920\beta^3)\lambda_i^3\lambda_j \\ &\quad + (564480\beta + 322560 + 241920\beta^2)\lambda_i^4, \\ \xi_{10}^{(1)}(\lambda_j; \lambda_i; \beta) &= (18869760\beta^2 + 73059840\beta + 111283200 + 1935360\beta^3)\lambda_j^3 \\ &\quad + (5443200\beta^3 - 74995200 + 26611200\beta + 30844800\beta^2)\lambda_i\lambda_j^2 \\ &\quad + (17902080\beta + 33868800 + 13063680\beta^2 + 2903040\beta^3)\lambda_i^2\lambda_j \\ &\quad + (7257600\beta + 3870720 + 2177280\beta^2 + 241920\beta^3)\lambda_i^3, \\ \xi_{10}^{(2)}(\lambda_j; \lambda_i; \beta) &= (56609280\beta^2 + 219179520\beta + 333849600 + 5806080\beta^3)\lambda_j^2 \\ &\quad + (10886400\beta^3 - 149990400 + 53222400\beta + 61689600\beta^2)\lambda_i\lambda_j \\ &\quad + (17902080\beta + 33868800 + 13063680\beta^2 + 2903040\beta^3)\lambda_i^2, \\ \xi_{10}^{(3)}(\lambda_j; \lambda_i; \beta) &= (113218560\beta^2 + 438359040\beta + 667699200 + 11612160\beta^3)\lambda_j \\ &\quad + (10886400\beta^3 - 149990400 + 53222400\beta + 61689600\beta^2)\lambda_i, \\ \xi_{10}^{(4)}(\lambda_j; \lambda_i; \beta) &= 113218560\beta^2 + 438359040\beta + 667699200 + 11612160\beta^3 \\ &\geq 0 \text{ for all } \beta \geq -3.7954.\end{aligned}$$

Now

$$\begin{aligned}\xi_{10}(\lambda_i; \lambda_i; \beta) &= 3991680\lambda_i^4(\beta + 3)(\beta + 2)(\beta + 1) > 0 \text{ for all } \beta > -1, \\ \xi_{10}^{(1)}(\lambda_i; \lambda_i; \beta) &= 362880\lambda_i^3(\beta + 3)(\beta + 2)(29\beta + 34) > 0 \text{ for all } \beta > -\frac{34}{29}, \\ \xi_{10}^{(2)}(\lambda_i; \lambda_i; \beta) &= 725760\lambda_i^2(\beta + 3)(27\beta^2 + 100\beta + 100) > 0 \text{ for all } \beta > -3, \\ \xi_{10}^{(3)}(\lambda_i; \lambda_i; \beta) &= 241920\lambda_i(723\beta^2 + 2032\beta + 2140 + 93\beta^3) > 0 \text{ for all } \beta > -3.1755.\end{aligned}$$

Hence, ξ_1 is positive for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$. This implies that $\xi(\lambda_l; \lambda_l; \lambda_j; \lambda_i; \beta)$ is positive for all $\lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Now

$$\begin{aligned} \xi_2(\lambda_i; \lambda_j; \lambda_i; \beta) &= (682560\beta^2 + 4337280\beta + 4008960 + 17280\beta^3)\lambda_i^4 \\ &\quad + (9227520\beta + 9400320 + 2989440\beta^2 + 207360\beta^3)(\lambda_i + \lambda_j)\lambda_i^3 \\ &\quad + (2488320\beta^2 + 9279360\beta + 18766080 + 311040\beta^3)(\lambda_j^2 + \lambda_i^2)\lambda_i^2 \\ &\quad + (-30412800 + 933120\beta^3 + 6376320\beta^2 + 2868480\beta)\lambda_i\lambda_j\lambda_i^2 \\ &\quad + (4371840\beta + 4354560 + 734400\beta^2 + 69120\beta^3)(\lambda_j^3 + \lambda_i^3)\lambda_i \\ &\quad + (207360 + 622080\beta^3 + 2980800\beta^2 + 3888000\beta)(\lambda_i\lambda_j^2 + \lambda_i^2\lambda_j)\lambda_i \\ &\quad + (1848960\beta + 2280960 + 475200\beta^2 + 51840\beta^3)(\lambda_i\lambda_j^3 + \lambda_i^3\lambda_j) \\ &\quad + (10264320 + 699840\beta^2 + 155520\beta^3 + 2177280\beta)\lambda_i^2\lambda_j^2 \\ &\quad + (60480\beta^2 + 259200\beta + 276480)(\lambda_j^4 + \lambda_i^4), \end{aligned}$$

and, hence,

$$\begin{aligned} \xi_2^{(3)}(\lambda_i; \lambda_j; \lambda_i; \beta) &= (16381440\beta^2 + 104094720\beta + 96215040 + 414720\beta^3)\lambda_i \\ &\quad + (55365120\beta + 56401920 + 17936640\beta^2 + 1244160\beta^3)(\lambda_j + \lambda_i) \\ &\geq 0 \text{ for all } \beta \geq -1.1141. \end{aligned}$$

Define $\xi_{22}(\lambda_j; \lambda_i; \beta) = \xi_2^{(2)}(\lambda_j; \lambda_j; \lambda_i; \beta)$. We wish to show that ξ_{22} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{22} with respect to λ_j gives

$$\begin{aligned} \xi_{22}(\lambda_j; \lambda_i; \beta) &= (31104000\beta^2 + 125971200\beta + 142041600 + 2073600\beta^3)\lambda_j^2 \\ &\quad + (61102080\beta - 4423680 + 30689280\beta^2 + 3110400\beta^3)\lambda_i\lambda_j \\ &\quad + (4976640\beta^2 + 18558720\beta + 37532160 + 622080\beta^3)\lambda_i^2, \\ \xi_{22}^{(1)}(\lambda_j; \lambda_i; \beta) &= (62208000\beta^2 + 251942400\beta + 284083200 + 4147200\beta^3)\lambda_j \\ &\quad + (61102080\beta - 4423680 + 30689280\beta^2 + 3110400\beta^3)\lambda_i, \\ \xi_{22}^{(2)}(\lambda_j; \lambda_i; \beta) &= 62208000\beta^2 + 251942400\beta + 284083200 + 4147200\beta^3 \\ &\geq 0 \text{ for all } \beta \geq -1.9255. \end{aligned}$$

Now

$$\begin{aligned} \xi_{22}(\lambda_i; \lambda_i; \beta) &= 483840\lambda_i^2(138\beta^2 + 425\beta + 362 + 12\beta^3) \geq 0 \text{ for all } \beta \geq -1.4432, \\ \xi_{22}^{(1)}(\lambda_i; \lambda_i; \beta) &= 483840\lambda_i(192\beta^2 + 647\beta + 578 + 15\beta^3) \geq 0 \text{ for all } \beta \geq -1.4382, \\ \xi_{22}^{(2)}(\lambda_i; \lambda_i; \beta) &= 62208000\beta^2 + 251942400\beta + 284083200 + 4147200\beta^3 \\ &\geq 0 \text{ for all } \beta \geq -1.9255. \end{aligned}$$

Hence, $\xi_2^{(2)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{21}(\lambda_j; \lambda_i; \beta) = \xi_2^{(1)}(\lambda_j; \lambda_j; \lambda_i; \beta)$. We wish to show that ξ_{21} is non-negative for

all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{21} with respect to λ_j gives

$$\begin{aligned} \xi_{21}(\lambda_j; \lambda_i; \beta) &= (17409600\beta^2 + 67962240\beta + 86123520 + 1382400\beta^3)\lambda_j^3 \\ &\quad + (37307520\beta - 32417280 + 24701760\beta^2 + 3110400\beta^3)\lambda_i\lambda_j^2 \\ &\quad + (7957440\beta^2 + 22446720\beta + 37739520 + 1244160\beta^3)\lambda_i^2\lambda_j \\ &\quad + (4371840\beta + 4354560 + 734400\beta^2 + 69120\beta^3)\lambda_i^3, \\ \xi_{21}^{(1)}(\lambda_j; \lambda_i; \beta) &= (52228800\beta^2 + 203886720\beta + 258370560 + 4147200\beta^3)\lambda_j^2 \\ &\quad + (74615040\beta - 64834560 + 49403520\beta^2 + 6220800\beta^3)\lambda_i\lambda_j \\ &\quad + (7957440\beta^2 + 22446720\beta + 37739520 + 1244160\beta^3)\lambda_i^2, \\ \xi_{21}^{(2)}(\lambda_j; \lambda_i; \beta) &= (104457600\beta^2 + 407773440\beta + 516741120 + 8294400\beta^3)\lambda_j \\ &\quad + (74615040\beta - 64834560 + 49403520\beta^2 + 6220800\beta^3)\lambda_i, \\ \xi_{21}^{(3)}(\lambda_j; \lambda_i; \beta) &= 104457600\beta^2 + 407773440\beta + 516741120 + 8294400\beta^3 \\ &\geq 0 \text{ for all } \beta \geq -6.5166. \end{aligned}$$

Now

$$\begin{aligned} \xi_{21}(\lambda_i; \lambda_i; \beta) &= 1451520\lambda_i^3(\beta + 3)(4\beta^2 + 23\beta + 22) \geq 0 \text{ for all } \beta \geq -1.2120, \\ \xi_{21}^{(1)}(\lambda_i; \lambda_i; \beta) &= 241920\lambda_i^2(453\beta^2 + 1244\beta + 956 + 48\beta^3) \geq 0 \text{ for all } \beta \geq -1.2965, \\ \xi_{21}^{(2)}(\lambda_i; \lambda_i; \beta) &= 483840\lambda_i(318\beta^2 + 997\beta + 934 + 30\beta^3) \geq 0 \text{ for all } \beta \geq -1.7649. \end{aligned}$$

Hence, $\xi_{21}^{(1)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{20}(\lambda_j; \lambda_i; \beta) = \xi_2(\lambda_j; \lambda_j; \lambda_i; \beta)$. We wish to show that ξ_{20} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{20} with respect to λ_j gives

$$\begin{aligned} \xi_{20}(\lambda_j; \lambda_i; \beta) &= (6955200\beta^2 + 27475200\beta + 36806400 + 604800\beta^3)\lambda_j^4 \\ &\quad + (-18524160 + 1814400\beta^3 + 12821760\beta^2 + 17832960\beta)\lambda_i\lambda_j^3 \\ &\quad + (6168960\beta^2 + 15344640\beta + 29237760 + 1088640\beta^3)\lambda_i^2\lambda_j^2 \\ &\quad + (6220800\beta + 6635520 + 1209600\beta^2 + 120960\beta^3)\lambda_i^3\lambda_j \\ &\quad + (60480\beta^2 + 259200\beta + 276480)\lambda_i^4, \\ \xi_{20}^{(1)}(\lambda_j; \lambda_i; \beta) &= (27820800\beta^2 + 109900800\beta + 147225600 + 2419200\beta^3)\lambda_j^3 \\ &\quad + (-55572480 + 5443200\beta^3 + 38465280\beta^2 + 53498880\beta)\lambda_i\lambda_j^2 \\ &\quad + (12337920\beta^2 + 30689280\beta + 58475520 + 2177280\beta^3)\lambda_i^2\lambda_j \\ &\quad + (6220800\beta + 6635520 + 1209600\beta^2 + 120960\beta^3)\lambda_i^3, \\ \xi_{20}^{(2)}(\lambda_j; \lambda_i; \beta) &= (83462400\beta^2 + 329702400\beta + 441676800 + 7257600\beta^3)\lambda_j^2 \\ &\quad + (-111144960 + 10886400\beta^3 + 76930560\beta^2 + 106997760\beta)\lambda_i\lambda_j \\ &\quad + (12337920\beta^2 + 30689280\beta + 58475520 + 2177280\beta^3)\lambda_i^2, \\ \xi_{20}^{(3)}(\lambda_j; \lambda_i; \beta) &= (166924800\beta^2 + 659404800\beta + 883353600 + 14515200\beta^3)\lambda_j \\ &\quad + (-111144960 + 10886400\beta^3 + 76930560\beta^2 + 106997760\beta)\lambda_i, \\ \xi_{20}^{(4)}(\lambda_j; \lambda_i; \beta) &= 166924800\beta^2 + 659404800\beta + 883353600 + 14515200\beta^3 \\ &\geq 0 \text{ for all } \beta \geq -3.4191. \end{aligned}$$

Now

$$\begin{aligned} \xi_{20}(\lambda_i; \lambda_i; \beta) &= 1814400\lambda_i^4(2\beta + 5)(\beta + 3)(\beta + 2) > 0 \text{ for all } \beta > -2, \\ \xi_{20}^{(1)}(\lambda_i; \lambda_i; \beta) &= 1451520\lambda_i^3(\beta + 3)(7\beta^2 + 34\beta + 36) > 0 \text{ for all } \beta > -1.5596, \\ \xi_{20}^{(2)}(\lambda_i; \lambda_i; \beta) &= 1451520\lambda_i^2(119\beta^2 + 322\beta + 268 + 14\beta^3) > 0 \text{ for all } \beta > -1.6020, \\ \xi_{20}^{(3)}(\lambda_i; \lambda_i; \beta) &= 725760\lambda_i(336\beta^2 + 1056\beta + 1064 + 35\beta^3) > 0 \text{ for all } \beta > -2.1389. \end{aligned}$$

Hence, ξ_2 is positive for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$. This implies that $\xi(\lambda_l; \lambda_l; \lambda_j; \lambda_i; \beta)$ is positive for all $\lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Now

$$\begin{aligned} \xi_3(\lambda_l; \lambda_j; \lambda_i; \beta) &= (984960\beta^2 + 5633280\beta + 5391360 + 17280\beta^3)\lambda_l^3 \\ &\quad + (9210240\beta + 7395840 + 3602880\beta^2 + 155520\beta^3)(\lambda_j + \lambda_i)\lambda_l^2 \\ &\quad + (2151360\beta^2 + 8657280\beta + 10160640 + 155520\beta^3)(\lambda_j^2 + \lambda_i^2)\lambda_l \\ &\quad + (6359040\beta - 23500800 + 5909760\beta^2 + 466560\beta^3)\lambda_i\lambda_j\lambda_l \\ &\quad + (17280\beta^3 + 259200\beta^2 + 2073600 + 2522880\beta)(\lambda_j^3 + \lambda_i^3) \\ &\quad + (5546880\beta + 6842880 + 1425600\beta^2 + 155520\beta^3)(\lambda_i\lambda_j^2 + \lambda_i^2\lambda_j), \\ \xi_3^{(1)}(\lambda_l; \lambda_j; \lambda_i; \beta) &= (2954880\beta^2 + 16899840\beta + 16174080 + 51840\beta^3)\lambda_l^2 \\ &\quad + ((18420480\beta + 14791680 + 7205760\beta^2 + 311040\beta^3)\lambda_j \\ &\quad + 2(9210240\beta + 7395840 + 3602880\beta^2 + 155520\beta^3)\lambda_i)\lambda_l \\ &\quad + (6359040\beta - 23500800 + 5909760\beta^2 + 466560\beta^3)\lambda_i\lambda_j \\ &\quad + (2151360\beta^2 + 8657280\beta + 10160640 + 155520\beta^3)\lambda_j^2 \\ &\quad + (2151360\beta^2 + 8657280\beta + 10160640 + 155520\beta^3)\lambda_i^2, \\ \xi_3^{(2)}(\lambda_l; \lambda_j; \lambda_i; \beta) &= (5909760\beta^2 + 33799680\beta + 32348160 + 103680\beta^3)\lambda_l \\ &\quad + (18420480\beta + 14791680 + 7205760\beta^2 + 311040\beta^3)(\lambda_i + \lambda_j) \\ &\geq 0 \text{ for all } \beta \geq -1.2060. \end{aligned}$$

Define $\xi_{31}(\lambda_j; \lambda_i; \beta) = \xi_3^{(1)}(\lambda_j; \lambda_j; \lambda_i; \beta)$. We wish to show that ξ_{31} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{31} with respect to λ_j gives

$$\begin{aligned} \xi_{31}(\lambda_j; \lambda_i; \beta) &= (12312000\beta^2 + 43977600\beta + 41126400 + 518400\beta^3)\lambda_j^2 \\ &\quad + (24779520\beta - 8709120 + 13115520\beta^2 + 777600\beta^3)\lambda_i\lambda_j \\ &\quad + (2151360\beta^2 + 8657280\beta + 10160640 + 155520\beta^3)\lambda_i^2, \\ \xi_{31}^{(1)}(\lambda_j; \lambda_i; \beta) &= (24624000\beta^2 + 87955200\beta + 82252800 + 1036800\beta^3)\lambda_j \\ &\quad + (24779520\beta - 8709120 + 13115520\beta^2 + 777600\beta^3)\lambda_i, \\ \xi_{31}^{(2)}(\lambda_j; \lambda_i; \beta) &= 24624000\beta^2 + 87955200\beta + 82252800 + 1036800\beta^3 \geq 0 \text{ for all } \beta \geq -1.6188. \end{aligned}$$

Now

$$\begin{aligned} \xi_{31}(\lambda_i; \lambda_i; \beta) &= 483840\lambda_i^2(57\beta^2 + 160\beta + 88 + 3\beta^3) \geq 0 \text{ for all } \beta \geq -0.7350, \\ \xi_{31}^{(1)}(\lambda_i; \lambda_i; \beta) &= 120960\lambda_i(312\beta^2 + 932\beta + 608 + 15\beta^3) \geq 0 \text{ for all } \beta \geq -0.9275. \end{aligned}$$

Hence, $\xi_3^{(1)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{30}(\lambda_j; \lambda_i; \beta) = \xi_3(\lambda_j; \lambda_j; \lambda_i; \beta)$. We wish to show that ξ_{30} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{30} with respect to λ_j gives

$$\begin{aligned} \xi_{30}(\lambda_j; \lambda_i; \beta) &= (6998400\beta^2 + 26023680\beta + 25021440 + 345600\beta^3)\lambda_j^3 \\ &\quad + (21116160\beta - 9262080 + 10938240\beta^2 + 777600\beta^3)\lambda_i\lambda_j^2 \\ &\quad + (3576960\beta^2 + 14204160\beta + 17003520 + 311040\beta^3)\lambda_i^2\lambda_j \\ &\quad + (17280\beta^3 + 259200\beta^2 + 2073600 + 2522880\beta)\lambda_i^3, \\ \xi_{30}^{(1)}(\lambda_j; \lambda_i; \beta) &= (20995200\beta^2 + 78071040\beta + 75064320 + 1036800\beta^3)\lambda_j^2 \\ &\quad + (42232320\beta - 18524160 + 21876480\beta^2 + 1555200\beta^3)\lambda_i\lambda_j \\ &\quad + (3576960\beta^2 + 14204160\beta + 17003520 + 311040\beta^3)\lambda_i^2, \\ \xi_{30}^{(2)}(\lambda_j; \lambda_i; \beta) &= (41990400\beta^2 + 156142080\beta + 150128640 + 2073600\beta^3)\lambda_j \\ &\quad + (42232320\beta - 18524160 + 21876480\beta^2 + 1555200\beta^3)\lambda_i, \\ \xi_{30}^{(3)}(\lambda_j; \lambda_i; \beta) &= 41990400\beta^2 + 156142080\beta + 150128640 + 2073600\beta^3 \\ &\geq 0 \text{ for all } \beta \geq -1.5818. \end{aligned}$$

Now

$$\begin{aligned} \xi_{30}(\lambda_i; \lambda_i; \beta) &= 1451520\lambda_i^3(\beta + 3)(\beta^2 + 12\beta + 8) > 0 \text{ for all } \beta > -0.7085, \\ \xi_{30}^{(1)}(\lambda_i; \lambda_i; \beta) &= 967680\lambda_i^2(48\beta^2 + 139\beta + 76 + 3\beta^3) > 0 \text{ for all } \beta > -0.7158, \\ \xi_{30}^{(2)}(\lambda_i; \lambda_i; \beta) &= 241920\lambda_i(264\beta^2 + 820\beta + 544 + 15\beta^3) > 0 \text{ for all } \beta > -0.9237. \end{aligned}$$

Hence, ξ_3 is positive for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$. This implies that $\xi(\lambda_l; \lambda_l; \lambda_j; \lambda_i; \beta)$ is positive for all $\lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Differentiating $\xi_4(\lambda_l; \lambda_j; \lambda_i; \beta)$ with respect to λ_l we obtain

$$\begin{aligned} \xi_4(\lambda_l; \lambda_j; \lambda_i; \beta) &= (725760\beta^2 + 3110400\beta + 3317760)\lambda_l^2 \\ &\quad + (3663360\beta + 552960 + 2177280\beta^2)(\lambda_j + \lambda_i)\lambda_l \\ &\quad + (3663360\beta + 552960 + 2177280\beta^2)\lambda_i\lambda_j \\ &\quad + (725760\beta^2 + 3110400\beta + 3317760)(\lambda_j^2 + \lambda_i^2), \\ \xi_4^{(1)}(\lambda_l; \lambda_j; \lambda_i; \beta) &= (1451520\beta^2 + 6220800\beta + 6635520)\lambda_l \\ &\quad + (3663360\beta + 552960 + 2177280\beta^2)\lambda_j \\ &\quad + (3663360\beta + 552960 + 2177280\beta^2)\lambda_i, \\ \xi_4^{(2)}(\lambda_l; \lambda_j; \lambda_i; \beta) &= 1451520\beta^2 + 6220800\beta + 6635520 \geq 0 \text{ for all } \beta \geq -2. \end{aligned}$$

Define $\xi_{41}(\lambda_j; \lambda_i; \beta) = \xi_4^{(1)}(\lambda_j; \lambda_j; \lambda_i; \beta)$. We wish to show that ξ_{41} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{41} with respect to λ_j gives

$$\begin{aligned} \xi_{41}(\lambda_j; \lambda_i; \beta) &= (3628800\beta^2 + 9884160\beta + 7188480)\lambda_j \\ &\quad + (3663360\beta + 552960 + 2177280\beta^2)\lambda_i, \\ \xi_{41}^{(1)}(\lambda_j; \lambda_i; \beta) &= 3628800\beta^2 + 9884160\beta + 7188480 \geq 0 \text{ for all } \beta. \end{aligned}$$

Now

$$\xi_{41}(\lambda_i; \lambda_i; \beta) = 1935360\lambda_i(3\beta + 4)(\beta + 1) \geq 0 \text{ for all } \beta \geq -1.$$

Hence, $\xi_4^{(1)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{40}(\lambda_j; \lambda_i; \beta) = \xi_4(\lambda_j; \lambda_j; \lambda_i; \beta)$. We wish to show that ξ_{40} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{40} with respect to λ_j gives

$$\begin{aligned} \xi_{40}(\lambda_j; \lambda_i; \beta) &= (3628800\beta^2 + 9884160\beta + 7188480)\lambda_j^2 \\ &\quad + (7326720\beta + 1105920 + 4354560\beta^2)\lambda_i\lambda_j \\ &\quad + (725760\beta^2 + 3110400\beta + 3317760)\lambda_i^2, \\ \xi_{40}^{(1)}(\lambda_j; \lambda_i; \beta) &= (7257600\beta^2 + 19768320\beta + 14376960)\lambda_j \\ &\quad + (7326720\beta + 1105920 + 4354560\beta^2)\lambda_i, \\ \xi_{40}^{(2)}(\lambda_j; \lambda_i; \beta) &= 7257600\beta^2 + 19768320\beta + 14376960 \geq 0 \text{ for all } \beta. \end{aligned}$$

Now

$$\begin{aligned} \xi_{40}(\lambda_i; \lambda_i; \beta) &= 2903040\lambda_i^2(3\beta + 4)(\beta + 1) > 0 \text{ for all } \beta > -1, \\ \xi_{40}^{(1)}(\lambda_i; \lambda_i; \beta) &= 3870720\lambda_i(3\beta + 4)(\beta + 1) > 0 \text{ for all } \beta > -1. \end{aligned}$$

Hence, ξ_4 is positive for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$. This implies that $\xi(\lambda_l; \lambda_l; \lambda_j; \lambda_i; \beta)$ is positive for all $\lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

□

Lemma B.18 Let

$$\begin{aligned} \xi(\lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta) &= 20160(322560 + 564480\beta + 241920\beta^2)t(0, 0, 1, 4) \\ &\quad + (60480\beta^3 + 3870720 + 3628800\beta + 725760\beta^2)t(0, 0, 2, 3) \\ &\quad + (1935360\beta^2 + 181440\beta^3 + 645120 + 4757760\beta)t(0, 1, 1, 3) \\ &\quad + (2903040\beta^2 + 544320\beta^3 + 13789440 + 6894720\beta)t(0, 1, 2, 2) \\ &\quad + (-3628800\beta + 6531840\beta^2 - 31933440 + 1632960\beta^3)t(1, 1, 1, 2), \end{aligned}$$

where

$$t(a, b, c, d) = \sum_{\substack{(n_1, n_2, n_3, n_4) \\ \in \text{perm}(a, b, c, d)}} \lambda_i^{n_1} \lambda_j^{n_2} \lambda_l^{n_3} \lambda_k^{n_4}.$$

The expression $\xi(\lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta)$ is positive for all $\lambda_k \geq \lambda_l \geq \lambda_j \geq \lambda_i$, and $\beta \geq -\frac{2}{5}$.

Proof.

Let

$$\hat{s}(a, b, c) = \sum_{\substack{(n_1, n_2, n_3) \\ \in \text{perm}(a, b, c)}} \lambda_i^{n_1} \lambda_j^{n_2} \lambda_l^{n_3}.$$

Differentiating $\xi(\lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta)$ with respect to λ_k we obtain

$$\begin{aligned} \xi^{(1)}(\lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta) &= (2257920\beta + 967680\beta^2 + 1290240)\lambda_k^3 \hat{s}(0, 0, 1) \\ &\quad + (1935360 + 14273280\beta + 544320\beta^3 + 5806080\beta^2)\lambda_k^2 \hat{s}(0, 1, 1) \\ &\quad + (11612160 + 2177280\beta^2 + 181440\beta^3 + 10886400\beta)\lambda_k^2 \hat{s}(0, 0, 2) \\ &\quad + (7741440 + 7257600\beta + 1451520\beta^2 + 120960\beta^3)\lambda_k \hat{s}(0, 0, 3) \\ &\quad + (-7257600\beta + 13063680\beta^2 - 63866880 + 3265920\beta^3)\lambda_k \hat{s}(1, 1, 1) \\ &\quad + (13789440\beta + 1088640\beta^3 + 5806080\beta^2 + 27578880)\lambda_k \hat{s}(0, 1, 2) \\ &\quad + (1632960\beta^3 - 3628800\beta - 31933440 + 6531840\beta^2)\hat{s}(1, 1, 2) \\ &\quad + (564480\beta + 322560 + 241920\beta^2)\hat{s}(0, 0, 4) \\ &\quad + (1935360\beta^2 + 181440\beta^3 + 4757760\beta + 645120)\hat{s}(0, 1, 3) \\ &\quad + (544320\beta^3 + 6894720\beta + 13789440 + 2903040\beta^2)\hat{s}(0, 2, 2), \\ \xi^{(2)}(\lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta) &= 3(2257920\beta + 967680\beta^2 + 1290240)\lambda_k^2 \hat{s}(0, 0, 1) \\ &\quad + 2(1935360 + 14273280\beta + 544320\beta^3 + 5806080\beta^2)\lambda_k \hat{s}(0, 1, 1) \\ &\quad + 2(11612160 + 2177280\beta^2 + 181440\beta^3 + 10886400\beta)\lambda_k \hat{s}(0, 0, 2) \\ &\quad + (7741440 + 7257600\beta + 1451520\beta^2 + 120960\beta^3)\hat{s}(0, 0, 3) \\ &\quad + (-7257600\beta + 13063680\beta^2 - 63866880 + 3265920\beta^3)\hat{s}(1, 1, 1) \\ &\quad + (13789440\beta + 1088640\beta^3 + 5806080\beta^2 + 27578880)\hat{s}(0, 1, 2), \\ \xi^{(3)}(\lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta) &= 6(2257920\beta + 967680\beta^2 + 1290240)\lambda_k \hat{s}(0, 0, 1) \\ &\quad + 2(1935360 + 14273280\beta + 544320\beta^3 + 5806080\beta^2)\hat{s}(0, 1, 1) \\ &\quad + 2(11612160 + 2177280\beta^2 + 181440\beta^3 + 10886400\beta)\hat{s}(0, 0, 2), \\ \xi^{(4)}(\lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta) &= 6(2257920\beta + 967680\beta^2 + 1290240)\hat{s}(0, 0, 1) \geq 0 \text{ for all } \beta \geq -1. \end{aligned}$$

Define

$$\begin{aligned} \xi_0(\lambda_l; \lambda_j; \lambda_i; \beta) &\stackrel{\text{def}}{=} \xi(\lambda_l; \lambda_l; \lambda_j; \lambda_i; \beta), \\ \xi_1(\lambda_l; \lambda_j; \lambda_i; \beta) &\stackrel{\text{def}}{=} \xi^{(1)}(\lambda_l; \lambda_l; \lambda_j; \lambda_i; \beta), \\ \xi_2(\lambda_l; \lambda_j; \lambda_i; \beta) &\stackrel{\text{def}}{=} \xi^{(2)}(\lambda_l; \lambda_l; \lambda_j; \lambda_i; \beta), \\ \xi_3(\lambda_l; \lambda_j; \lambda_i; \beta) &\stackrel{\text{def}}{=} \xi^{(3)}(\lambda_l; \lambda_l; \lambda_j; \lambda_i; \beta). \end{aligned}$$

We will prove that the above equations are non-negative for all $\lambda_k \geq \lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$ and, hence, deduce that $\xi(\lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta)$ is positive for all $\lambda_k \geq \lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Now

$$\begin{aligned} \xi_0(\lambda_l; \lambda_j; \lambda_i; \beta) = & (8386560\beta + 1935360\beta^2 + 8386560 + 120960\beta^3)\lambda_l^5 \\ & +(15724800 + 17539200\beta + 907200\beta^3 + 7257600\beta^2)(\lambda_j + \lambda_i)\lambda_l^4 \\ & +(35320320 + 1209600\beta^3 + 7257600\beta^2 + 21047040\beta)(\lambda_j^2 + \lambda_i^2)\lambda_l^3 \\ & +(16934400\beta^2 + 3628800\beta^3 - 62576640 + 2257920\beta)\lambda_i\lambda_j\lambda_l^3 \\ & +(10160640\beta - 4354560 + 2721600\beta^3 + 12337920\beta^2)(\lambda_i\lambda_j^2 + \lambda_i^2\lambda_j)\lambda_l^2 \\ & +(8386560 + 12015360\beta + 3386880\beta^2 + 302400\beta^3)(\lambda_j^3 + \lambda_i^3)\lambda_l^2 \\ & +(483840\beta^2 + 645120 + 1128960\beta)(\lambda_j^4 + \lambda_i^4)\lambda_l \\ & +(9515520\beta + 362880\beta^3 + 1290240 + 3870720\beta^2)(\lambda_i\lambda_j^3 + \lambda_i^3\lambda_j)\lambda_l \\ & +(5806080\beta^2 + 13789440\beta + 1088640\beta^3 + 27578880)\lambda_i^2\lambda_j^2\lambda_l \\ & +(725760\beta^2 + 60480\beta^3 + 3870720 + 3628800\beta)(\lambda_i^3\lambda_j^2 + \lambda_i^2\lambda_j^3) \\ & +(564480\beta + 322560 + 241920\beta^2)(\lambda_i\lambda_j^4 + \lambda_i^4\lambda_j), \end{aligned}$$

and, hence,

$$\begin{aligned} \xi_0^{(4)}(\lambda_l; \lambda_j; \lambda_i; \beta) = & (1006387200\beta + 232243200\beta^2 + 1006387200 + 14515200\beta^3)\lambda_l \\ & +(377395200 + 420940800\beta + 21772800\beta^3 + 174182400\beta^2)(\lambda_i + \lambda_j) \\ \geq & 0 \text{ for all } \beta \geq -1.4294. \end{aligned}$$

Define $\xi_{03}(\lambda_j; \lambda_i; \beta) = \xi_0^{(3)}(\lambda_j; \lambda_j; \lambda_i; \beta)$. We wish to show that ξ_{03} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{03} with respect to λ_j gives

$$\begin{aligned} \xi_{03}(\lambda_j; \lambda_i; \beta) = & (1050416640\beta + 333849600\beta^2 + 1092510720 + 36288000\beta^3)\lambda_j^2 \\ & +(1935360 + 434488320\beta + 43545600\beta^3 + 275788800\beta^2)\lambda_i\lambda_j \\ & +(211921920 + 7257600\beta^3 + 43545600\beta^2 + 126282240\beta)\lambda_i^2, \\ \xi_{03}^{(1)}(\lambda_j; \lambda_i; \beta) = & (2100833280\beta + 667699200\beta^2 + 2185021440 + 72576000\beta^3)\lambda_j \\ & +(1935360 + 434488320\beta + 43545600\beta^3 + 275788800\beta^2)\lambda_i, \\ \xi_{03}^{(2)}(\lambda_j; \lambda_i; \beta) = & 2100833280\beta + 667699200\beta^2 + 2185021440 + 72576000\beta^3 \\ \geq & 0 \text{ for all } \beta \geq -2.3123. \end{aligned}$$

Now

$$\begin{aligned} \xi_{03}(\lambda_i; \lambda_i; \beta) = & 43545600\lambda_i^2(2\beta + 5)(\beta + 3)(\beta + 2) \geq 0 \text{ for all } \beta \geq -2, \\ \xi_{03}^{(1)}(\lambda_i; \lambda_i; \beta) = & 4838400\lambda_i(524\beta + 195\beta^2 + 452 + 24\beta^3) \geq 0 \text{ for all } \beta \geq -1.8116. \end{aligned}$$

Hence, $\xi_0^{(3)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{02}(\lambda_j; \lambda_i; \beta) = \xi_0^{(2)}(\lambda_j; \lambda_j; \lambda_i; \beta)$. We wish to show that ξ_{02} is non-negative for

all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{02} with respect to λ_j gives

$$\begin{aligned} \xi_{02}(\lambda_j; \lambda_i; \beta) &= (528514560\beta + 176117760\beta^2 + 585123840 + 21168000\beta^3)\lambda_j^3 \\ &\quad + (213373440\beta^2 + 38102400\beta^3 - 195471360 + 244339200\beta)\lambda_i\lambda_j^2 \\ &\quad + (203212800 + 12700800\beta^3 + 68221440\beta^2 + 146603520\beta)\lambda_i^2\lambda_j \\ &\quad + (16773120 + 24030720\beta + 6773760\beta^2 + 604800\beta^3)\lambda_i^3, \\ \xi_{02}^{(1)}(\lambda_j; \lambda_i; \beta) &= (1585543680\beta + 528353280\beta^2 + 1755371520 + 63504000\beta^3)\lambda_j^2 \\ &\quad + (426746880\beta^2 + 76204800\beta^3 - 390942720 + 488678400\beta)\lambda_i\lambda_j \\ &\quad + (203212800 + 12700800\beta^3 + 68221440\beta^2 + 146603520\beta)\lambda_i^2, \\ \xi_{02}^{(2)}(\lambda_j; \lambda_i; \beta) &= (3171087360\beta + 1056706560\beta^2 + 3510743040 + 127008000\beta^3)\lambda_j \\ &\quad + (426746880\beta^2 + 76204800\beta^3 - 390942720 + 488678400\beta)\lambda_i, \\ \xi_{02}^{(3)}(\lambda_j; \lambda_i; \beta) &= 3171087360\beta + 1056706560\beta^2 + 3510743040 + 127008000\beta^3 \\ &\geq 0 \text{ for all } \beta \geq -3.2688. \end{aligned}$$

Now

$$\begin{aligned} \xi_{02}(\lambda_i; \lambda_i; \beta) &= 14515200\lambda_i^3(5\beta + 7)(\beta + 3)(\beta + 2) \geq 0 \text{ for all } \beta \geq -2, \\ \xi_{02}^{(1)}(\lambda_i; \lambda_i; \beta) &= 21772800\lambda_i^2(\beta + 3)(\beta + 2)(7\beta + 12) \geq 0 \text{ for all } \beta \geq -\frac{12}{7}, \\ \xi_{02}^{(2)}(\lambda_i; \lambda_i; \beta) &= 967680\lambda_i(3782\beta + 1533\beta^2 + 3224 + 210\beta^3) \geq 0 \text{ for all } \beta \geq -3.0192. \end{aligned}$$

Hence, $\xi_0^{(2)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{01}(\lambda_j; \lambda_i; \beta) = \xi_0^{(1)}(\lambda_j; \lambda_j; \lambda_i; \beta)$. We wish to show that ξ_{01} is non-negative for

all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{01} with respect to λ_j gives

$$\begin{aligned} \xi_{01}(\lambda_j; \lambda_i; \beta) &= (200390400\beta + 67737600\beta^2 + 228211200 + 8467200\beta^3)\lambda_j^4 \\ &\quad + (-132249600 + 106767360\beta + 20321280\beta^3 + 108380160\beta^2)\lambda_i\lambda_j^3 \\ &\quad + (124830720 + 10160640\beta^3 + 52254720\beta^2 + 97251840\beta)\lambda_i^2\lambda_j^2 \\ &\quad + (18063360 + 33546240\beta + 10644480\beta^2 + 967680\beta^3)\lambda_i^3\lambda_j \\ &\quad + (483840\beta^2 + 645120 + 1128960\beta)\lambda_i^4, \\ \xi_{01}^{(1)}(\lambda_j; \lambda_i; \beta) &= (801561600\beta + 270950400\beta^2 + 912844800 + 33868800\beta^3)\lambda_j^3 \\ &\quad + (-396748800 + 320302080\beta + 60963840\beta^3 + 325140480\beta^2)\lambda_i\lambda_j^2 \\ &\quad + (249661440 + 20321280\beta^3 + 104509440\beta^2 + 194503680\beta)\lambda_i^2\lambda_j \\ &\quad + (18063360 + 33546240\beta + 10644480\beta^2 + 967680\beta^3)\lambda_i^3, \\ \xi_{01}^{(2)}(\lambda_j; \lambda_i; \beta) &= (2404684800\beta + 812851200\beta^2 + 2738534400 + 101606400\beta^3)\lambda_j^2 \\ &\quad + (-793497600 + 640604160\beta + 121927680\beta^3 + 650280960\beta^2)\lambda_i\lambda_j \\ &\quad + (249661440 + 20321280\beta^3 + 104509440\beta^2 + 194503680\beta)\lambda_i^2, \\ \xi_{01}^{(3)}(\lambda_j; \lambda_i; \beta) &= (4809369600\beta + 1625702400\beta^2 + 5477068800 + 203212800\beta^3)\lambda_j \\ &\quad + (-793497600 + 640604160\beta + 121927680\beta^3 + 650280960\beta^2)\lambda_i, \\ \xi_{01}^{(4)}(\lambda_j; \lambda_i; \beta) &= 4809369600\beta + 1625702400\beta^2 + 5477068800 + 203212800\beta^3 \\ &\geq 0 \text{ for all } \beta \geq -3.3100. \end{aligned}$$

Now

$$\begin{aligned} \xi_{01}(\lambda_i; \lambda_i; \beta) &= 39916800\lambda_i^4(\beta + 3)(\beta + 2)(\beta + 1) \geq 0 \text{ for all } \beta \geq -1, \\ \xi_{01}^{(1)}(\lambda_i; \lambda_i; \beta) &= 14515200\lambda_i^3(8\beta + 9)(\beta + 3)(\beta + 2) \geq 0 \text{ for all } \beta \geq -\frac{9}{8}, \\ \xi_{01}^{(2)}(\lambda_i; \lambda_i; \beta) &= 34836480\lambda_i^2(\beta + 3)(7\beta^2 + 24\beta + 21) \geq 0 \text{ for all } \beta \geq -3, \\ \xi_{01}^{(3)}(\lambda_i; \lambda_i; \beta) &= 7741440\lambda_i(704\beta + 294\beta^2 + 605 + 42\beta^3) \geq 0 \text{ for all } \beta \geq -3.0629. \end{aligned}$$

Hence, $\xi_0^{(1)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{00}(\lambda_j; \lambda_i; \beta) = \xi_0(\lambda_j; \lambda_j; \lambda_i; \beta)/\lambda_j$. We wish to show that ξ_{00} is non-negative for

all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{00} with respect to λ_j gives

$$\begin{aligned} \xi_{00}(\lambda_j; \lambda_i; \beta) &= (60117120\beta + 20321280\beta^2 + 68463360 + 2540160\beta^3)\lambda_j^4 \\ &\quad + (40642560\beta^2 + 7620480\beta^3 - 49593600 + 40037760\beta)\lambda_i\lambda_j^3 \\ &\quad + (62415360 + 5080320\beta^3 + 26127360\beta^2 + 48625920\beta)\lambda_i^2\lambda_j^2 \\ &\quad + (13547520 + 25159680\beta + 7983360\beta^2 + 725760\beta^3)\lambda_i^3\lambda_j \\ &\quad + (725760\beta^2 + 967680 + 1693440\beta)\lambda_i^4, \\ \xi_{00}^{(1)}(\lambda_j; \lambda_i; \beta) &= (240468480\beta + 81285120\beta^2 + 273853440 + 10160640\beta^3)\lambda_j^3 \\ &\quad + (121927680\beta^2 + 22861440\beta^3 - 148780800 + 120113280\beta)\lambda_i\lambda_j^2 \\ &\quad + (124830720 + 10160640\beta^3 + 52254720\beta^2 + 97251840\beta)\lambda_i^2\lambda_j \\ &\quad + (13547520 + 25159680\beta + 7983360\beta^2 + 725760\beta^3)\lambda_i^3, \\ \xi_{00}^{(2)}(\lambda_j; \lambda_i; \beta) &= (721405440\beta + 243855360\beta^2 + 821560320 + 30481920\beta^3)\lambda_j^2 \\ &\quad + (243855360\beta^2 + 45722880\beta^3 - 297561600 + 240226560\beta)\lambda_i\lambda_j \\ &\quad + (124830720 + 10160640\beta^3 + 52254720\beta^2 + 97251840\beta)\lambda_i^2, \\ \xi_{00}^{(3)}(\lambda_j; \lambda_i; \beta) &= (1442810880\beta + 487710720\beta^2 + 1643120640 + 60963840\beta^3)\lambda_j \\ &\quad + (243855360\beta^2 + 45722880\beta^3 - 297561600 + 240226560\beta)\lambda_i, \\ \xi_{00}^{(4)}(\lambda_j; \lambda_i; \beta) &= 1442810880\beta + 487710720\beta^2 + 1643120640 + 60963840\beta^3 \\ &\geq 0 \text{ for all } \beta \geq -3.3100. \end{aligned}$$

Now

$$\begin{aligned} \xi_{00}(\lambda_i; \lambda_i; \beta) &= 15966720\lambda_i^4(\beta + 3)(\beta + 2)(\beta + 1) > 0 \text{ for all } \beta > -1, \\ \xi_{00}^{(1)}(\lambda_i; \lambda_i; \beta) &= 43908480\lambda_i^3(\beta + 3)(\beta + 2)(\beta + 1) > 0 \text{ for all } \beta > -1, \\ \xi_{00}^{(2)}(\lambda_i; \lambda_i; \beta) &= 725760\lambda_i^2(\beta + 3)(\beta + 2)(119\beta + 149) > 0 \text{ for all } \beta > -\frac{149}{119}, \\ \xi_{00}^{(3)}(\lambda_i; \lambda_i; \beta) &= 2177280\lambda_i(\beta + 3)(49\beta^2 + 189\beta + 206) > 0 \text{ for all } \beta > -3. \end{aligned}$$

Hence, ξ_0 is positive for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$. This implies that $\xi(\lambda_l; \lambda_l; \lambda_j; \lambda_i; \beta)$ is positive for all $\lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Now

$$\begin{aligned} \xi_1(\lambda_l; \lambda_j; \lambda_i; \beta) &= (20966400 + 302400\beta^3 + 20966400\beta + 4838400\beta^2)\lambda_l^4 \\ &\quad + (31449600 + 1814400\beta^3 + 14515200\beta^2 + 35078400\beta)(\lambda_i + \lambda_j)\lambda_l^3 \\ &\quad + (31570560\beta + 52980480 + 1814400\beta^3 + 10886400\beta^2)(\lambda_j^2 + \lambda_i^2)\lambda_l^2 \\ &\quad + (25401600\beta^2 + 3386880\beta + 5443200\beta^3 - 93864960)\lambda_i\lambda_j\lambda_l^2 \\ &\quad + (10160640\beta - 4354560 + 2721600\beta^3 + 12337920\beta^2)(\lambda_i\lambda_j^2 + \lambda_i^2\lambda_j)\lambda_l \\ &\quad + (8386560 + 12015360\beta + 3386880\beta^2 + 302400\beta^3)(\lambda_j^3 + \lambda_i^3)\lambda_l \\ &\quad + (544320\beta^3 + 6894720\beta + 13789440 + 2903040\beta^2)\lambda_i^2\lambda_j^2 \\ &\quad + (564480\beta + 322560 + 241920\beta^2)(\lambda_j^4 + \lambda_i^4) \\ &\quad + (1935360\beta^2 + 181440\beta^3 + 4757760\beta + 645120)(\lambda_i\lambda_j^3 + \lambda_i^3\lambda_j), \end{aligned}$$

and, hence,

$$\begin{aligned} \xi_1^{(3)}(\lambda_i; \lambda_j; \lambda_i; \beta) &= (503193600 + 7257600\beta^3 + 503193600\beta + 116121600\beta^2)\lambda_i \\ &\quad + (188697600 + 10886400\beta^3 + 87091200\beta^2 + 210470400\beta)(\lambda_i + \lambda_j) \\ &\geq 0 \text{ for all } \beta \geq -1.4294. \end{aligned}$$

Define $\xi_{12}(\lambda_j; \lambda_i; \beta) = \xi_1^{(2)}(\lambda_j; \lambda_j; \lambda_i; \beta)$. We wish to show that ξ_{12} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{12} with respect to λ_j gives

$$\begin{aligned} \xi_{12}(\lambda_j; \lambda_i; \beta) &= (546255360 + 18144000\beta^3 + 525208320\beta + 166924800\beta^2)\lambda_j^2 \\ &\quad + (967680 + 21772800\beta^3 + 137894400\beta^2 + 217244160\beta)\lambda_i\lambda_j \\ &\quad + (63141120\beta + 105960960 + 3628800\beta^3 + 21772800\beta^2)\lambda_i^2, \\ \xi_{12}^{(1)}(\lambda_j; \lambda_i; \beta) &= (1092510720 + 36288000\beta^3 + 1050416640\beta + 333849600\beta^2)\lambda_j \\ &\quad + (967680 + 21772800\beta^3 + 137894400\beta^2 + 217244160\beta)\lambda_i, \\ \xi_{12}^{(2)}(\lambda_j; \lambda_i; \beta) &= 1092510720 + 36288000\beta^3 + 1050416640\beta + 333849600\beta^2 \\ &\geq 0 \text{ for all } \beta \geq -2.3123. \end{aligned}$$

Now

$$\begin{aligned} \xi_{12}(\lambda_i; \lambda_i; \beta) &= 21772800\lambda_i^2(2\beta + 5)(\beta + 3)(\beta + 2) \geq 0 \text{ for all } \beta \geq -2, \\ \xi_{12}^{(1)}(\lambda_i; \lambda_i; \beta) &= 2419200\lambda_i(452 + 24\beta^3 + 524\beta + 195\beta^2) \geq 0 \text{ for all } \beta \geq -1.8116. \end{aligned}$$

Hence, $\xi_1^{(2)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{11}(\lambda_j; \lambda_i; \beta) = \xi_1^{(1)}(\lambda_j; \lambda_j; \lambda_i; \beta)$. We wish to show that ξ_{11} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{11} with respect to λ_j gives

$$\begin{aligned} \xi_{11}(\lambda_j; \lambda_i; \beta) &= (292561920 + 10584000\beta^3 + 264257280\beta + 88058880\beta^2)\lambda_j^3 \\ &\quad + (106686720\beta^2 + 122169600\beta + 19051200\beta^3 - 97735680)\lambda_i\lambda_j^2 \\ &\quad + (73301760\beta + 101606400 + 6350400\beta^3 + 34110720\beta^2)\lambda_i^2\lambda_j \\ &\quad + (8386560 + 12015360\beta + 3386880\beta^2 + 302400\beta^3)\lambda_i^3, \\ \xi_{11}^{(1)}(\lambda_j; \lambda_i; \beta) &= (877685760 + 31752000\beta^3 + 792771840\beta + 264176640\beta^2)\lambda_j^2 \\ &\quad + (213373440\beta^2 + 38102400\beta^3 - 195471360 + 244339200\beta)\lambda_i\lambda_j \\ &\quad + (73301760\beta + 101606400 + 6350400\beta^3 + 34110720\beta^2)\lambda_i^2, \\ \xi_{11}^{(2)}(\lambda_j; \lambda_i; \beta) &= (1755371520 + 63504000\beta^3 + 1585543680\beta + 528353280\beta^2)\lambda_j \\ &\quad + (213373440\beta^2 + 38102400\beta^3 - 195471360 + 244339200\beta)\lambda_i, \\ \xi_{11}^{(3)}(\lambda_j; \lambda_i; \beta) &= 1755371520 + 63504000\beta^3 + 1585543680\beta + 528353280\beta^2 \\ &\geq 0 \text{ for all } \beta \geq -3.2688. \end{aligned}$$

Now

$$\begin{aligned}\xi_{11}(\lambda_i; \lambda_i; \beta) &= 7257600\lambda_i^3(5\beta + 7)(\beta + 3)(\beta + 2) \geq 0 \text{ for all } \beta \geq -\frac{7}{5}, \\ \xi_{11}^{(1)}(\lambda_i; \lambda_i; \beta) &= 10886400\lambda_i^2(\beta + 3)(\beta + 2)(7\beta + 12) \geq 0 \text{ for all } \beta \geq -\frac{12}{7}, \\ \xi_{11}^{(2)}(\lambda_i; \lambda_i; \beta) &= 483840\lambda_i(3224 + 210\beta^3 + 3782\beta + 1533\beta^2) \geq 0 \text{ for all } \beta \geq -3.0192.\end{aligned}$$

Hence, $\xi_1^{(1)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{10}(\lambda_j; \lambda_i; \beta) = \xi_1(\lambda_j; \lambda_j; \lambda_i; \beta)$. We wish to show that ξ_{10} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{10} with respect to λ_j gives

$$\begin{aligned}\xi_{10}(\lambda_j; \lambda_i; \beta) &= (114105600 + 4233600\beta^3 + 100195200\beta + 33868800\beta^2)\lambda_j^4 \\ &\quad + (-66124800 + 10160640\beta^3 + 54190080\beta^2 + 53383680\beta)\lambda_i\lambda_j^3 \\ &\quad + (48625920\beta + 62415360 + 5080320\beta^3 + 26127360\beta^2)\lambda_i^2\lambda_j^2 \\ &\quad + (9031680 + 16773120\beta + 5322240\beta^2 + 483840\beta^3)\lambda_i^3\lambda_j \\ &\quad + (564480\beta + 322560 + 241920\beta^2)\lambda_i^4, \\ \xi_{10}^{(1)}(\lambda_j; \lambda_i; \beta) &= (456422400 + 16934400\beta^3 + 400780800\beta + 135475200\beta^2)\lambda_j^3 \\ &\quad + (-198374400 + 30481920\beta^3 + 162570240\beta^2 + 160151040\beta)\lambda_i\lambda_j^2 \\ &\quad + (97251840\beta + 124830720 + 10160640\beta^3 + 52254720\beta^2)\lambda_i^2\lambda_j \\ &\quad + (9031680 + 16773120\beta + 5322240\beta^2 + 483840\beta^3)\lambda_i^3, \\ \xi_{10}^{(2)}(\lambda_j; \lambda_i; \beta) &= (1369267200 + 50803200\beta^3 + 1202342400\beta + 406425600\beta^2)\lambda_j^2 \\ &\quad + (-396748800 + 60963840\beta^3 + 325140480\beta^2 + 320302080\beta)\lambda_i\lambda_j \\ &\quad + (97251840\beta + 124830720 + 10160640\beta^3 + 52254720\beta^2)\lambda_i^2, \\ \xi_{10}^{(3)}(\lambda_j; \lambda_i; \beta) &= (2738534400 + 101606400\beta^3 + 2404684800\beta + 812851200\beta^2)\lambda_j \\ &\quad + (-396748800 + 60963840\beta^3 + 325140480\beta^2 + 320302080\beta)\lambda_i, \\ \xi_{10}^{(4)}(\lambda_j; \lambda_i; \beta) &= 2738534400 + 101606400\beta^3 + 2404684800\beta + 812851200\beta^2 \\ &\geq 0 \text{ for all } \beta \geq -3.3100.\end{aligned}$$

Now

$$\begin{aligned}\xi_{10}(\lambda_i; \lambda_i; \beta) &= 19958400\lambda_i^4(\beta + 3)(\beta + 2)(\beta + 1) > 0 \text{ for all } \beta > -1, \\ \xi_{10}^{(1)}(\lambda_i; \lambda_i; \beta) &= 7257600\lambda_i^3(8\beta + 9)(\beta + 3)(\beta + 2) > 0 \text{ for all } \beta > -\frac{9}{8}, \\ \xi_{10}^{(2)}(\lambda_i; \lambda_i; \beta) &= 17418240\lambda_i^2(\beta + 3)(7\beta^2 + 24\beta + 21) > 0 \text{ for all } \beta > -3, \\ \xi_{10}^{(3)}(\lambda_i; \lambda_i; \beta) &= 3870720\lambda_i(605 + 42\beta^3 + 704\beta + 294\beta^2) > 0 \text{ for all } \beta > -3.0629.\end{aligned}$$

Hence, ξ_1 is positive for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$. This implies that $\xi(\lambda_l; \lambda_l; \lambda_j; \lambda_i; \beta)$ is positive for all $\lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Now

$$\begin{aligned} \xi_2(\lambda_i; \lambda_j; \lambda_i; \beta) &= (34836480 + 8709120\beta^2 + 35804160\beta + 483840\beta^3)\lambda_i^3 \\ &\quad + (2177280\beta^3 + 20321280\beta^2 + 49109760\beta + 35320320)(\lambda_j + \lambda_i)\lambda_i^2 \\ &\quad + (10160640\beta^2 + 35562240\beta + 50803200 + 1451520\beta^3)(\lambda_j^2 + \lambda_i^2)\lambda_i \\ &\quad + (-59996160 + 4354560\beta^3 + 24675840\beta^2 + 21288960\beta)\lambda_i\lambda_j\lambda_i \\ &\quad + (7741440 + 7257600\beta + 1451520\beta^2 + 120960\beta^3)(\lambda_j^3 + \lambda_i^3) \\ &\quad + (5806080\beta^2 + 13789440\beta + 1088640\beta^3 + 27578880)(\lambda_i^2\lambda_j + \lambda_i\lambda_j^2), \end{aligned}$$

and, hence,

$$\begin{aligned} \xi_2^{(3)}(\lambda_i; \lambda_j; \lambda_i; \beta) &= 209018880 + 52254720\beta^2 + 214824960\beta + 2903040\beta^3 \\ &\geq 0 \text{ for all } \beta \geq -1.4322. \end{aligned}$$

Define $\xi_{22}(\lambda_j; \lambda_i; \beta) = \xi_2^{(2)}(\lambda_j; \lambda_j; \lambda_i; \beta)$. We wish to show that ξ_{22} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{22} with respect to λ_j gives

$$\begin{aligned} \xi_{22}(\lambda_j; \lambda_i; \beta) &= (279659520 + 92897280\beta^2 + 313044480\beta + 7257600\beta^3)\lambda_j \\ &\quad + (4354560\beta^3 + 40642560\beta^2 + 98219520\beta + 70640640)\lambda_i, \\ \xi_{22}^{(1)}(\lambda_j; \lambda_i; \beta) &= 279659520 + 92897280\beta^2 + 313044480\beta + 7257600\beta^3 \\ &\geq 0 \text{ for all } \beta \geq -1.4382. \end{aligned}$$

Now

$$\xi_{22}(\lambda_i; \lambda_i; \beta) = 967680\lambda_i(362 + 138\beta^2 + 425\beta + 12\beta^3) \geq 0 \text{ for all } \beta \geq -1.4432.$$

Hence, $\xi_2^{(2)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{21}(\lambda_j; \lambda_i; \beta) = \xi_2^{(1)}(\lambda_j; \lambda_j; \lambda_i; \beta)$. We wish to show that ξ_{21} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{21} with respect to λ_j gives

$$\begin{aligned} \xi_{21}(\lambda_j; \lambda_i; \beta) &= (225953280 + 76930560\beta^2 + 241194240\beta + 7257600\beta^3)\lambda_j^2 \\ &\quad + (8709120\beta^3 + 65318400\beta^2 + 119508480\beta + 10644480)\lambda_i\lambda_j \\ &\quad + (10160640\beta^2 + 35562240\beta + 50803200 + 1451520\beta^3)\lambda_i^2, \\ \xi_{21}^{(1)}(\lambda_j; \lambda_i; \beta) &= (451906560 + 153861120\beta^2 + 482388480\beta + 14515200\beta^3)\lambda_j \\ &\quad + (8709120\beta^3 + 65318400\beta^2 + 119508480\beta + 10644480)\lambda_i, \\ \xi_{21}^{(2)}(\lambda_j; \lambda_i; \beta) &= 451906560 + 153861120\beta^2 + 482388480\beta + 14515200\beta^3 \\ &\geq 0 \text{ for all } \beta \geq -1.7649. \end{aligned}$$

Now

$$\begin{aligned} \xi_{21}(\lambda_i; \lambda_i; \beta) &= 4354560\lambda_i^2(\beta + 3)(4\beta^2 + 23\beta + 22) \geq 0 \text{ for all } \beta \geq -1.2120, \\ \xi_{21}^{(1)}(\lambda_i; \lambda_i; \beta) &= 483840\lambda_i(956 + 453\beta^2 + 1244\beta + 48\beta^3) \geq 0 \text{ for all } \beta \geq -1.2965. \end{aligned}$$

Hence, $\xi_2^{(1)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{20}(\lambda_j; \lambda_i; \beta) = \xi_2(\lambda_j; \lambda_j; \lambda_i; \beta)$. We wish to show that ξ_{20} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{20} with respect to λ_j gives

$$\begin{aligned}\xi_{20}(\lambda_j; \lambda_i; \beta) &= (128701440 + 40642560\beta^2 + 127733760\beta + 4233600\beta^3)\lambda_j^3 \\ &\quad + (2903040 + 7620480\beta^3 + 50803200\beta^2 + 84188160\beta)\lambda_i\lambda_j^2 \\ &\quad + (15966720\beta^2 + 49351680\beta + 78382080 + 2540160\beta^3)\lambda_i^2\lambda_j \\ &\quad + (7741440 + 7257600\beta + 1451520\beta^2 + 120960\beta^3)\lambda_i^3, \\ \xi_{20}^{(1)}(\lambda_j; \lambda_i; \beta) &= (386104320 + 121927680\beta^2 + 383201280\beta + 12700800\beta^3)\lambda_j^2 \\ &\quad + (5806080 + 15240960\beta^3 + 101606400\beta^2 + 168376320\beta)\lambda_i\lambda_j \\ &\quad + (15966720\beta^2 + 49351680\beta + 78382080 + 2540160\beta^3)\lambda_i^2, \\ \xi_{20}^{(2)}(\lambda_j; \lambda_i; \beta) &= (772208640 + 243855360\beta^2 + 766402560\beta + 25401600\beta^3)\lambda_j \\ &\quad + (5806080 + 15240960\beta^3 + 101606400\beta^2 + 168376320\beta)\lambda_i, \\ \xi_{20}^{(3)}(\lambda_j; \lambda_i; \beta) &= 772208640 + 243855360\beta^2 + 766402560\beta + 25401600\beta^3 \\ &\geq 0 \text{ for all } \beta \geq -2.1389.\end{aligned}$$

Now

$$\begin{aligned}\xi_{20}(\lambda_i; \lambda_i; \beta) &= 7257600\lambda_i^3(2\beta + 5)(\beta + 3)(\beta + 2) > 0 \text{ for all } \beta > -2, \\ \xi_{20}^{(1)}(\lambda_i; \lambda_i; \beta) &= 4354560\lambda_i^2(\beta + 3)(7\beta^2 + 34\beta + 36) > 0 \text{ for all } \beta > -1.5596, \\ \xi_{20}^{(2)}(\lambda_i; \lambda_i; \beta) &= 2903040\lambda_i(268 + 119\beta^2 + 322\beta + 14\beta^3) > 0 \text{ for all } \beta > -1.6020.\end{aligned}$$

Hence, ξ_2 is positive for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$. This implies that $\xi(\lambda_l; \lambda_l; \lambda_j; \lambda_i; \beta)$ is positive for all $\lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Now

$$\begin{aligned}\xi_3(\lambda_l; \lambda_j; \lambda_i; \beta) &= (362880\beta^3 + 10160640\beta^2 + 30965760 + 35320320\beta)\lambda_l^2 \\ &\quad + (1088640\beta^3 + 42094080\beta + 11612160 + 17418240\beta^2)(\lambda_j + \lambda_i)\lambda_l \\ &\quad + (4354560\beta^2 + 21772800\beta + 362880\beta^3 + 23224320)\lambda_j^2 \\ &\quad + (1088640\beta^3 + 28546560\beta + 3870720 + 11612160\beta^2)\lambda_i\lambda_j \\ &\quad + (4354560\beta^2 + 21772800\beta + 362880\beta^3 + 23224320)\lambda_i^2,\end{aligned}$$

and, hence,

$$\begin{aligned}\xi_3^{(2)}(\lambda_l; \lambda_j; \lambda_i; \beta) &= 725760\beta^3 + 20321280\beta^2 + 61931520 + 70640640\beta \\ &\geq 0 \text{ for all } \beta \geq -1.4510.\end{aligned}$$

Define $\xi_{31}(\lambda_j; \lambda_i; \beta) = \xi_3^{(1)}(\lambda_j; \lambda_j; \lambda_i; \beta)$. We wish to show that ξ_{31} is non-negative for

all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{31} with respect to λ_j gives

$$\begin{aligned} \xi_{31}(\lambda_j; \lambda_i; \beta) &= (1814400\beta^3 + 37739520\beta^2 + 73543680 + 112734720\beta)\lambda_j \\ &\quad + (1088640\beta^3 + 42094080\beta + 11612160 + 17418240\beta^2)\lambda_i, \\ \xi_{31}^{(1)}(\lambda_j; \lambda_i; \beta) &= 1814400\beta^3 + 37739520\beta^2 + 73543680 + 112734720\beta \\ &\geq 0 \text{ for all } \beta \geq -0.9275. \end{aligned}$$

Now

$$\xi_{31}(\lambda_i; \lambda_i; \beta) = 967680\lambda_i(3\beta^3 + 57\beta^2 + 88 + 160\beta) \geq 0 \text{ for all } \beta \geq -0.7350.$$

Hence, $\xi_3^{(1)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{30}(\lambda_j; \lambda_i; \beta) = \xi_3(\lambda_j; \lambda_j; \lambda_i; \beta)$. We wish to show that ξ_{30} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{30} with respect to λ_j gives

$$\begin{aligned} \xi_{30}(\lambda_j; \lambda_i; \beta) &= (1814400\beta^3 + 31933440\beta^2 + 65802240 + 99187200\beta)\lambda_j^2 \\ &\quad + (2177280\beta^3 + 70640640\beta + 15482880 + 29030400\beta^2)\lambda_i\lambda_j \\ &\quad + (4354560\beta^2 + 21772800\beta + 362880\beta^3 + 23224320)\lambda_i^2, \\ \xi_{30}^{(1)}(\lambda_j; \lambda_i; \beta) &= (3628800\beta^3 + 63866880\beta^2 + 131604480 + 198374400\beta)\lambda_j \\ &\quad + (2177280\beta^3 + 70640640\beta + 15482880 + 29030400\beta^2)\lambda_i, \\ \xi_{30}^{(2)}(\lambda_j; \lambda_i; \beta) &= 3628800\beta^3 + 63866880\beta^2 + 131604480 + 198374400\beta \\ &\geq 0 \text{ for all } \beta \geq -0.9237. \end{aligned}$$

Now

$$\begin{aligned} \xi_{30}(\lambda_i; \lambda_i; \beta) &= 4354560\lambda_i^2(\beta + 3)(\beta^2 + 12\beta + 8) > 0 \text{ for all } \beta > -0.7085, \\ \xi_{30}^{(1)}(\lambda_i; \lambda_i; \beta) &= 1935360\lambda_i(3\beta^3 + 48\beta^2 + 76 + 139\beta) > 0 \text{ for all } \beta > -0.7158. \end{aligned}$$

Hence, ξ_3 is positive for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$. This implies that $\xi(\lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta)$ is positive for all $\lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

□

Lemma B.19 Let

$$\begin{aligned} \xi(\lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta) = & (725760\beta^2 + 967680 + 1693440\beta)t(0, 0, 0, 4) \\ & + (10321920 + 483840\beta^3 + 19031040\beta + 6289920\beta^2)t(0, 0, 1, 3) \\ & + (38223360\beta + 23950080\beta^2 + 25159680 + 4354560\beta^3)t(0, 1, 1, 2) \\ & + (50803200 + 10160640\beta^2 + 35562240\beta + 1451520\beta^3)t(0, 0, 2, 2) \\ & + (-29030400\beta + 52254720\beta^2 - 255467520 + 13063680\beta^3)t(1, 1, 1, 1), \end{aligned}$$

where

$$t(a, b, c, d) = \sum_{\substack{(n_1, n_2, n_3, n_4) \\ \in \text{perm}(a, b, c, d)}} \lambda_i^{n_1} \lambda_j^{n_2} \lambda_l^{n_3} \lambda_l^{n_4}.$$

The expression $\xi(\lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta)$ is positive for all $\lambda_k \geq \lambda_l \geq \lambda_j \geq \lambda_i$, and $\beta \geq -\frac{2}{5}$.

Proof.

Let

$$\hat{s}(a, b, c) = \sum_{\substack{(n_1, n_2, n_3) \\ \in \text{perm}(a, b, c)}} \lambda_i^{n_1} \lambda_j^{n_2} \lambda_l^{n_3}.$$

Differentiating $\xi(\lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta)$ with respect to λ_k we obtain

$$\begin{aligned} \xi^{(1)}(\lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta) = & (6773760\beta + 3870720 + 2903040\beta^2)\lambda_k^3 \\ & + ((1451520\beta^3 + 30965760 + 18869760\beta^2 + 57093120\beta)\lambda_k^2 \hat{s}(0, 0, 2) \\ & + (101606400 + 71124480\beta + 20321280\beta^2 + 2903040\beta^3)\lambda_l^2 \lambda_k \hat{s}(0, 0, 2) \\ & + (8709120\beta^3 + 47900160\beta^2 + 76446720\beta + 50319360)\lambda_k \hat{s}(0, 1, 1) \\ & + (-29030400\beta + 52254720\beta^2 + 13063680\beta^3 - 255467520)\hat{s}(1, 1, 1) \\ & + (10321920 + 19031040\beta + 6289920\beta^2 + 483840\beta^3)\hat{s}(0, 0, 3) \\ & + (4354560\beta^3 + 23950080\beta^2 + 25159680 + 38223360\beta)\hat{s}(0, 1, 2), \\ \xi^{(2)}(\lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta) = & 3(6773760\beta + 3870720 + 2903040\beta^2)\lambda_k^2 \\ & + 2((1451520\beta^3 + 30965760 + 18869760\beta^2 + 57093120\beta)\lambda_k \hat{s}(0, 0, 2) \\ & + (101606400 + 71124480\beta + 20321280\beta^2 + 2903040\beta^3)\lambda_l^2 \hat{s}(0, 0, 2) \\ & + (8709120\beta^3 + 47900160\beta^2 + 76446720\beta + 50319360)\hat{s}(0, 1, 1) \\ \geq & 0 \text{ for all } \beta \geq -0.6924. \end{aligned}$$

Define

$$\begin{aligned} \xi_0(\lambda_l; \lambda_j; \lambda_i; \beta) & \stackrel{\text{def}}{=} \xi(\lambda_l; \lambda_l; \lambda_j; \lambda_i; \beta), \\ \xi_1(\lambda_l; \lambda_j; \lambda_i; \beta) & \stackrel{\text{def}}{=} \xi^{(1)}(\lambda_l; \lambda_l; \lambda_j; \lambda_i; \beta). \end{aligned}$$

We will prove that the above equations are non-negative for all $\lambda_k \geq \lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$ and, hence, deduce that $\xi(\lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta)$ is positive for all $\lambda_k \geq \lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Now

$$\begin{aligned} \xi_0(\lambda_l; \lambda_j; \lambda_i; \beta) &= (24192000\beta^2 + 73382400 + 77011200\beta + 2419200\beta^3)\lambda_l^4 \\ &\quad + (9676800\beta^3 + 60480000\beta^2 + 70963200 + 114508800\beta)(\lambda_i + \lambda_j)\lambda_l^3 \\ &\quad + (7257600\beta^3 + 44271360\beta^2 + 126766080 + 109347840\beta)(\lambda_j^2 + \lambda_i^2)\lambda_l^2 \\ &\quad + (21772800\beta^3 + 100154880\beta^2 - 205148160 + 47416320\beta)\lambda_i\lambda_j\lambda_l^2 \\ &\quad + 2(10321920 + 19031040\beta + 6289920\beta^2 + 483840\beta^3)(\lambda_j^3 + \lambda_i^3)\lambda_l \\ &\quad + 2(4354560\beta^3 + 23950080\beta^2 + 25159680 + 38223360\beta)(\lambda_i\lambda_j^2 + \lambda_i^2\lambda_j)\lambda_l \\ &\quad + (725760\beta^2 + 1693440\beta + 967680)\lambda_j^4 \\ &\quad + (10321920 + 19031040\beta + 6289920\beta^2 + 483840\beta^3)\lambda_i\lambda_j^3 \\ &\quad + (10321920 + 19031040\beta + 6289920\beta^2 + 483840\beta^3)\lambda_i^3\lambda_j \\ &\quad + (10160640\beta^2 + 50803200 + 35562240\beta + 1451520\beta^3)\lambda_i^2\lambda_j^2 \\ &\quad + (725760\beta^2 + 1693440\beta + 967680)\lambda_i^4, \end{aligned}$$

and, hence,

$$\begin{aligned} \xi_0^{(3)}(\lambda_l; \lambda_j; \lambda_i; \beta) &= (580608000\beta^2 + 1761177600 + 1848268800\beta + 58060800\beta^3)\lambda_l \\ &\quad + (58060800\beta^3 + 362880000\beta^2 + 425779200 + 687052800\beta)(\lambda_i + \lambda_j) \\ &\geq 0 \text{ for all } \beta \geq -1.7321. \end{aligned}$$

Define $\xi_{02}(\lambda_j; \lambda_i; \beta) = \xi_0^{(2)}(\lambda_j; \lambda_j; \lambda_i; \beta)$. We wish to show that ξ_{02} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{02} with respect to λ_j gives

$$\begin{aligned} \xi_{02}(\lambda_j; \lambda_i; \beta) &= (741726720\beta^2 + 1559900160 + 1829882880\beta + 101606400\beta^3)\lambda_j^2 \\ &\quad + (101606400\beta^3 + 563189760\beta^2 + 15482880 + 781885440\beta)\lambda_i\lambda_j \\ &\quad + (14515200\beta^3 + 88542720\beta^2 + 253532160 + 218695680\beta)\lambda_i^2, \\ \xi_{02}^{(1)}(\lambda_j; \lambda_i; \beta) &= (1483453440\beta^2 + 3119800320 + 3659765760\beta + 203212800\beta^3)\lambda_j \\ &\quad + (101606400\beta^3 + 563189760\beta^2 + 15482880 + 781885440\beta)\lambda_i, \\ \xi_{02}^{(2)}(\lambda_j; \lambda_i; \beta) &= 1483453440\beta^2 + 3119800320 + 3659765760\beta + 203212800\beta^3 \\ &\geq 0 \text{ for all } \beta \geq -3.0192. \end{aligned}$$

Now

$$\begin{aligned} \xi_{02}(\lambda_i; \lambda_i; \beta) &= 43545600\lambda_i^2(5\beta + 7)(\beta + 3)(\beta + 2) \geq 0 \text{ for all } \beta \geq -\frac{7}{5}, \\ \xi_{02}^{(1)}(\lambda_i; \lambda_i; \beta) &= 43545600\lambda_i(\beta + 3)(\beta + 2)(7\beta + 12) \geq 0 \text{ for all } \beta \geq -\frac{12}{7}. \end{aligned}$$

Hence, $\xi_0^{(2)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{01}(\lambda_j; \lambda_i; \beta) = \xi_0^{(1)}(\lambda_j; \lambda_j; \lambda_i; \beta)$. We wish to show that ξ_{01} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{01} with respect to λ_j gives

$$\begin{aligned} \xi_{01}(\lambda_j; \lambda_i; \beta) &= (379330560\beta^2 + 780595200 + 908328960\beta + 54190080\beta^3)\lambda_j^3 \\ &\quad + (81285120\beta^3 + 429649920\beta^2 - 147087360 + 514805760\beta)\lambda_i\lambda_j^2 \\ &\quad + (23224320\beta^3 + 136442880\beta^2 + 303851520 + 295142400\beta)\lambda_i^2\lambda_j \\ &\quad + (20643840 + 38062080\beta + 12579840\beta^2 + 967680\beta^3)\lambda_i^3, \\ \xi_{01}^{(1)}(\lambda_j; \lambda_i; \beta) &= (1137991680\beta^2 + 2341785600 + 2724986880\beta + 162570240\beta^3)\lambda_j^2 \\ &\quad + (162570240\beta^3 + 859299840\beta^2 - 294174720 + 1029611520\beta)\lambda_i\lambda_j \\ &\quad + (23224320\beta^3 + 136442880\beta^2 + 303851520 + 295142400\beta)\lambda_i^2, \\ \xi_{01}^{(2)}(\lambda_j; \lambda_i; \beta) &= (2275983360\beta^2 + 4683571200 + 5449973760\beta + 325140480\beta^3)\lambda_j \\ &\quad + (162570240\beta^3 + 859299840\beta^2 - 294174720 + 1029611520\beta)\lambda_i, \\ \xi_{01}^{(3)}(\lambda_j; \lambda_i; \beta) &= 2275983360\beta^2 + 4683571200 + 5449973760\beta + 325140480\beta^3 \\ &\geq 0 \text{ for all } \beta \geq -3.0629. \end{aligned}$$

Now

$$\begin{aligned} \xi_{01}(\lambda_i; \lambda_i; \beta) &= 159667200\lambda_i^3(\beta + 3)(\beta + 2)(\beta + 1) \geq 0 \text{ for all } \beta \geq -1, \\ \xi_{01}^{(1)}(\lambda_i; \lambda_i; \beta) &= 43545600\lambda_i^2(8\beta + 9)(\beta + 3)(\beta + 2) \geq 0 \text{ for all } \beta \geq -\frac{9}{8}, \\ \xi_{01}^{(2)}(\lambda_i; \lambda_i; \beta) &= 69672960\lambda_i(\beta + 3)(7\beta^2 + 24\beta + 21) \geq 0 \text{ for all } \beta \geq -3. \end{aligned}$$

Hence, $\xi_0^{(1)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{00}(\lambda_j; \lambda_i; \beta) = \xi_0(\lambda_j; \lambda_j; \lambda_i; \beta)$. We wish to show that ξ_{00} is non-negative for all

$\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{00} with respect to λ_j gives

$$\begin{aligned} \xi_{00}(\lambda_j; \lambda_i; \beta) &= (142248960\beta^2 + 292723200 + 340623360\beta + 20321280\beta^3)\lambda_j^4 \\ &\quad + (40642560\beta^3 + 214824960\beta^2 - 73543680 + 257402880\beta)\lambda_i\lambda_j^3 \\ &\quad + (17418240\beta^3 + 102332160\beta^2 + 227888640 + 221356800\beta)\lambda_i^2\lambda_j^2 \\ &\quad + (30965760 + 57093120\beta + 18869760\beta^2 + 1451520\beta^3)\lambda_i^3\lambda_j \\ &\quad + (725760\beta^2 + 1693440\beta + 967680)\lambda_i^4, \\ \xi_{00}^{(1)}(\lambda_j; \lambda_i; \beta) &= (568995840\beta^2 + 1170892800 + 1362493440\beta + 81285120\beta^3)\lambda_j^3 \\ &\quad + (121927680\beta^3 + 644474880\beta^2 - 220631040 + 772208640\beta)\lambda_i\lambda_j^2 \\ &\quad + (34836480\beta^3 + 204664320\beta^2 + 455777280 + 442713600\beta)\lambda_i^2\lambda_j \\ &\quad + (30965760 + 57093120\beta + 18869760\beta^2 + 1451520\beta^3)\lambda_i^3, \\ \xi_{00}^{(2)}(\lambda_j; \lambda_i; \beta) &= (1706987520\beta^2 + 3512678400 + 4087480320\beta + 243855360\beta^3)\lambda_j^2 \\ &\quad + (243855360\beta^3 + 1288949760\beta^2 - 441262080 + 1544417280\beta)\lambda_i\lambda_j \\ &\quad + (34836480\beta^3 + 204664320\beta^2 + 455777280 + 442713600\beta)\lambda_i^2, \\ \xi_{00}^{(3)}(\lambda_j; \lambda_i; \beta) &= (3413975040\beta^2 + 7025356800 + 8174960640\beta + 487710720\beta^3)\lambda_j \\ &\quad + (243855360\beta^3 + 1288949760\beta^2 - 441262080 + 1544417280\beta)\lambda_i, \\ \xi_{00}^{(4)}(\lambda_j; \lambda_i; \beta) &= 3413975040\beta^2 + 7025356800 + 8174960640\beta + 487710720\beta^3 \\ &> 0 \quad \text{for all } \beta > -3.0629. \end{aligned}$$

Now

$$\begin{aligned} \xi_{00}(\lambda_i; \lambda_i; \beta) &= 79833600\lambda_i^4(\beta + 3)(\beta + 2)(\beta + 1) > 0 \text{ for all } \beta > -1, \\ \xi_{00}^{(1)}(\lambda_i; \lambda_i; \beta) &= 239500800\lambda_i^3(\beta + 3)(\beta + 2)(\beta + 1) > 0 \text{ for all } \beta > -1, \\ \xi_{00}^{(2)}(\lambda_i; \lambda_i; \beta) &= 65318400\lambda_i^2(8\beta + 9)(\beta + 3)(\beta + 2) > 0 \text{ for all } \beta > -\frac{9}{8}, \\ \xi_{00}^{(3)}(\lambda_i; \lambda_i; \beta) &= 104509440\lambda_i(\beta + 3)(7\beta^2 + 24\beta + 21) > 0 \text{ for all } \beta > -3. \end{aligned}$$

Hence, ξ_0 is positive for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$. This implies that $\xi(\lambda_l; \lambda_l; \lambda_j; \lambda_i; \beta)$ is positive for all $\lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Now

$$\begin{aligned} \xi_1(\lambda_l; \lambda_j; \lambda_i; \beta) &= (154022400\beta + 146764800 + 48384000\beta^2 + 4838400\beta^3)\lambda_l^3 \\ &\quad + (106444800 + 171763200\beta + 90720000\beta^2 + 14515200\beta^3)(\lambda_j + \lambda_i)\lambda_l^2 \\ &\quad + (7257600\beta^3 + 44271360\beta^2 + 126766080 + 109347840\beta)(\lambda_j^2 + \lambda_i^2)\lambda_l \\ &\quad + (21772800\beta^3 + 100154880\beta^2 - 205148160 + 47416320\beta)\lambda_i\lambda_j\lambda_l \\ &\quad + (10321920 + 19031040\beta + 6289920\beta^2 + 483840\beta^3)\lambda_j^3 \\ &\quad + (4354560\beta^3 + 23950080\beta^2 + 25159680 + 38223360\beta)\lambda_i\lambda_j^2 \\ &\quad + (10321920 + 19031040\beta + 6289920\beta^2 + 483840\beta^3)\lambda_i^3 \\ &\quad + (4354560\beta^3 + 23950080\beta^2 + 25159680 + 38223360\beta)\lambda_i^2\lambda_j, \end{aligned}$$

and, hence,

$$\begin{aligned}\xi_1^{(2)}(\lambda_i; \lambda_j; \lambda_i; \beta) &= (924134400\beta + 880588800 + 290304000\beta^2 + 29030400\beta^3)\lambda_i \\ &\quad + (29030400\beta^3 + 181440000\beta^2 + 212889600 + 343526400\beta)(\lambda_i + \lambda_j) \\ &\geq 0 \text{ for all } \beta \geq -1.7321.\end{aligned}$$

Define $\xi_{11}(\lambda_j; \lambda_i; \beta) = \xi_1^{(1)}(\lambda_j; \lambda_j; \lambda_i; \beta)$. We wish to show that ξ_{11} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{11} with respect to λ_j gives

$$\begin{aligned}\xi_{11}(\lambda_j; \lambda_i; \beta) &= (914941440\beta + 779950080 + 370863360\beta^2 + 50803200\beta^3)\lambda_j^2 \\ &\quad + (50803200\beta^3 + 281594880\beta^2 + 7741440 + 390942720\beta)\lambda_i\lambda_j \\ &\quad + (7257600\beta^3 + 44271360\beta^2 + 126766080 + 109347840\beta)\lambda_i^2, \\ \xi_{11}^{(1)}(\lambda_j; \lambda_i; \beta) &= (741726720\beta^2 + 1559900160 + 1829882880\beta + 101606400\beta^3)\lambda_j \\ &\quad + (50803200\beta^3 + 281594880\beta^2 + 7741440 + 390942720\beta)\lambda_i, \\ \xi_{11}^{(2)}(\lambda_j; \lambda_i; \beta) &= 741726720\beta^2 + 1559900160 + 1829882880\beta + 101606400\beta^3 \\ &\geq 0 \text{ for all } \beta \geq -3.0192.\end{aligned}$$

Now

$$\begin{aligned}\xi_{11}(\lambda_i; \lambda_i; \beta) &= 21772800\lambda_i^2(5\beta + 7)(\beta + 3)(\beta + 2) \geq 0 \text{ for all } \beta \geq -\frac{7}{5}, \\ \xi_{11}^{(1)}(\lambda_i; \lambda_i; \beta) &= 21772800\lambda_i(\beta + 3)(\beta + 2)(7\beta + 12) \geq 0 \text{ for all } \beta \geq -\frac{12}{7}.\end{aligned}$$

Hence, $\xi_1^{(1)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{10}(\lambda_j; \lambda_i; \beta) = \xi_1(\lambda_j; \lambda_j; \lambda_i; \beta)$. We wish to show that ξ_{10} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{10} with respect to λ_j gives

$$\begin{aligned}\xi_{10}(\lambda_j; \lambda_i; \beta) &= (454164480\beta + 390297600 + 189665280\beta^2 + 27095040\beta^3)\lambda_j^3 \\ &\quad + (40642560\beta^3 + 214824960\beta^2 - 73543680 + 257402880\beta)\lambda_i\lambda_j^2 \\ &\quad + (11612160\beta^3 + 68221440\beta^2 + 151925760 + 147571200\beta)\lambda_i^2\lambda_j \\ &\quad + (10321920 + 19031040\beta + 6289920\beta^2 + 483840\beta^3)\lambda_i^3, \\ \xi_{10}^{(1)}(\lambda_j; \lambda_i; \beta) &= (1362493440\beta + 1170892800 + 568995840\beta^2 + 81285120\beta^3)\lambda_j^2 \\ &\quad + (81285120\beta^3 + 429649920\beta^2 - 147087360 + 514805760\beta)\lambda_i\lambda_j \\ &\quad + (11612160\beta^3 + 68221440\beta^2 + 151925760 + 147571200\beta)\lambda_i^2, \\ \xi_{10}^{(2)}(\lambda_j; \lambda_i; \beta) &= (2724986880\beta + 2341785600 + 1137991680\beta^2 + 162570240\beta^3)\lambda_j \\ &\quad + (81285120\beta^3 + 429649920\beta^2 - 147087360 + 514805760\beta)\lambda_i, \\ \xi_{10}^{(3)}(\lambda_j; \lambda_i; \beta) &= 2724986880\beta + 2341785600 + 1137991680\beta^2 + 162570240\beta^3 \\ &> 0 \text{ for all } \beta > -3.0629.\end{aligned}$$

Now

$$\begin{aligned} \xi_{10}(\lambda_i; \lambda_i; \beta) &= 79833600\lambda_i^3(\beta + 3)(\beta + 2)(\beta + 1) > 0 \text{ for all } \beta > -1, \\ \xi_{10}^{(1)}(\lambda_i; \lambda_i; \beta) &= 21772800\lambda_i^2(8\beta + 9)(\beta + 3)(\beta + 2) > 0 \text{ for all } \beta > -\frac{9}{8}, \\ \xi_{10}^{(2)}(\lambda_i; \lambda_i; \beta) &= 34836480\lambda_i(\beta + 3)(7\beta^2 + 24\beta + 21) > 0 \text{ for all } \beta > -3. \end{aligned}$$

Hence, ξ_1 is positive for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$. This implies that $\xi(\lambda_l; \lambda_l; \lambda_j; \lambda_i; \beta)$ is positive for all $\lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

□

Lemma B.20 Let

$$\begin{aligned} \xi(\lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta) &= 1451520(15\beta^2 + 44\beta + \beta^3 + 24)t(0, 0, 0, 3) \\ &\quad + 4354560(3\beta^3 + 42 + 47\beta + 20\beta^2)t(0, 0, 1, 2) \\ &\quad + 4354560(9\beta^3 + 45\beta^2 - 24 + 46\beta)t(0, 1, 1, 1), \end{aligned}$$

where

$$t(a, b, c, d) = \sum_{\substack{(n_1, n_2, n_3, n_4) \\ \in \text{perm}(a, b, c, d)}} \lambda_i^{n_1} \lambda_j^{n_2} \lambda_l^{n_3} \lambda_l^{n_4}.$$

The expression $\xi(\lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta)$ is positive for all $\lambda_k \geq \lambda_l \geq \lambda_j \geq \lambda_i$, and $\beta \geq -\frac{2}{5}$.

Proof.

Let

$$\hat{s}(a, b, c) = \sum_{\substack{(n_1, n_2, n_3) \\ \in \text{perm}(a, b, c)}} \lambda_i^{n_1} \lambda_j^{n_2} \lambda_l^{n_3}.$$

Differentiating $\xi(\lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta)$ with respect to λ_k we obtain

$$\begin{aligned} \xi^{(1)}(\lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta) &= (104509440 + 4354560\beta^3 + 65318400\beta^2 + 191600640\beta)\lambda_k^2 \\ &\quad + (409328640\beta + 174182400\beta^2 + 365783040 + 26127360\beta^3)\lambda_k\hat{s}(0, 0, 1) \\ &\quad + (39191040\beta^3 + 195955200\beta^2 - 104509440 + 200309760\beta)\hat{s}(0, 1, 1) \\ &\quad + (13063680\beta^3 + 204664320\beta + 87091200\beta^2 + 182891520)\hat{s}(0, 0, 2), \end{aligned}$$

$$\begin{aligned} \xi^{(2)}(\lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta) &= (209018880 + 8709120\beta^3 + 130636800\beta^2 + 383201280\beta)\lambda_k \\ &\quad + (409328640\beta + 174182400\beta^2 + 365783040 + 26127360\beta^3)\hat{s}(0, 0, 1) \\ &\geq 0 \text{ for all } \beta \geq -0.7085. \end{aligned}$$

Define

$$\begin{aligned}\xi_0(\lambda_l; \lambda_j; \lambda_i; \beta) &\stackrel{\text{def}}{=} \xi(\lambda_l; \lambda_l; \lambda_j; \lambda_i; \beta), \\ \xi_1(\lambda_l; \lambda_j; \lambda_i; \beta) &\stackrel{\text{def}}{=} \xi^{(1)}(\lambda_l; \lambda_l; \lambda_j; \lambda_i; \beta).\end{aligned}$$

We will prove that the above equations are non-negative for all $\lambda_k \geq \lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$ and, hence, deduce that $\xi(\lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta)$ is positive for all $\lambda_k \geq \lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Differentiating $\xi_0(\lambda_l; \lambda_j; \lambda_i; \beta)$ with respect to λ_l we obtain

$$\begin{aligned}\xi_0(\lambda_l; \lambda_j; \lambda_i; \beta) &= (217728000\beta^2 + 537062400\beta + 29030400\beta^3 + 435456000)\lambda_l^3 \\ &\quad + (65318400\beta^3 + 609638400\beta + 370137600\beta^2 + 261273600)(\lambda_j + \lambda_i)\lambda_l^2 \\ &\quad + (409328640\beta + 174182400\beta^2 + 365783040 + 26127360\beta^3)(\lambda_j^2 + \lambda_i^2)\lambda_l \\ &\quad + 2(39191040\beta^3 + 195955200\beta^2 - 104509440 + 200309760\beta)\lambda_i\lambda_j\lambda_l \\ &\quad + (21772800\beta^2 + 63866880\beta + 1451520\beta^3 + 34836480)\lambda_j^3 \\ &\quad + (13063680\beta^3 + 204664320\beta + 87091200\beta^2 + 182891520)\lambda_i\lambda_j^2 \\ &\quad + (21772800\beta^2 + 63866880\beta + 1451520\beta^3 + 34836480)\lambda_i^3 \\ &\quad + (13063680\beta^3 + 204664320\beta + 87091200\beta^2 + 182891520)\lambda_i^2\lambda_j, \\ \xi_0^{(1)}(\lambda_l; \lambda_j; \lambda_i; \beta) &= (653184000\beta^2 + 1611187200\beta + 87091200\beta^3 + 1306368000)\lambda_l^2 \\ &\quad + (130636800\beta^3 + 1219276800\beta + 740275200\beta^2 + 522547200)(\lambda_j + \lambda_i)\lambda_l \\ &\quad + (78382080\beta^3 + 391910400\beta^2 - 209018880 + 400619520\beta)\lambda_i\lambda_j \\ &\quad + (409328640\beta + 174182400\beta^2 + 365783040 + 26127360\beta^3)(\lambda_j^2 + \lambda_i^2), \\ \xi_0^{(2)}(\lambda_l; \lambda_j; \lambda_i; \beta) &= (1306368000\beta^2 + 3222374400\beta + 174182400\beta^3 + 2612736000)\lambda_l \\ &\quad + (130636800\beta^3 + 1219276800\beta + 740275200\beta^2 + 522547200)(\lambda_i + \lambda_j) \\ &\geq 0 \text{ for all } \beta \geq -\frac{2}{3}.\end{aligned}$$

Define $\xi_{01}(\lambda_j; \lambda_i; \beta) = \xi_0^{(1)}(\lambda_j; \lambda_j; \lambda_i; \beta)$. We wish to show that ξ_{01} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{01} with respect to λ_j gives

$$\begin{aligned}\xi_{01}(\lambda_j; \lambda_i; \beta) &= (1567641600\beta^2 + 3239792640\beta + 243855360\beta^3 + 2194698240)\lambda_j^2 \\ &\quad + (209018880\beta^3 + 1619896320\beta + 1132185600\beta^2 + 313528320)\lambda_i\lambda_j \\ &\quad + (409328640\beta + 174182400\beta^2 + 365783040 + 26127360\beta^3)\lambda_i^2, \\ \xi_{01}^{(1)}(\lambda_j; \lambda_i; \beta) &= (3135283200\beta^2 + 6479585280\beta + 487710720\beta^3 + 4389396480)\lambda_j \\ &\quad + (209018880\beta^3 + 1619896320\beta + 1132185600\beta^2 + 313528320)\lambda_i, \\ \xi_{01}^{(2)}(\lambda_j; \lambda_i; \beta) &= 3135283200\beta^2 + 6479585280\beta + 487710720\beta^3 + 4389396480 \\ &\geq 0 \text{ for all } \beta \geq -3.\end{aligned}$$

Now

$$\begin{aligned} \xi_{01}(\lambda_i; \lambda_i; \beta) &= 479001600\lambda_i^2(\beta + 3)(\beta + 2)(\beta + 1) \geq 0 \text{ for all } \beta \geq -1, \\ \xi_{01}^{(1)}(\lambda_i; \lambda_i; \beta) &= 87091200\lambda_i(8\beta + 9)(\beta + 3)(\beta + 2) \geq 0 \text{ for all } \beta \geq -\frac{9}{8}. \end{aligned}$$

Hence, $\xi_0^{(1)}$ is non-negative for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Define $\xi_{00}(\lambda_j; \lambda_i; \beta) = \xi_0(\lambda_j; \lambda_j; \lambda_i; \beta)$. We wish to show that ξ_{00} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{00} with respect to λ_j gives

$$\begin{aligned} \xi_{00}(\lambda_j; \lambda_i; \beta) &= (783820800\beta^2 + 1619896320\beta + 121927680\beta^3 + 1097349120)\lambda_j^3 \\ &\quad + (156764160\beta^3 + 849139200\beta^2 + 235146240 + 1214922240\beta)\lambda_i\lambda_j^2 \\ &\quad + (613992960\beta + 261273600\beta^2 + 548674560 + 39191040\beta^3)\lambda_i^2\lambda_j \\ &\quad + (21772800\beta^2 + 63866880\beta + 1451520\beta^3 + 34836480)\lambda_i^3, \\ \xi_{00}^{(1)}(\lambda_j; \lambda_i; \beta) &= (235146240\beta^2 + 4859688960\beta + 365783040\beta^3 + 3292047360)\lambda_j^2 \\ &\quad + (313528320\beta^3 + 1698278400\beta^2 + 470292480 + 2429844480\beta)\lambda_i\lambda_j \\ &\quad + (613992960\beta + 261273600\beta^2 + 548674560 + 39191040\beta^3)\lambda_i^2, \\ \xi_{00}^{(2)}(\lambda_j; \lambda_i; \beta) &= (470292480\beta^2 + 9719377920\beta + 731566080\beta^3 + 6584094720)\lambda_j \\ &\quad + (313528320\beta^3 + 1698278400\beta^2 + 470292480 + 2429844480\beta)\lambda_i, \\ \xi_{00}^{(3)}(\lambda_j; \lambda_i; \beta) &= 104509440(\beta + 3)(7\beta^2 + 24\beta + 21) > 0 \text{ for all } \beta > -3. \end{aligned}$$

Now

$$\begin{aligned} \xi_{00}(\lambda_i; \lambda_i; \beta) &= 319334400\lambda_i^3(\beta + 3)(\beta + 2)(\beta + 1) > 0 \text{ for all } \beta > -1, \\ \xi_{00}^{(1)}(\lambda_i; \lambda_i; \beta) &= 718502400\lambda_i^2(\beta + 3)(\beta + 2)(\beta + 1) > 0 \text{ for all } \beta > -1, \\ \xi_{00}^{(2)}(\lambda_i; \lambda_i; \beta) &= 130636800\lambda_i(8\beta + 9)(\beta + 3)(\beta + 2) > 0 \text{ for all } \beta > -\frac{9}{8}. \end{aligned}$$

Hence, ξ_0 is positive for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$. This implies that $\xi(\lambda_l; \lambda_l; \lambda_j; \lambda_i; \beta)$ is positive for all $\lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

Differentiating $\xi_1(\lambda_l; \lambda_j; \lambda_i; \beta)$ with respect to λ_l we obtain

$$\begin{aligned} \xi_1(\lambda_l; \lambda_j; \lambda_i; \beta) &= (43545600\beta^3 + 805593600\beta + 326592000\beta^2 + 653184000)\lambda_l^2 \\ &\quad + ((65318400\beta^3 + 609638400\beta + 370137600\beta^2 + 261273600)\lambda_j \\ &\quad + (65318400\beta^3 + 609638400\beta + 370137600\beta^2 + 261273600)\lambda_i)\lambda_l \\ &\quad + (13063680\beta^3 + 204664320\beta + 87091200\beta^2 + 182891520)\lambda_i^2 \\ &\quad + (13063680\beta^3 + 204664320\beta + 87091200\beta^2 + 182891520)\lambda_j^2 \\ &\quad + (39191040\beta^3 + 195955200\beta^2 - 104509440 + 200309760\beta)\lambda_i\lambda_j, \end{aligned}$$

$$\begin{aligned}\xi_1^{(1)}(\lambda_l; \lambda_j; \lambda_i; \beta) &= (87091200\beta^3 + 1611187200\beta + 653184000\beta^2 + 1306368000)\lambda_l \\ &\quad + (65318400\beta^3 + 609638400\beta + 370137600\beta^2 + 261273600)\lambda_j \\ &\quad + (65318400\beta^3 + 609638400\beta + 370137600\beta^2 + 261273600)\lambda_i \\ &\geq 0 \text{ for all } \beta \geq -\frac{2}{3}.\end{aligned}$$

Define $\xi_{10}(\lambda_j; \lambda_i; \beta) = \xi_1(\lambda_j; \lambda_j; \lambda_i; \beta)$. We wish to show that ξ_{10} is non-negative for all $\lambda_j \geq \lambda_i > 0$. Differentiating ξ_{10} with respect to λ_j gives

$$\begin{aligned}\xi_{10}(\lambda_j; \lambda_i; \beta) &= (121927680\beta^3 + 1619896320\beta + 783820800\beta^2 + 1097349120)\lambda_j^2 \\ &\quad + (104509440\beta^3 + 809948160\beta + 566092800\beta^2 + 156764160)\lambda_i\lambda_j \\ &\quad + (13063680\beta^3 + 204664320\beta + 87091200\beta^2 + 182891520)\lambda_i^2, \\ \xi_{10}^{(1)}(\lambda_j; \lambda_i; \beta) &= (243855360\beta^3 + 3239792640\beta + 1567641600\beta^2 + 2194698240)\lambda_j \\ &\quad + (104509440\beta^3 + 809948160\beta + 566092800\beta^2 + 156764160)\lambda_i, \\ \xi_{10}^{(2)}(\lambda_j; \lambda_i; \beta) &= 34836480(\beta + 3)(7\beta^2 + 24\beta + 21) > 0 \text{ for all } \beta > -3.\end{aligned}$$

Now

$$\begin{aligned}\xi_{10}(\lambda_i; \lambda_i; \beta) &= 239500800\lambda_i^2(\beta + 3)(\beta + 2)(\beta + 1) > 0 \text{ for all } \beta > -1, \\ \xi_{10}^{(1)}(\lambda_i; \lambda_i; \beta) &= 43545600\lambda_i(8\beta + 9)(\beta + 3)(\beta + 2) > 0 \text{ for all } \beta > -\frac{9}{8}.\end{aligned}$$

Hence, ξ_1 is positive for all $\lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$. This implies that $\xi(\lambda_l; \lambda_l; \lambda_j; \lambda_i; \beta)$ is positive for all $\lambda_l \geq \lambda_j \geq \lambda_i > 0$ and $\beta \geq -\frac{2}{5}$.

□

Lemma B.21 Let

$$\begin{aligned}\xi(\lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta) &= (653184000 + 326592000\beta^2 + 43545600\beta^3 + 805593600\beta)t(0, 0, 0, 2) \\ &\quad + (130636800\beta^3 + 1219276800\beta + 522547200 + 740275200\beta^2)t(0, 0, 1, 1),\end{aligned}$$

where

$$t(a, b, c, d) = \sum_{\substack{(n_1, n_2, n_3, n_4) \\ \in \text{perm}(a, b, c, d)}} \lambda_i^{n_1} \lambda_j^{n_2} \lambda_l^{n_3} \lambda_k^{n_4}.$$

The expression $\xi(\lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta)$ is positive for all $\lambda_k \geq \lambda_l \geq \lambda_j \geq \lambda_i > 0$, and $\beta \geq -\frac{2}{5}$.

Proof.

Now

$$\begin{aligned}653184000 + 326592000\beta^2 + 43545600\beta^3 + 805593600\beta &> 0 \text{ for all } \beta \geq -2, \\130636800\beta^3 + 1219276800\beta + 522547200 + 740275200\beta^2 &> 0 \text{ for all } \beta \geq -\frac{2}{3}.\end{aligned}$$

Hence, $\xi(\lambda_k; \lambda_l; \lambda_j; \lambda_i; \beta)$ is positive for all $\lambda_k \geq \lambda_l \geq \lambda_j \geq \lambda_i$, and $\beta \geq -\frac{2}{5}$.

□

References

- [1] H. S. DOLLAR, N. I. M. GOULD, AND D. P. ROBINSON, *On solving trust-region and other regularized subproblems in optimization*, Tech. Rep. RAL-TR-2008-0xx, Rutherford Appleton Laboratory, 2008.
- [2] J. J. MORÉ AND D. C. SORENSEN, *Computing a trust region step*, SIAM J. Sci. Statist. Comput., 4 (1983), pp. 553–572.