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Convergence of inexact adaptive finite element solvers for elliptic problems

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We address the question of convergence of practical adaptive finite element solvers for symmetric elliptic problems, in the case where the resulting linear systems on each level of the algorithm are solved approximately. In particular, we show that if a particular smallness criterion (involving residuals of the linear solver and the a-posteriori bounds used in the adaptive finite element algorithm) is satisfied, then the adaptive algorithm satisfies a contraction property between two consecutive levels of refinement. This generalises the current convergence analyses of adaptive algorithms for elliptic problems whereby the resulting linear systems are assumed to be solved exactly. Moreover, based on known and new results for the estimation of the residual of the conjugate gradient method, we show that the smallness criterion gives rise to a practical stopping criterion for the iterations of the linear solver, which guarantees that the (inexact) adaptive algorithm converges. A series of numerical experiments highlights the practicality of the theoretical developments.

Keywords: Adaptative finite-element method, Conjugate gradient method, stopping criteria,
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Contents

1 Introduction

Adaptive finite element methods (AFEM), based on a-posteriori error estimates, have become established procedures for computing efficient and accurate approximations to the solution of partial differential equations (see, e.g., [35,1] and the references therein). The typical algorithm is essentially a simple four-step procedure: solve–estimate–mark–refine. Namely, given a subdivision of the computational domain, one solves the finite element problem (which involves the assembly of the stiffness matrix and the solution of the resulting linear system), then computes an a posteriori error estimate; based on this estimate, a marking strategy is applied to identify for further refinement a set of elements in the current subdivision. Invariably, it is assumed that the solution of the linear systems on each level is exact. For large problems, however, exact solution of the linear systems may not be obtained efficiently (using, e.g., sparse direct solvers), if at all. In such cases, a competitive, or only, alternative is provided by iterative linear solvers (such as the conjugate gradient method) which compute approximations to the exact finite element solution. The key issue in this context is accuracy: a highly accurate approximation of the solution to the linear system at each level is inefficient and, most likely, unnecessary, while a poor approximation may affect the quality of the a posteriori estimators, thereby yielding different, possibly extensive refinements; the latter, in turn, affects the quality of the solution at the next step and of the adaptive algorithm itself.

The question of accuracy of an iterative method employed to solve a finite element system of equations has been considered both for the case of a single problem see, e.g., [2,16,4] and for a sequence of problems in an adaptive context, see [24]. It is now known that, in order to bring the finite element error measured in the energy norm below a certain tolerance in an efficient way, one needs to monitor the norm of the algebraic residual dual to the energy norm. This task is non-trivial. Furthermore, to the best of our knowledge, the integration of this approach in an adaptive procedure has only been effected in an empirical manner, since the tolerance has been only set empirically [24]. As such, it has not been clear how the a posteriori error estimate (step two) could be used to advise the required accuracy of iterative linear solver.

Recent years have seen a growing interest in establishing convergence results for AFEM algoritms. The focus has been mainly second order elliptic problems; for this class of problems results are currently available for conforming finite element methods [17,30,8,34,32,14,31,10,7], for classical non-conforming methods [13] and for discontinuous Galerkin methods [25,23,9]. In the light of these recent results, it is desirable to provide new stopping criteria for iterative methods which maintain the convergence of the adaptive procedure.

The contribution of this work is two-fold. First, it is shown that, given an exact convergent adaptive strategy, one can compute a sequence of inexact solutions which will yield a convergent adaptive algorithm as long as the algebraic residual satisfies a certain criterion. This criterion involves both the energy-dual norm of the algebraic residual and the a posteriori error estimate. Second, it is shown how to estimate and implement this criterion practically in an adaptive context, yielding an automatic stopping criterion for the accuracy of iterative solutions for the adaptive finite element algorithm, where each level is solved iteratively using the Conjugate Gradient (CG) method.

In this work, we only consider iterative solution methods of Krylov subspace type. This class of methods has been analysed extensively over the last three decades and convergence and applicability are largely understood. Furthermore, specific results are available for linear systems arising from finite element discretisations; in particular, it has been shown that convergence should be monitored in the energy norm for reasons of efficiency [2,16,4,3]. This applies even more stringently in the case of adaptive finite element algorithms where the size of the problem grows with each step. Moreover, one expects an adaptive, possibly hierarchical procedure, such as AFEM, to provide certain a posteriori information or even recycling capabilities at each step which would aid the iterative method at the next step. While this is stated in [24], we show that this is indeed the case and that efficiency can be dramatically improved.

For the sake of simplicity of exposition we illustrate our approach on a standard symmetric second-order elliptic problem. Some of the intermediate results presented below are inspired and influenced by the works of Karakashian and Pascal [25], Cascon, Kreuzer, Nochetto and Siebert [14] and Stevenson [32] in the context of second order elliptic problems. We stress, however, that generalisations of the results (in various settings) presented in this work are expected to be possible.

The paper is organised as follows. In Section 2 we introduce the model problem and the adaptive finite element algorithm in exact form. Section 3 contains an analysis of the inexact version of the AFEM method. In particular we derive a sufficient criterion to ensure convergence of the inexact algorithm. Section 4 discusses the practical implementation of this criterion which requires the evaluation of the dual norm of the algebraic residual. This can be achieved by employing the Conjugate Gradient method for solving the linear systems. Finally, in Section 5 we illustrate the efficiency of our approach on a range of two- and three-dimensional problems.

2 Problem formulation

In this section we introduce notation and a description of the model problem together with an posteriori error bound of residual type. We also include a description of an adaptive finite element algorithm together with the corresponding convergence result.

2.1 Model problem and the finite element method

The standard Lebesgue spaces are denoted by $L^p(\omega)$, $1 \leq p \leq +\infty$, $\omega \subset \mathbb{R}^d$; when $p = 2$ the corresponding inner product is denoted by $\langle \cdot, \cdot, \rangle_\omega$ and norm $\| \cdot \|_\omega$; when $\omega = \Omega$, we shall drop the subindex for brevity, writing $\langle \cdot, \cdot, \rangle$ and $\| \cdot \|$, respectively. We also denote by $H_0^1(\omega)$ the standard Sobolev space of functions with zero trace on $\partial \omega$.

Let Ω be a bounded open polyhedral domain in $\mathbb{R}^d$, $d = 2, 3$ and let $\partial \Omega$ denote its boundary. We consider the second order equation

$$-\nabla \cdot (a \nabla u) = f \quad \text{in } \Omega, \tag{2.1}$$

where $a \in [L^\infty(\Omega)]^{d \times d}$ is a positive definite tensor and $f \in L^2(\Omega)$. For simplicity of the presentation, we impose homogeneous Dirichlet boundary condition $u = 0$ on $\partial \Omega$, although this appears not to be an essential restriction. We shall denote by $\| \cdot \|_a := \|\sqrt{a} \nabla(\cdot)\|$ the, so-called, energy norm.

Let $\mathcal{T}$ be a conforming subdivision of Ω into disjoint simplicial elements $\kappa \in \mathcal{T}$. We assume that the subdivision $\mathcal{T}$ is shape-regular (see, e.g., p.124 in [15]) and that it is constructed via affine mappings F_κ, where $F_\kappa : \hat{\kappa} \to \kappa$, with non-singular Jacobian, where $\hat{\kappa}$ is the reference simplex.

For a nonnegative integer r, we denote by $\mathcal{P}_r(\hat{\kappa})$, the set of all polynomials of total degree at most r, defined on $\hat{\kappa}$. We consider the finite element space

$$\mathcal{V} := \{V \in H_0^1(\Omega) : \ V|_\kappa \circ F_\kappa \in \mathcal{P}_r(\hat{\kappa}), \ \kappa \in \mathcal{T}\}. \tag{2.2}$$

By Γ we denote the union of all $(d-1)$-dimensional element faces associated with the subdivision $\mathcal{T}$ (including the boundary). Further we decompose Γ into two disjoint subsets $\Gamma = \partial \Omega \cup \Gamma_{\text{int}}$, where $\Gamma_{\text{int}} := \Gamma \backslash \partial \Omega$. We define $h_\kappa := (\mu_d(\kappa))^{1/d}$, $\kappa \in \mathcal{T}$, where μ_d is the d-dimensional Lebesgue measure. Also, for two (generic) elements κ^+, κ^- sharing a face $e := \partial \kappa^+ \cap \partial \kappa^- \subset \Gamma_{\text{int}}$ we define $h_e := \mu_{d-1}(e)$. We collect these quantities into the element-wise constant function $\mathbf{h} : \Omega \to \mathbb{R}$, with $\mathbf{h}|_\kappa = h_\kappa$, $\kappa \in \mathcal{T}$ and $\mathbf{h}|_e = h_e$, $e \in \Gamma$. The families of meshes constructed by the algorithms presented in this work will be conforming and shape-regular.

We assume throughout that the diffusion tensor a is element-wise constant; this has been done in the interest of simplicity of the exposition only. This restriction can be lifted by combining the a posteriori bounds from [14] (where the case of variable diffusivity is presented) with the developments described below.

The finite element method reads:

$$\text{find } U \in \mathcal{V} \text{ such that} \quad a(U, V) = l(V) \quad \forall V \in \mathcal{V}, \tag{2.3}$$

where the bilinear form $a : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ and the linear form $l : H_0^1(\Omega) \to \mathbb{R}$ are given by

$$a(w, v) := \int_\Omega a \nabla w \cdot \nabla v \, \mathrm{d}x \qquad \text{and} \qquad l(v) := \int_\Omega f v \, \mathrm{d}x, \tag{2.4}$$

respectively, for $w, v \in H_0^1(\Omega)$.

We note the projection property of the finite element method in the following lemma.

Lemma 1 *Let u be the (weak) solution to problem (2.1) with homogeneous Dirichlet boundary conditions and $U \in \mathcal{V}$ be the finite element solution to (2.3). Then, for any $V \in \mathcal{V}$, we have*

$$||u - U||_a^2 = ||u - V||_a^2 - ||V - U||_a^2. \tag{2.5}$$

Proof We have, respectively,

$$||u - U||_a^2 = a(u - U, u - U) = a(u - V, u - V) + a(u - V, V - U)$$
$$= ||u - V||_a^2 + a(U - V, V - U) = ||u - V||_a^2 - ||U - V||_a^2,$$

using Galerkin orthogonality.

Let now $\{\phi_i\}_{1 \le i \le N}$ denote a set of basis functions for $\mathcal{V}$ so that

$$U = \sum_{i=1}^{N} \mathbf{u}_i \phi_i,$$

and let $A_{ij} = a(\phi_j, \phi_i)$, $\mathbf{b}_k = l(\phi_k)$, $i, j, k = 1, \cdots, N$. With this notation, the linear system corresponding to (2.3) is

$$A\mathbf{u} = \mathbf{b}, \tag{2.6}$$

where $A \in \mathbb{R}^{N \times N}$ is the stiffness matrix corresponding to a set of basis functions $\{\phi_i\}_{1 \le i \le N}$.

2.2 A posteriori error bounds of residual type

For every face $e \in \Gamma_{\text{int}}$, we define the *jump* across e of a scalar function w, defined in an open neighbourhood of e, by

$$[w](x) = \lim_{t \to 0} \big(w(x - t\mathbf{n}_e) - w(x + t\mathbf{n}_e) \big),$$

for $x \in e$, where $\mathbf{n}_e$ denotes a normal vector to e. (Note that the jump is only uniquely defined up to a sign, which is unimportant for the discussion below.) For any subset $\mathcal{M} \subset \mathcal{T}$ (i.e., $\mathcal{M}$ is a collection of elements of $\mathcal{T}$), we define the local estimator by

$$\eta_{\mathcal{T}}(U, \mathcal{M}) := \Big(\sum_{\kappa \in \mathcal{M}} \Big(h_\kappa^2 \|f + \nabla \cdot (a\nabla U)\|_\kappa^2 + \sum_{e \in \Gamma_{\text{int}} \cap \partial\kappa} h_e \|[a\nabla U \cdot \mathbf{n}_e]\|_e^2 \Big) \Big)^{1/2}.$$

We recall a standard residual-type a posteriori reliability bound of the energy-norm error (see, e.g., [35,1]).

Theorem 1 *Let $u \in H_0^1(\Omega)$ be the solution to (2.1), with homogeneous Dirichlet boundary conditions, $U \in \mathcal{V}$ be the finite element approximation, associated with the mesh $\mathcal{T}$. Then there exists a positive constant C_1, independent of $\mathcal{T}$, $\mathbf{h}$, u and U, so that*

$$\|u - U\|_a^2 \le C_1 \, \eta_{\mathcal{T}}^2(U, \mathcal{T}). \tag{2.7}$$

2.3 AFEM algorithms

We describe now briefly the adaptive finite element algorithm analyzed in [14]. Henceforth, all objects indexed by $m \in \mathbb{N}$ refer to the object in the m-th iteration of the adaptive algorithm.

Each iteration of the algorithm comprises four steps which are summarized in the workflow below:

$$\text{SOLVE} \to \text{ESTIMATE} \to \text{MARK} \to \text{REFINE} \tag{2.8}$$

The first step involves computing the finite element solution U_m^e to the problem:

$$\text{find } U_m^e \in \mathcal{V}_m \quad \text{such that} \quad a(U_m^e, V_m) = l(V_m) \quad \forall V_m \in \mathcal{V}_m. \tag{2.9}$$

(The superscript "e" in this section signifies that here we refer to the "exact" algorithm, i.e., all calculations are performed exactly on all levels.)

In the second step, for each element $\kappa \in \mathcal{T}_m$ we calculate the local estimators

$$\eta_{\mathcal{T}_m}^2(U_m^e, \kappa) := h_\kappa^2 \|f + \nabla \cdot (a\nabla U_m^e)\|_\kappa^2 + \sum_{e \in \Gamma_{\text{int},m} \cap \partial\kappa} h_e \|[a\nabla U_m^e \cdot \mathbf{n}]\|_e^2.$$

The third step identifies a subset $\mathcal{M}_m$ of elements of the mesh $\mathcal{T}_m$ which are marked for refinement, using a Dörfler-type marking strategy (see [17]); more precisely, for a user-defined parameter $0 < \theta \le 1$ (from now on termed the *Dörfler marking parameter*), we find $\mathcal{M}_m$, subset of $\mathcal{T}_m$, such that

$$\eta_{\mathcal{T}_m}^2(U_m^e, \mathcal{M}_m) \ge \theta \, \eta_{\mathcal{T}_m}^2(U_m^e, \mathcal{T}_m); \tag{2.10}$$

the elements $\kappa \in \mathcal{M}_m$ are called *marked elements*.

In the fourth step, the marked elements are refined by newest vertex bisection (see, e.g., [14] for details); if the resulting mesh is not conforming, it is made into a conforming mesh by sufficient additional refinement (again

by newest vertex bisection) of elements possessing hanging nodes. These elements are then added to the marked elements $\mathcal{M}_m$, thereby arriving to the new mesh $\mathcal{T}_{m+1}$.

Summarising, the adaptive algorithm constructs a sequence of objects $\{\mathcal{T}_m, \mathcal{V}_m, U_m^e\}_{m \geq 1}$, starting with a (given) conforming mesh $\mathcal{T}_0$; the corresponding pseudo-code is given below.

Algorithm 1. AFEM algorithm

Set parameter $0 < \theta \leq 1$. Set $m = 0$.
While convergence criterion not satisfied
    1. Solve exactly (2.9) to obtain U_m^e.
    2. Compute local estimators $\eta_{\mathcal{T}_m}(U_m^e, \kappa)$, $\kappa \in \mathcal{T}_m$.
    3. Mark elements $\mathcal{M}_m$ for refinement in $\mathcal{T}_m$ using (2.10).
    4. Refine $\mathcal{M}_m$ to obtain new mesh $\mathcal{T}_{m+1}$. Set $m \leftarrow m + 1$.
End

The above procedure assumes that exact integration is employed and that the solution of the corresponding linear system in the first step is exact at each iteration of the adaptive algorithm. This may not be possible for large problems and the next section describes an inexact version of this algorithm. We conclude this section with the main convergence result in [14].

Theorem 2 *There exist constants $\xi > 0$ and $0 < \alpha < 1$ such that*

$$\|u - U_{m+1}^e\|_a^2 + \xi \eta_{\mathcal{T}_{m+1}}^2(U_{m+1}^e, \mathcal{T}_{m+1}) \leq \alpha \left(\|u - U_m^e\|_a^2 + \xi \eta_{\mathcal{T}_m}^2(U_m^e, \mathcal{T}_m) \right).$$

The above result can be extended to the inexact case, as shown next.

3 Inexact AFEM algorithms

We introduce now an inexact AFEM algorithm with the same workflow (2.8) as the exact algorithm described above. In general, the subdivisions arising in an inexact context are different from the exact case; we will denote the subdivisions and associated finite element spaces by $\tilde{\mathcal{T}}_m$ and by $\tilde{\mathcal{V}}_m$, respectively.

In the first step we solve inexactly (2.9) to obtain an *inexact finite element solution* $\tilde{U}_m \in \tilde{\mathcal{V}}_m$, which is an approximation to the *exact finite element solution* U_m *on the space* $\tilde{\mathcal{V}}_m$ (i.e., U_m is the solution to the finite element problem (2.3) posed on the finite element space $\mathcal{V} = \tilde{\mathcal{V}}_m$). The theory presented in this section is not dependent on the specific nature of inexactness of $\tilde{U}_m$, so we keep the discussion in an abstract setting.

In the remainder, we shall use the abbreviations $\eta_m(V, \mathcal{M}) \equiv \eta_{\tilde{\mathcal{T}}_m}(V, \mathcal{M})$ for $V \in \tilde{\mathcal{V}}_m$, $\mathcal{M} \subset \tilde{\mathcal{T}}_m$ and $\eta_m(V) \equiv \eta_m(V, \tilde{\mathcal{T}}_m)$, for $m = 0, 1, \ldots$. We shall assume that the inexact solution $\tilde{U}_m$ satisfies the following inequality:

$$||\tilde{U}_{m-1} - U_{m-1}||_a^2 + \mu ||\tilde{U}_m - U_m||_a^2 \leq \nu \eta_{m-1}^2(\tilde{U}_{m-1}), \tag{3.11}$$

for some values μ and ν (to be made precise later).

In the second step we calculate local estimators $\eta_m^2(\tilde{U}_m, \kappa)$ based on $\tilde{U}_m$, for each element $\kappa \in \tilde{\mathcal{T}}_m$.

The third step uses the same Dörfler-type marking strategy as in the exact case, with U_m replaced by $\tilde{U}_m$: we find $\tilde{\mathcal{M}}_m$, subset of $\tilde{\mathcal{T}}_m$, such that

$$\eta_m^2(\tilde{U}_m, \tilde{\mathcal{M}}_m) \geq \theta \ \eta_m^2(\tilde{U}_m). \tag{3.12}$$

Finally, the marked elements are refined as in the exact case, by newest vertex bisection

The adaptive algorithm constructs a sequence of objects $\{\tilde{\mathcal{T}}_m, \tilde{\mathcal{V}}_m, \tilde{U}_m\}_{m \geq 1}$, starting with a (given) conforming mesh $\mathcal{T}_0$ and it is summarised as follows.

Algorithm 2. Inexact AFEM

Set parameters $0 < \theta \leq 1$, μ and ν. Initialise $\tilde{U}_0$. Set $m = 1$.
While convergence criterion not satisfied
    1. Solve inexactly (2.9) to obtain $\tilde{U}_m$ so that (3.11) is satisfied.
    2. Compute local estimators $\eta_{\tilde{\mathcal{T}}_m}(\tilde{U}_m, \kappa), \kappa \in \tilde{\mathcal{T}}_m$.
    3. Mark elements $\tilde{\mathcal{M}}_m$ for refinement in $\tilde{\mathcal{T}}_m$ using (2.10).
    4. Refine $\tilde{\mathcal{M}}_m$ to obtain new mesh $\tilde{\mathcal{T}}_{m+1}$. Set $m \leftarrow m + 1$.
End

3.1 Convergence of the inexact adaptive finite element solver

We first need the following lemmata.

Lemma 2 *Let $V, W \in \mathcal{V}$. Then, there exists $C_2 > 1$, depending only on the polynomial degree r and the shape-regularity of the mesh $\mathcal{T}$, such that*

$$\eta^2(V) \leq (1+\gamma)\,\eta^2(W) + C_2(1+\gamma^{-1})\|W - V\|_a^2, \tag{3.13}$$

for any $\gamma > 0$.

Proof Using standard inverse estimates, one can show that

$$\|\mathbf{h}\nabla \cdot (a\nabla(W - V))\|^2 + \|\mathbf{h}^{1/2}[a\nabla(W - V) \cdot \mathbf{n}]\|_{\Gamma_{\text{int}}}^2 \leq C\|W - V\|_a^2, \tag{3.14}$$

for a constant $C > 1$, depending on r and on the shape-regularity of $\mathcal{T}$. Using the triangle inequality, Young's inequality and the bound (3.14), we have

$$
\begin{aligned}
\eta(V)^2 =& \|\mathbf{h}(f + \nabla \cdot (a\nabla(W - V + V)))\|^2 + \|\mathbf{h}^{1/2}[a\nabla(W - V + V) \cdot \mathbf{n}]\|_{\Gamma_{\text{int}}}^2 \\
\leq& (1+\gamma)\big(\|\mathbf{h}(f + \nabla \cdot (a\nabla W)\|^2 + \|\mathbf{h}^{1/2}[a\nabla W \cdot \mathbf{n}]\|_{\Gamma_{\text{int}}}^2\big) \\
& + C(1+\gamma^{-1})\|W - V\|_a^2.
\end{aligned}
\tag{3.15}
$$

The following lemma is inspired from a result in [14, Corollary 4.4].

Lemma 3 *For any $V_{m+1} \in \mathcal{V}_{m+1}$, we have*

$$\eta_{m+1}^2(V_{m+1}) \leq (1 - \tau\theta)(1+\delta)\eta_m^2(\tilde{U}_m) + (1+\delta^{-1})C_2\|V - \tilde{U}_m\|_a^2, \tag{3.16}$$

where $\theta \in (0,1)$ is the Dörfler marking parameter, and $\tau = 1 - 2^{-1/d}$.

Proof Using standard inverse estimates, one can show that

$$\|\mathbf{h}_{m+1}\nabla \cdot (a\nabla(V_{m+1} - \tilde{U}_m))\|^2 + \|\mathbf{h}_{m+1}^{1/2}[a\nabla(V_{m+1} - \tilde{U}_m) \cdot \mathbf{n}]\|_{\Gamma_{\text{int},m+1}}^2 \leq C\|V_{m+1} - \tilde{U}_m\|_a^2, \tag{3.17}$$

for C as in (3.17), with $\mathbf{h}_m$ and $\Gamma_{\text{int},m}$ denoting the mesh-size function and the union of interior faces on the mesh $\tilde{\mathcal{T}}_m$, $m = 0, 1, \ldots$, respectively. Using the triangle inequality, Young's inequality and the bound (3.14), we have

$$
\begin{aligned}
\eta_{m+1}^2(V_{m+1}) =& \|\mathbf{h}_{m+1}(f + \nabla \cdot (a\nabla V_{m+1}))\|^2 + \|\mathbf{h}_{m+1}^{1/2}[a\nabla V_{m+1} \cdot \mathbf{n}]\|_{\Gamma_{\text{int},m+1}}^2 \\
\leq& (1+\delta)\big(\|\mathbf{h}_{m+1}(f + \nabla \cdot (a\nabla \tilde{U}_m)\|^2 + \|\mathbf{h}_{m+1}^{1/2}[a\nabla \tilde{U}_m \cdot \mathbf{n}]\|_{\Gamma_{\text{int},m+1}}^2\big) \\
& + (1+\delta^{-1})C\|V_{m+1} - \tilde{U}_m\|_a^2 \\
=& (1+\delta)\eta_{m+1}^2(V_{m+1}) + (1+\delta^{-1})C\|V_{m+1} - \tilde{U}_m\|_a^2,
\end{aligned}
\tag{3.18}
$$

for any $\delta > 0$.

An element $\kappa \in \tilde{\mathcal{M}}_m$ is bisected into elements

$$\mathcal{R}_\kappa := \{\kappa' \in \tilde{\mathcal{T}}_{m+1} : \kappa' \subset \kappa\},$$

due to the refinement strategy of the adaptive algorithm. Observing that, for all $\kappa' \in \mathcal{R}_\kappa$, we have $h_{\kappa'} \leq 2^{-1/d}h_\kappa$, we deduce

$$
\begin{aligned}
\eta_{m+1}^2(\tilde{U}_m) &= \eta_m^2(\tilde{U}_m, \tilde{\mathcal{T}}_m \setminus \tilde{\mathcal{M}}_m) + \eta_{m+1}^2(\tilde{U}_m, \{\mathcal{R}_\kappa : \kappa \in \tilde{\mathcal{M}}_m\}) \\
&\leq \eta_m^2(\tilde{U}_m, \tilde{\mathcal{T}}_m \setminus \tilde{\mathcal{M}}_m) + 2^{-1/d}\eta_m^2(\tilde{U}_m, \tilde{\mathcal{M}}_m),
\end{aligned}
\tag{3.19}
$$

since $[a\nabla \tilde{U}_m \cdot \mathbf{n}] = 0$ almost everywhere on $\Gamma_{\text{int},m+1} \setminus \Gamma_{\text{int},m}$. Combining (3.18) with (3.19), the result readily follows by making use of the marking strategy (2.10).

We are now in the position to prove a contraction property for the error of the inexact adaptive algorithm.

Theorem 3 *Let u and $\tilde{U}_m$, $m \geq 1$ as above, so that*

$$||\tilde{U}_m - U_m||_a^2 + \mu||\tilde{U}_{m+1} - U_{m+1}||_a^2 \leq \nu\eta_m^2(\tilde{U}_m), \tag{3.20}$$

with

$$\mu := \frac{1 + \xi C_2(1 + \gamma^{-1})}{\epsilon\xi(1 + 2C_1C_2)}, \quad \nu := \frac{\beta}{\epsilon(1 + 2C_1C_2)}, \tag{3.21}$$

where $0 < \epsilon < 1$, $\xi := \left(C_2(1 + \gamma)(1 + \delta^{-1})\right)^{-1}$, and β, γ, δ and ϵ are chosen small enough, so that $(1 - \tau\theta)(1 + \gamma)(1 + \delta) + 2\epsilon C_1 + \beta < 1$. Then, there exist a constant $0 < \alpha < 1$, depending only on the shape regularity of $\tilde{\mathcal{T}}_1$ and on the marking parameter θ, such that

$$||u - \tilde{U}_{m+1}||_a^2 + \xi\eta_{m+1}^2(\tilde{U}_{m+1}) \leq \alpha\left(||u - \tilde{U}_m||_a^2 + \xi\eta_m^2(\tilde{U}_m)\right). \tag{3.22}$$

Proof The projection property of the finite element method (2.5) for the (exact) finite element solution U_m with $V = \tilde{U}_m \in \tilde{\mathcal{V}}_m$, for all $m \geq 1$, reads

$$||u - \tilde{U}_m||_a^2 = ||u - U_m||_a^2 + ||\tilde{U}_m - U_m||_a^2. \tag{3.23}$$

Also, (2.5) for U_{m+1} with $V = \tilde{U}_m$ (noting that $\tilde{U}_m \in \tilde{\mathcal{V}}_{m+1}$, as the adaptive algorithm involves only refinement of elements), for all $m \geq 1$, yielding

$$||u - U_{m+1}||_a^2 = ||u - \tilde{U}_m||_a^2 - ||\tilde{U}_m - U_{m+1}||_a^2. \tag{3.24}$$

Combining (3.23) (for $m + 1$) with (3.24), we deduce

$$||u - \tilde{U}_{m+1}||_a^2 = ||u - \tilde{U}_m||_a^2 - ||\tilde{U}_m - U_{m+1}||_a^2 + ||\tilde{U}_{m+1} - U_{m+1}||_a^2. \tag{3.25}$$

Focusing on estimating the term $\eta_{m+1}^2(\tilde{U}_{m+1})$, we combine (3.13) (with $V = \tilde{U}_{m+1}$ and $W = U_{m+1}$), with (3.16) (with $V_{m+1} = U_{m+1}$), to arrive to:

$$\eta_{m+1}^2(\tilde{U}_{m+1}) \leq (1 - \tau\theta)(1 + \gamma)(1 + \delta)\eta_m^2(\tilde{U}_m) + C_2(1 + \gamma)(1 + \delta^{-1})||\tilde{U}_m - U_{m+1}||_a^2 \\ + C_2(1 + \gamma^{-1})||\tilde{U}_{m+1} - U_{m+1}||_a^2. \tag{3.26}$$

Setting $\xi = \left(C_2(1 + \gamma)(1 + \delta^{-1})\right)^{-1}$, (3.25) and (3.26) imply

$$||u - \tilde{U}_{m+1}||_a^2 + \xi\eta_{m+1}^2(\tilde{U}_{m+1}) \leq ||u - \tilde{U}_m||_a^2 + \left(1 + \xi C_2(1 + \gamma^{-1})\right)||\tilde{U}_{m+1} - U_{m+1}||_a^2 \\ + \xi(1 - \tau\theta)(1 + \gamma)(1 + \delta)\eta_m^2(\tilde{U}_m). \tag{3.27}$$

To estimate further the first term on the right-hand side of (3.27), we use (3.23), working as follows:

$$||u - \tilde{U}_m||_a^2 = (1 - \epsilon\xi)||u - \tilde{U}_m||_a^2 + \epsilon\xi||u - U_m||_a^2 + \epsilon\xi||\tilde{U}_m - U_m||_a^2 \\ \leq (1 - \epsilon\xi)||u - \tilde{U}_m||_a^2 + \epsilon\xi C_1\eta_m^2(U_m) + \epsilon\xi||\tilde{U}_m - U_m||_a^2 \\ \leq (1 - \epsilon\xi)||u - \tilde{U}_m||_a^2 + 2\epsilon\xi C_1\eta_m^2(\tilde{U}_m) + \epsilon\xi(1 + 2C_1C_2)||\tilde{U}_m - U_m||_a^2, \tag{3.28}$$

for any $0 < \epsilon < 1$, using Theorem 1 and (3.13) (with $V = U_m$, $W = \tilde{U}_m$ and $\gamma = 1$). Applying (3.28) on (3.27), we arrive to

$$||u - \tilde{U}_{m+1}||_a^2 + \xi\eta_{m+1}^2(\tilde{U}_{m+1}) \leq (1 - \epsilon\xi)||u - \tilde{U}_m||_a^2 + \epsilon\xi(1 + 2C_1C_2)||\tilde{U}_m - U_m||_a^2 \\ + \xi\left((1 - \tau\theta)(1 + \gamma)(1 + \delta) + 2\epsilon C_1\right)\eta_m^2(\tilde{U}_m) \\ + \left(1 + \xi C_2(1 + \gamma^{-1})\right)||\tilde{U}_{m+1} - U_{m+1}||_a^2. \tag{3.29}$$

From hypothesis, we have

$$\epsilon\xi(1 + 2C_1C_2)||\tilde{U}_m - U_m||_a^2 + \left(1 + \xi C_2(1 + \gamma^{-1})\right)||\tilde{U}_{m+1} - U_{m+1}||_a^2 \leq \beta\xi\eta_m^2(\tilde{U}_m), \tag{3.30}$$

for some $\beta > 0$, we get

$$||u - \tilde{U}_{m+1}||_a^2 + \xi\eta_{m+1}^2(\tilde{U}_{m+1}) \leq (1 - \epsilon)||u - \tilde{U}_m||_a^2 + \xi\left((1 - \tau\theta)(1 + \gamma)(1 + \delta) + 2\epsilon C_1 + \beta\right)\eta_m^2(\tilde{U}_m). \tag{3.31}$$

Choosing β, γ, δ and ϵ small enough, so that $(1 - \tau\theta)(1 + \gamma)(1 + \delta) + 2\epsilon C_1 + \beta < 1$, the result follows choosing

$$\alpha := \max\left\{1 - \epsilon\xi, (1 - \tau\theta)(1 + \gamma)(1 + \delta) + 2\epsilon C_1 + \beta\right\}.$$

Criterion (3.21) involves algebraic errors alone; as such, it represents essentially an *adaptive stopping criterion* for an iterative method. However, its implementation is non-trivial since it involves the exact solution U_m. In the next section we show how (3.21) can be implemented efficiently for the Conjugate Gradient method.

4 Stopping criteria for the Conjugate Gradient method

We now turn to the practical implementation of criterion (3.21). Our choice of solver is the Conjugate Gradient (CG) method, due to its convenient properties (see below). At each step m we need to solve iteratively a linear system of the form (2.6)

$$A_m \mathbf{u}_m = \mathbf{b}_m. \tag{4.32}$$

The matrices A_m have size $N_m \times N_m$ with N_m an increasing sequence. We denote by $\mathbf{u}_m^k \in \mathbb{R}^{N_m}$ the k-th CG iterate at step m of the adaptive algorithm and by U_m^k the corresponding function in $\tilde{\mathcal{V}}_m$. We denote the residual by $\mathbf{r}_m^k := \mathbf{b}_m - A_m \mathbf{u}_m^k$ and note that the energy norm of the error can be expressed as a dual norm of the residual:

$$\|U_m - U_m^k\|_a = \|\mathbf{u}_m - \mathbf{u}_m^k\|_{A_m} = \|\mathbf{r}_m^k\|_{A_m^{-1}},$$

where $\langle x, y \rangle_A := x^T A y$, $x, y \in \mathbb{R}^N$, $A \in \mathbb{R}^{N \times N}$, denotes the standard inner product weighted by A in $\mathbb{R}^N$, with the corresponding norm $\|x\|_A := \sqrt{\langle x, x \rangle_A}$. A candidate for $\tilde{U}_m$ is U_m^k for any k for which criterion (3.21) is satisfied. Our aim is to find an automatic way of choosing k so that the overall solution process is efficient.

It is well-known that the Conjugate Gradient method minimises the energy norm of the error, namely

$$\mathbf{u}_m^k = \arg \min_{\mathbf{u} \in \mathcal{K}_k(\mathbf{r}_m^0, A_m)} \|\mathbf{u}_m - \mathbf{u}\|_{A_m},$$

where $\mathcal{K}_k(\mathbf{r}_m^0, A_m) := \left\{ \mathbf{r}_m^0, A_m \mathbf{r}_m^0, \cdots, A_m^{k-1} \mathbf{r}_m^0 \right\}$ is the Krylov subspace of dimension k. Thus, the energy norm of the error decreases monotonically and criterion (3.21) will be satisfied for all U_m^k with $k > k^*$ for some k^*. This makes CG an attractive choice for enforcing criterion (3.21) efficiently. In addition, there are various established numerical techniques that provide bounds or estimates for the energy norm of the error at each step. As such, the method is both optimal and practical. We note that these properties do not hold in general and for non-symmetric problems the best choice of iterative method remains undecided.

We review below some of the existing bounds and estimates for the energy norm of the error. We also discuss the practical implementation of the only existing guaranteed upper bound for the CG error. Finally, we derive adaptive stopping criteria for the CG method.

4.1 Error bounds for the Conjugate Gradient method

The CG method is included below for the generic problem

$$A\mathbf{u} = \mathbf{b}.$$

Algorithm 3. Conjugate Gradient Algorithm

Set $\mathbf{u}^0 := 0; \mathbf{p}^0 := \mathbf{r}^0 := \mathbf{b}; \sigma_0 := \|\mathbf{r}^0\|^2$;
For $j = 0, 1, \ldots$ until convergence do
$\quad \mathbf{v}^j = A\mathbf{p}^j; \ \gamma_j = \sigma_j/(\mathbf{r}^j \cdot \mathbf{v}^j)$;
$\quad \mathbf{u}^{j+1} = \mathbf{u}^j + \gamma_j \mathbf{p}^j; \ \mathbf{r}^{j+1} = \mathbf{r}^j - \gamma_j \mathbf{u}^j; \ \sigma_{j+1} = \|\mathbf{r}^{j+1}\|^2$;
$\quad \chi_{j+1} = \sigma_{j+1}/\sigma_j; \ \mathbf{p}^{j+1} = \mathbf{r}^{j+1} + \chi_{j+1}\mathbf{p}^j$;
End

The above algorithm constructs implicitly a Lanczos tridiagonalisation which we write in the form

$$V_k^T A V_k = T_k,$$

where V_k is an orthogonal matrix and T_k is a symmetric and positive tridiagonal matrix with entries computable from the CG coefficients. The explicit form of $T_k \in \mathbb{R}^{k \times k}$ is given below

$$T_k = \begin{pmatrix} \alpha_1 & \beta_1 & & & 0 \\ \beta_1 & \alpha_2 & \beta_2 & & \\ & \ddots & \ddots & \ddots & \\ & & & \alpha_{k-1} & \beta_{k-1} \\ 0 & & & \beta_{k-1} & \alpha_k \end{pmatrix}.$$

where for $j = 1, \ldots, k$,

$$\alpha_j = \frac{1}{\gamma_{j-1}} + \frac{\chi_{j-1}}{\gamma_{j-2}}, \quad \beta_j = \frac{\sqrt{\chi_j}}{|\gamma_{j-1}|},$$

with $\gamma_{-1} = 1, \chi_0 = 0$.

Several authors have proposed rules that compute error bounds for the Conjugate Gradient method [5, 6, 11, 12, 20, 21, 27–29, 33]. Some of these rules compute estimate of the error in Euclidean norm and others compute estimates related to the energy norm. We review below some of the existing results; we also introduce new estimates suitable to the adaptive finite element context.

4.1.1 The Hestenes and Stiefel estimate

In their historical paper, Hestenes and Stiefel [22] proposed a method for the estimate of the error energy norm that use the values computed during the Conjugate Gradient method. Strakovs and Tichý, [33], studied the relations between the estimates proposed in [22, 20, 21, 27–29] and they proved that the Hestenes-Stiefel estimate [22] is numerically stable. The method uses the fact that the error can be written as a linear combinatinon of residual norms:

$$\|\mathbf{u} - \mathbf{u}^k\|_A^2 = \sum_{k+1}^{N} \gamma_j \|\mathbf{r}^j\|^2.$$

Under the assumption that $e_A^{(k+d)} << e_A^{(k)}$, where the integer d denotes a suitable delay, the Hestenes and Stiefel estimate is given by the formula

$$\|\mathbf{u} - \mathbf{u}^k\|_A^2 \approx \sum_{j=k+1}^{k+d} \gamma_j \|\mathbf{r}^j\|^2. \tag{4.33}$$

In [20] $d = 10$ is indicated as a successful compromise, and numerical experiments support this conclusion ([20, 2]). In Section 5, we will indicate that the cheaper choice $d = 5$ can be reliable if the solution u of (2.1) is reasonably regular; in general, one can expect d to be required to be larger for ill-conditioned problems.

4.1.2 The Golub and Meurant bounds

The A-norm of the error at each CG step can be written in the following way, using the orthogonality $\mathbf{r}_k^T \mathbf{u}^k = 0$,

$$\|\mathbf{u} - \mathbf{u}^k\|_A^2 = \|\mathbf{r}_k\|_{A^{-1}}^2 = \mathbf{b}^T A^{-1} \mathbf{b} - \mathbf{b}^T \mathbf{u}^k. \tag{4.34}$$

Thus, the main difficulty in evaluating the above quantity is in the evaluation of the first term on the right. One can write this term as an integral

$$F(A) = \mathbf{b}^T A^{-1} \mathbf{b} = \int_0^\infty \frac{1}{t} d\omega(t),$$

where the measure ω is a non-decreasing step function with jump discontinuities at the eigenvalues of A. Golub and Meurant used this formulation to provide upper and lower bounds on the CG error, by employing Gauss-Radau and Gauss-Lobatto quadrature rules, respectively. The latter approach can be shown to be equivalent to the Hestnes and Stiefel estimate above.

The only guaranteed upper bound for the A-norm of the CG error uses a Gauss-Radau associated with the measure ω and with one node prescribed at $\lambda < \lambda_{\min}(A)$. Let

$$\hat{T}_{k+1} = \begin{pmatrix} \alpha_1 & \beta_1 & & & 0 \\ \beta_1 & \alpha_2 & \beta_2 & & \\ & \ddots & \ddots & \ddots & \\ & & & \alpha_k & \beta_k \\ 0 & & & \beta_k & \hat{\alpha}_{k+1} \end{pmatrix}.$$

where

$$\hat{\alpha}_{k+1} = \lambda + \beta_k^2 \mathbf{e}_k^T (T_k - \lambda I_k)^{-1} \mathbf{e}_k$$

with $\mathbf{e}_k$ the k-th column of the $k \times k$ identity matrix. Under the assumption $0 < \lambda < \lambda_{\min}(A)$, the Cholesky decomposition $\hat{T}_{k+1} = \hat{R}_{k+1}^T \hat{R}_{k+1}$ can be shown to exist. Let now $\hat{\mathbf{y}}^{k+1}$ be the solution of

$$\hat{R}_{k+1}^T \hat{\mathbf{y}}^{k+1} = \|\mathbf{b}\| \hat{\mathbf{e}}_1$$

where $\hat{\mathbf{e}}_1$ denotes the first column of the identity matrix of size $k+1$. Then an upper bound on the CG error is given by [20]

$$\|\mathbf{u} - \mathbf{u}_k\|_A \leq \left|\hat{\mathbf{y}}_{k+1}^{k+1}\right|. \tag{4.35}$$

It is clear that in order to compute this bound, the lower bound λ is required. In fact, experiments show [29] that a close lower bound on the smallest eigenvalue of A yields tight upper bounds for the CG error. We consider this issue at the end of this section.

4.1.3 Other estimates

Another estimate for the CG error was derived in [11]. The derivation employs an anti-Gauss rule to evaluate the integral $F(A)$. We include a brief description here. Let

$$\tilde{T}_k = \begin{pmatrix} \alpha_1 & \beta_1 & & & & 0 \\ \beta_1 & \alpha_2 & \beta_2 & & & \\ & \ddots & \ddots & \ddots & & \\ & & & \alpha_{k-2} & \beta_{k-2} & \\ & & & \beta_{k-2} & \alpha_{k-1} & \sqrt{2}\beta_{k-1} \\ 0 & & & & \sqrt{2}\beta_{k-1} & \alpha_k \end{pmatrix}.$$

Assuming $\tilde{T}_k$ is positive definite, we consider as before the Cholesky factorisation $\tilde{T}_k = \tilde{R}_k^T \tilde{R}_k$ and similarly we denote by $\mathbf{y}^k$, $\tilde{\mathbf{y}}^k$ the solution of the linear systems

$$R^k \mathbf{y}^k = \|\mathbf{b}\|\mathbf{e}_1, \quad \tilde{R}_k^T \tilde{\mathbf{y}}^k = \|\mathbf{b}\|\mathbf{e}_1,$$

where $T_k = R_k^T R_k$. One can use these vectors to derive an upper bound on the error, under the additional assumption of rapid decay of the coefficients of $f(t) = 1/t$ in the expansion in terms of orthonormal polynomials with respect to the inner product $(g, h) := \int_0^\infty gh\mathrm{d}\omega$ (see [11] for details). The bound reads

$$\|\mathbf{u} - \mathbf{u}_k\|_A \leq \left[(\tilde{\mathbf{y}}_k^k + \mathbf{y}_k^k)(\tilde{\mathbf{y}}_k^k - \mathbf{y}_k^k)\right]^{1/2}. \tag{4.36}$$

We remark here that the above expression may not exist, if $\tilde{T}_k$ is not positive definite, or if it exists, it is not guaranteed to be an upper bound.

4.2 Adaptive stopping criteria for CG

Criterion (3.21) cannot be employed in a practical context. Instead, the following generic criterion will be considered:

$$E(\tilde{U}_m)^2 + \mu E(\tilde{U}_{m+1})^2 \leq \nu \eta_m^2(\tilde{U}_m) \tag{4.37}$$

where $E(\tilde{U}_m)$ denotes an estimate or bound for the error $\|U_m - \tilde{U}_m\|_a$. Note that if $E(\tilde{U}_m)$ is an upper bound, then the result of Thm 3 holds and the inexact AFEM algorithm is guaranteed to converge. In general, estimates will not provide this guarantee, though a tight estimate or lower bound could also ensure the contraction result of Thm 3, possibly at a different rate. For such cases, further analysis is required.

We discuss now the only *guaranteed* upper bound available – the Golub and Meurant bound (4.35) for the case where a in model problem (2.1) has minimum eigenvalue $a_{\min}$. As described above, this bound requires a lower bound on the smallest eigenvalue of the system matrix. This information is not readily available and in general expensive to compute. We introduce below two bounds and an estimate for this quantity.

4.2.1 Eigenvalue bounds based on Poincaré inequalities.

To find a lower bound on the smallest eigenvalue of a discrete Laplacian, one could employ the Poincaré inequality

$$\|v\|_{L^2(\Omega)} \leq C_P |v|_{H^1(\Omega)}, \quad \forall v \in \mathcal{V}_m \subset H^1_\Gamma(\Omega),$$

where $\Gamma \subseteq \partial\Omega$ is Lipschitz continuous. The constant $C_P = C_P(\Omega)$ depends on the domain Ω only and can be estimated for polygonal domains. Since

$$\|v\|_a^2 \geq a_{\min} |v|_{H^1(\Omega)}^2$$

the Poincaré inequality yields the following lower bound on Rayleigh quotients involving A_m:

$$\frac{a_{\min}}{C_P^2} \frac{\mathbf{v}^T M_m \mathbf{v}}{\mathbf{v}^T \mathbf{v}} \leq \frac{\mathbf{v}^T A_m \mathbf{v}}{\mathbf{v}^T \mathbf{v}}, \quad \forall \mathbf{v} \in \mathbb{R}^{N_m},$$

where M_m is the mass matrix corresponding to the m level of the AFEM algorithm. We conclude that

$$\frac{a_{\min}}{C_P^2} \lambda_{\min}(M_m) \leq \lambda_{\min}(A_m)$$

and the task is now to find a lower bound on the smallest eigenvalue of M_m. This is easily done in a finite element context as shown in [18], [19], [36] and we include the result here

$$\min_{\kappa \in \mathcal{T}_m} \lambda_{\min}(M_\kappa) \leq \lambda_{\min}(M_m), \tag{4.38}$$

where M_κ is the elemental mass matrix assembled on the element κ. The resulting bound on the smallest eigenvalue of A_m is

$$\lambda_{\min}(A_m) \geq \frac{a_{\min}}{C_P^2} \min_{\kappa \in \mathcal{T}_m} \lambda_{\min}(M_\kappa); \tag{4.39}$$

we call this bound *the global Poincaré bound*. Bound (4.38) is remarkably tight for matrices assembled on quasi-uniform or isotropic subdivisions. However, it can be fairly poor in the case of adaptive refinements. Improvements are described in [26]. We consider below an adaptive procedure which refines the Poincaré bound (see [26] for other possibilities).

Let $\tilde{\mathcal{M}}_m$ be the set of triangles marked for bisection at step m and let $\mathcal{D}_m \supseteq \tilde{\mathcal{M}}_m$ denote the region comprising all the refined elements. Let us further assume that following condition holds:

$$\partial\Omega \neq \mathcal{D}_m \cap \partial\Omega \neq \emptyset. \tag{4.40}$$

If $\mathcal{D}_m$ is a multiply-connected set, then we will assume that the above condition holds for each disjoint region it is comprised of. The following result can be found in [26].

Proposition 1 *Let $\mathcal{D}_m$ satisfy (4.40) and let C_P^m denote its Poincaré constant. Let A_m, A_{m+1} denote the matrices assembled on consecutive subdivisions in the inexact AFEM algorithm 2. Then*

$$\lambda_{\min}(A_{m+1}) \geq \min \left\{ \lambda_{\min}(A_m), \frac{a_{\min}}{(C_P^m)^2} \min_{\kappa \in \mathcal{D}_m} \lambda_{\min}(M_\kappa) \right\}. \tag{4.41}$$

The bound (4.41) is essentially an *adaptive Poincaré* lower bound for the smallest eigenvalue of the stiffness matrix. Given its form, the above bound is updated at each step, by taking the minimum of the current lower bound and the bound expressing the Poincare inequality on the region $\mathcal{D}_m$.

Remark 1 The advantage of this approach is that the lower bound on $\lambda_{\min}(\tilde{A}_{m+1})$ is tight since it employs a smaller Poincaré constant, corresponding to the smaller region $\mathcal{D}_m$ which in addition has an isotropic subdivision.

Remark 2 If the region $\mathcal{D}_m$ is multiply-connected, with each subregion satisfying condition (4.40), the same approach applies, with the second term in (4.41) replaced by the set of bounds corresponding to each subregion.

4.2.2 Estimates using the Lanczos algorithm

It is known that the underlying Lanczos tridiagonalization constructed by the CG algorithm provides also estimates to the eigenvalues of the system matrix. In particular, there holds $\lambda_{\min}(A) \leq \lambda_{\min}(T_k)$ with the bound getting tighter with growing k. This suggests the following estimate at step $m+1$

$$\lambda := \lambda^{m+1} := c\lambda_{\min}(T_{k+1}^m),\tag{4.42}$$

with $c < 1$ a constant that will account for the reduction in $\lambda_{\min}(A)$ with refinement but also for the poor approximation of $\lambda_{\min}(A_m)$ by $\lambda_{\min}(T_{k+1}^m)$. If corresponding to this choice of λ the Cholesky factorisation of $\hat{T}_{k+1}$ does not exist, we reduce c until the factorisation and hence (4.35) exist. Note that this is always possible.

We end this section with a remark regarding preconditioning. The implementation of the stopping criteria presented above can be extended to the case where a preconditioning routine is employed to accelerate convergence of the Conjugate Gradient method. Most criteria can be adapted directly to preconditioning contexts; however, the Golub and Meurant bounds require estimates of the smallest eigenvalue of the *preconditioned system*, which may not be straightfoward to obtain. Some results regarding this situation can be found in [4]. We note here that the estimate (4.42) using the Lanczos algorithm generalises naturally in this case.

5 Numerical experiments

We investigate now the usefulness of criterion (3.21) and corresponding approximations of the form (4.37) as they are applied to the CG method to solve some standard adaptive finite element problems. We are interested in the following comparisons:

- 'inexact' vs. 'exact' meshes;
- comparison of energy errors for exact and inexact cases;
- comparison of order of convergence for exact and inexact cases;
- comparison with Euclidean stopping criteria;
- comparions between bounds and estimators of the CG error.

We used both the exact and inexact versions of the AFEM algorithm with 10 iterations and various starting meshes. We used for illustration purposes the ideal criterion (3.21); we also employed a number of practical approximations to the ideal criterion of the form (4.37) corresponding to various bounds or estimators for the CG error. We summarize these below.

1. DNR: the ideal bound (3.21) using the exact dual norm of the residual;
2. GM0: the ideal Golub-Meurant upper bound (4.35) using the exact $\lambda_{\min}(A_m)$;
3. GM1: the Golub-Meurant upper bound (4.35) with adaptive Poincaré bound (4.39) for $\lambda_{\min}(A_m)$;
4. GM2: the Golub-Meurant upper bound (4.35) with global Poincaré bound (4.41) for $\lambda_{\min}(A_m)$;
5. GM3: the Golub-Meurant criterion (4.35) with the estimator (4.42) for $\lambda_{\min}(A_m)$ with $c = 1/2$;
6. HS: the Hestenes-Stiefel estimator (4.33) with a delay of $d = 5$ steps.
7. AG: the anti-Gauss estimator (4.36);
8. ER($|\log tol|$): the standard Euclidean residual with various stopping tolerances *tol*.

We note that only the first two criteria are of theoretical interest and are not available generally in a practical context. The following two criteria are the only guaranteed upper bounds for the CG error and thus the only bounds for which the convergence result of Thm 3 applies. Criteria 5-7 are estimators, while the criterion based on the Euclidean residual is also an empirical estimator. In order to compare in a fair and explicit fashion the performance of the CG algorithm equipped with the above stopping criteria we chose to transform the computational cost on each level into units corresponding to matrix-vector products on the last level. Since each CG iteration has a cost proportional to the number of nonzeros in the system matrix, the formula employed (and the tabulated variable) is

$$mv := \text{matvecs}(m) = \sum_{k=1}^m \frac{\text{nnz}(A_k)}{\text{nnz}(A_m)} \cdot \text{its}(k),\tag{5.43}$$

where $\text{nnz}(A_k)$ denotes the number of nonzero elements of A_k and $\text{its}(k)$ represents the number of CG iterations on level k.

Finally, another important ingredient in our iterative solver is the starting guess. In all cases this choice was provided by the solution obtained at the previous refinement level interpolated onto the current (refined) level.

(a) Solution of test problem 1   (b) Adaptive meshes: $m = 4, 6, 8; N_0 = 83$.
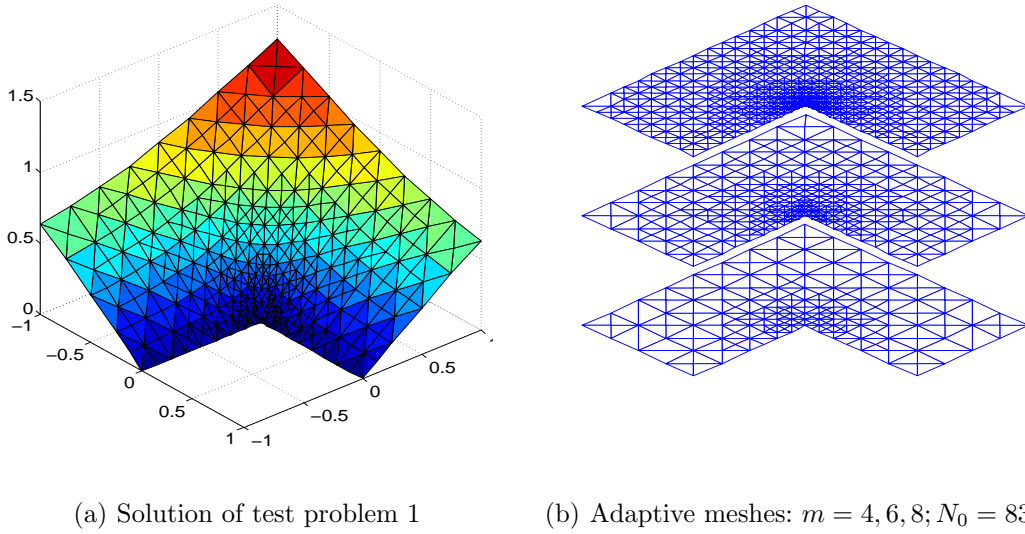
Fig. 5.1 Solution and adaptive meshes for test problem 1

### 5.1 Experiments in two dimensions

5.1.1 Test Problem 1

We solved problem (2.1) on an L-shaped domain with the right-hand side chosen so that the analytic solution is $u = r^{2/3} \sin \frac{2\theta}{3}$. The AFEM algorithms were implemented with a Dörfler marking parameter $\theta = 0.75$. The solution and typical meshes are shown in Figure 5.1. The results are displayed in Table 5.1. For each N_0 we give the final size of the problem N_m (where in fact $m = 10$), the energy norm of the exact error corresponding to various approximations $\tilde{U}_m$ and the number of matvecs (5.43) over the 10 AFEM iterations.

We notice first that the difference between exact and inexact meshes is negligible while the error level is similar in all cases. We also see that the ideal stopping criterion DNR achieves roughly the same level of accuracy with a very low number of matvecs. The ideal bound GM0 appears to exhibit the same behaviour with a few more matvecs and consequently a more accurate approximation. The practical bounds generally require more matvecs for the same level of accuracy. The guaranteed upper bounds GM1 and GM2 are the most accurate (excepting the ER

method	$N_0 = 29$			$N_0 = 83$			$N_0 = 262$		
	N_m	$\|u - \tilde{U}_m\|_a$	mv	N_m	$\|u - \tilde{U}_m\|_a$	mv	N_m	$\|u - \tilde{U}_m\|_a$	mv
exact	1,140	0.025028	–	1,764	0.020155	–	4,208	0.013033	–
DNR	1,125	0.025144	44	1,765	0.020158	55	4,191	0.013053	76
GM0	1,137	0.025039	52	1,810	0.020057	61	4,179	0.013066	83
GM1	1,134	0.025010	91	1,776	0.020124	115	4,206	0.013033	163
GM2	1,140	0.025028	116	1,776	0.020124	147	4,208	0.013033	232
GM3	1,132	0.025054	41	1,775	0.020157	59	4,179	0.013061	66
HS	1,134	0.025061	36	1,792	0.020169	34	4,194	0.013207	39
AG	1,132	0.025036	49	1,807	0.020051	57	4,190	0.013066	75
ER(6)	1,140	0.025028	159	1,764	0.020155	207	4,208	0.013033	320
ER(8)	1,140	0.025028	198	1,764	0.020155	282	4,208	0.013033	436
ER(10)	1,140	0.025028	226	1,764	0.020155	338	4,208	0.013033	518

Table 5.1 Performance of stopping criteria: matvecs (mv) for Test Problem 1 ($m = 10$).

(a) Errors $\|u - U_m^k\|_a$ for $m = 10$.   (b) Convergence rates for inexact AFEM for $m = 20$.
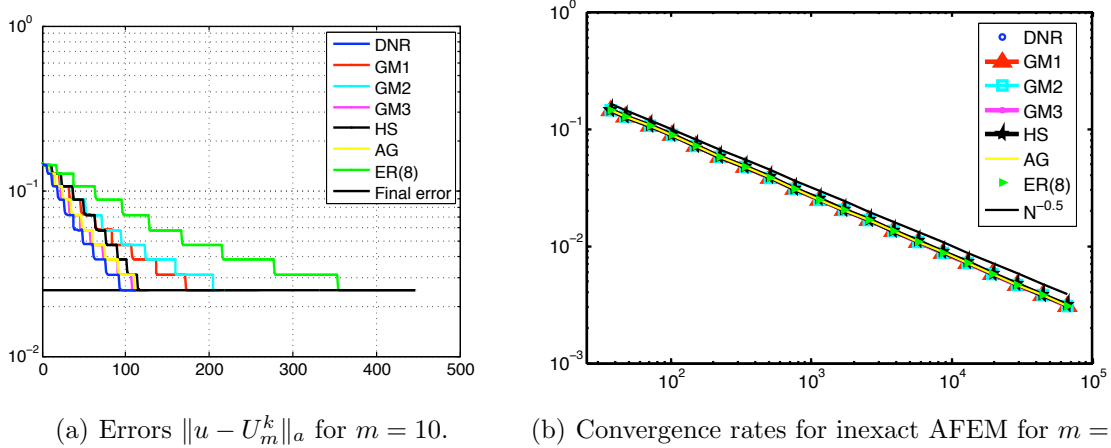
**Fig. 5.2** Convergence of inexact AFEM: CG with various stopping criteria.

bounds) and also require the largest number of matvecs among the computable bounds. The adaptive bound GM1 is indeed an improvement over GM2 at essentially no cost. Among the estimators, GM3 and the anti-Gauss (AG) criterion provided competitive choices. The Hestenes-Stiefel estimator appears to loose accuracy which indicates that the delay needs to be increased with the mesh size.

The results of Table 5.1 are also displayed in Figure 5.2. The first plot shows the energy norm of the error corresponding to CG iterates produced by the above stopping criteria throughout the adaptive process. We see that wasteful criteria such as ER exhibit long plateaux, spending a lot of iterations to achieve an insignificant reduction in the energy norm of the error. At the same time, in Figure 5.2(b) we see that the convergence rates of all the inexact algorithms remain virtually unchanged from the exact case, which for this test problem is $O(N_m^{-1/2})$.

method	$N_0 = 29$			$N_0 = 83$			$N_0 = 262$		
	N_m	$\|u - \tilde{U}_m\|_a$	mv	N_m	$\|u - \tilde{U}_m\|_a$	mv	N_m	$\|u - \tilde{U}_m\|_a$	mv
exact	66,115	0.0030648	–	101,292	0.0024657	–	235,574	0.0016016	–
DNR	65,910	0.0030697	176	101,334	0.0024670	206	234,934	0.0016040	287
GM1	66,080	0.0030656	280	101,432	0.0024650	338	235,531	0.0016018	394
GM2	66,081	0.0030655	890	101,444	0.0024646	1,099	235,574	0.0016016	1,319
GM3	66,028	0.0030708	118	101,821	0.0024689	112	235,622	0.0016053	177
HS	66,433	0.0036571	42	102,381	0.0032227	35	237,132	0.0026592	90
AG	66,042	0.0030796	118	101,403	0.0024764	143	234,893	0.0016137	201
ER(6)	66,114	0.0030648	1,041	101,292	0.0024657	1,262	235,574	0.0016016	1,840
ER(8)	66,115	0.0030648	1,390	101,292	0.0024657	1,709	235,574	0.0016016	2,563
ER(10)	66,115	0.0030648	1,796	101,292	0.0024657	2,077	235,574	0.0016016	3,124

Table 5.2 Performance of stopping criteria: matvecs (mv) for Test Problem 1 ($m = 20$).

The same problem was solved with 20 AFEM iterations, starting from the same three initial meshes. The results are shown in Table 5.2. The performance of the ideal estimator GM0 is not included. In this case also, the resulting meshes exhibit relatively small differences, with the level of error achieved similar to the exact case. There is one exception: estimator HS is very poor for large problems; this is due to the fact that this estimator is a lower bound which is known to be tight after a suitable delay and in our case, this delay is insufficient ($d = 5$). The guaranteed bounds GM1, GM2 provide again the most accurate approximations; they may appear expensive compared to the performance of the other estimators, but are certainly an improvement over the standard Euclidean criteria. In particular, the performance of GM1 is improving relative to GM2 and to the standard Euclidean criteria. Finally,

(a) $\epsilon = 1$.

(b) $\epsilon = 1/2$.

(c) $\epsilon = 1$.

(d) $\epsilon = 1/2$.
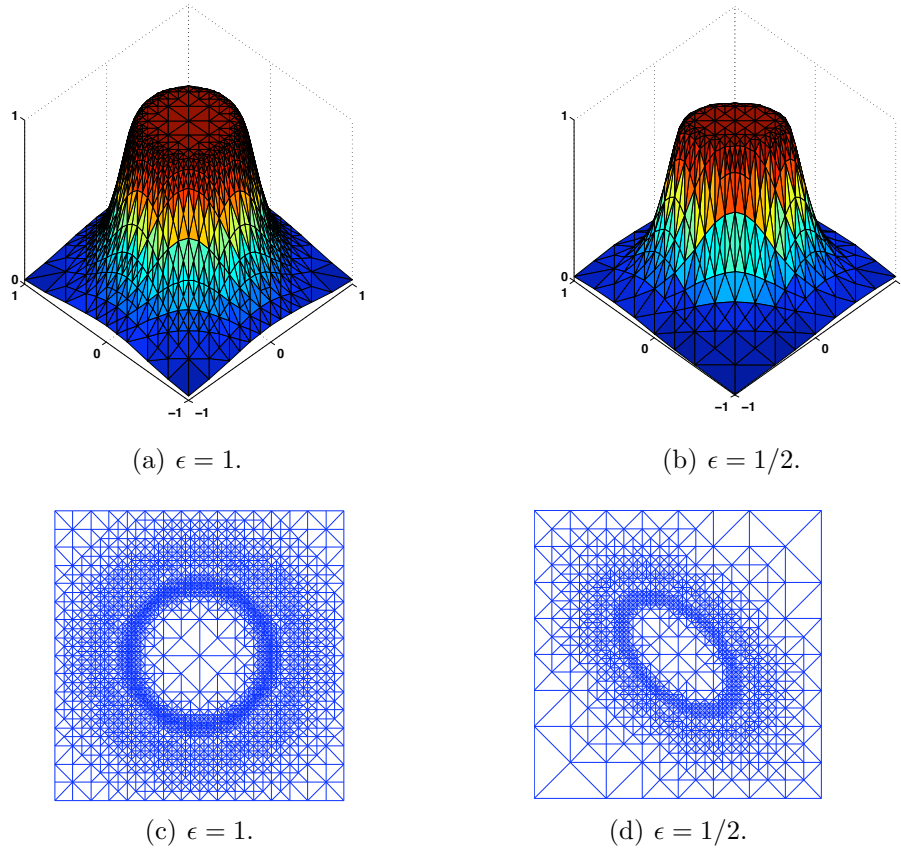
Fig. 5.3 Solutions and adaptive meshes ($m = 10$) for test problem 2

the estimators GM3 and AG provide convenient alternatives which are cheap and, for many practical applications, sufficiently accurate.

5.1.2 Test Problem 2

We solved again problem (2.1) posed on $\Omega = (-1, 1) \times (-1, 1)$ and with the choice of diffusion coefficient

$$a = \frac{1}{\epsilon} \begin{pmatrix} 1 & \epsilon - 1 \\ \epsilon - 1 & 1 \end{pmatrix},$$

with $0 < \epsilon \leq 1$ and eigenvalues $\{1, 2/\epsilon - 1\}$ so that $a_{\min} = 1$. The right-hand side was chosen so that the exact solution is

$$u(x, y) = \tanh\left(\frac{0.1}{r^4 + 10^{-4}}\right), \quad r^2 = \frac{1}{\epsilon}\left(x^2 - 2(\epsilon - 1)xy + y^2\right).$$

The solution corresponding to $\epsilon = 1$ and $\epsilon = 1/2$ and the corresponding meshes are displayed in Figure 5.3.

The results are displayed in Table 5.3 for $\epsilon \in \{1, 1/2, 1/5\}$. We first note that bound GM1 is not applicable here since (4.40) does not hold, as evident from the mesh plots in Figure 5.3. The performance of GM2, the only guaranteed bound, is an improvement over, but essentially of the order of the Euclidean criteria. The estimator HS with a delay $d = 5$ remains poor, while the estimator AG performs poorly for $\epsilon = 1/5$. Estimator GM3 remains consistent, with the best performance overall.

5.2 Experiments in three dimensions

5.2.1 Test Problem 3

For our final test, we solved problem (2.1) with $a = 1$ in $\Omega = (-1,\ 1)^3$ and the forcing function chosen so that the exact solution is $u = e^{-10r^2}$. We used the same Dörfler parameter $\theta = 0.75$ and started the adaptive algorithm

method	$\epsilon = 1$			$\epsilon = 1/2$			$\epsilon = 1/5$		
	N_m	$\|u - \tilde{U}_m\|_a$	mv	N_m	$\|u - \tilde{U}_m\|_a$	mv	N_m	$\|u - \tilde{U}_m\|_a$	mv
exact	328,355	0.00929582	–	264,033	0.02334634	–	274,668	0.05653274	–
DNR	325,613	0.00933259	327	263,115	0.02341392	429	274,712	0.05653893	640
GM1	–	–	–	–	–	–	–	–	–
GM2	328,334	0.00929578	824	264,075	0.02334562	1,141	274,657	0.05653038	1,160
GM3	329,618	0.01087563	84	263,421	0.02349577	280	274,581	0.05657660	712
HS	324,894	0.01665924	13	263,759	0.04047418	53	274,306	0.09741096	49
AG	325,327	0.00967695	18	263,801	0.02342935	119	237,654	1.5098438	15
ER(6)	328,077	0.00929576	1,100	264,033	0.02334676	1,218	239,128	0.06003255	1,190
ER(8)	328,355	0.00929582	1,172	264,033	0.02334676	1,264	274,666	0.05652907	1,201
ER(10)	328,355	0.00929582	1,209	264,033	0.02334676	1,276	274,667	0.05652919	1,205

Table 5.3 Performance of stopping criteria: matvecs (mv) for Test Problem 2 ($m = 20$, $N_0 = 17$).
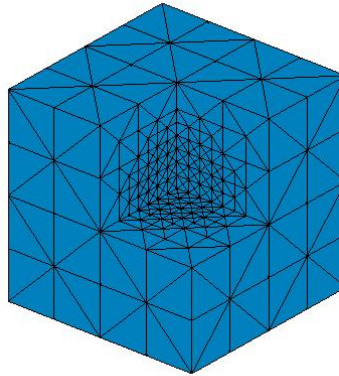
Fig. 5.4 Adaptive meshing for test problem 3

from a range of initial regular meshes of tetrahedra and ran the procedure for $m = 10$ iterations. The refinement is concentrated near the origin, as illustrated in Figure 5.4, where the solution exhibits a sharp exponential decay.

The results are displayed in Table 5.4 and are similar to those obtained for the previous 2D experiment. As was the case for Test Problem 2, the bound GM1 does not apply. Likewise, the criterion AG turns out not to be defined for any level and this is indicated by a "∗". The bound GM2 is cheaper than, but of the same order as, the ER criteria, while criteria GM3 and HS appear to yield consistently good quality approximations. In particular, criterion GM3 appears to yield marginally better solutions, though at a higher cost than criterion HS.

6 Concluding remarks

This work was motivated by the need to solve intractable large-scale problems arising in the context of adaptivity. For such problems, iterative methods are mandatory; however, classical stopping criteria, such as the Euclidean norm of the residual are known to be wasteful in practice. Our focus was on establishing how accurate approximations need to be in an adaptive context. Thus, we considered inexact versions of adaptive finite element algorithms which are known to be convergent. We showed that convergence holds provided the approximations replacing the exact solution satisfy a certain bound involving the energy norm of only algebraic errors on both current and previous level. This result yields stopping criteria for iterative methods such as the Conjugate Gradient method. Consequently, we looked at versions of the CG method endowed with various stopping criteria arising from the analysis. We found that Golub-Meurant type bounds are guaranteed to yield useful and at times efficient results. We introduced and employed adaptive Poincaré inequalities which allow for better estimation of the energy norm

method	$N_0 = 142$			$N_0 = 779$			$N_0 = 5,191$		
	N_m	$\|u - \tilde{U}_m\|_a$	mv	N_m	$\|u - \tilde{U}_m\|_a$	mv	N_m	$\|u - \tilde{U}_m\|_a$	mv
exact	19,579	0.08967018	–	131,250	0.047497978	–	†	†	–
DNR	19,543	0.09042983	121	131,984	0.047802189	291	†	†	†
GM1	–	–	–	–	–	–	–	–	–
GM2	19,574	0.08966619	169	131,276	0.047490362	407	951,011	0.024697507	988
GM3	19,597	0.08969471	140	131,331	0.047486755	344	950,833	0.024700263	873
HS	19,528	0.09013577	126	132,443	0.049701492	275	956,033	0.027038043	670
AG	*	*	*	*	*	*	*	*	*
ER(6)	19,587	0.08967714	237	131,246	0.047497979	465	951,239	0.024692998	958
ER(8)	19,579	0.08967018	332	131,250	0.047497978	650	950,954	0.024697519	1,505
ER(10)	19,579	0.08967018	412	131,250	0.047497978	840	950,932	0.02469783	2,001

Table 5.4 Performance of stopping criteria: matvecs (mv) for Test Problem 3 ($m = 10$) for various N_0. Legend: †: out of memory; −: does not apply; ∗: does not exist.

of the algebraic errors. Other estimates, such as the Hestenes and Stiefel criterion or Lanczos approximations can also be competitive.

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