NSDE 1: LECTURE 1

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Examples in presentation:

- Moon landing and code
- Finding Dory
- Driverless cars

In the examples we've seen, and most other 'real world' systems, the differential equations cannot be solved analytically, and instead they must be solved approximately using numerical algorithms.

Questions that this course will answer:

- **How** do I solve an initial value problem (IVP) for an ordinary differential equation (ODE), or an IVP for a parabolic partial differential equation?
- **How** do I analyse the accuracy of the solution I get?
- **How** can I know if the algorithm I'm using is stable?

1. Initial value problems for ODEs. Here is the basic form that we will use throughout the course:

$$\begin{aligned} \mathbf{u}' &= \mathbf{f}(t, \mathbf{u}), \qquad \text{for } t \in [t_0, T_M] \\ \mathbf{u}(t_0) &= \mathbf{u}_0 \end{aligned}$$

where $\mathbf{u}, \mathbf{u}_0, \mathbf{f}(\cdot) \in \mathbb{R}^k$ for some k.

First we need to know: is this a sensible question to ask?

Example 1.1.

$$\begin{aligned} u' &= 3u^{2/3} \\ u(0) &= 0. \end{aligned}$$

What are the solutions to this problem?

A-level maths tells us that

$$\begin{aligned} du/dt &= 3u^{2/3} \\ \int \frac{1}{3} u^{-2/3}\, du &= \int dt \\ u^{1/3} & \qquad\qquad = t + K \\ u &= (t+K)^3 \end{aligned}$$

and, since $u(0) = 0$, we have that $u = t^3$.

However, *there is another solution – $u = 0$!*

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In fact, it's worse than that – as

$$u(t) = \begin{cases} 0 & 0 \le t \le c \\ (t-c)^3 & c < t < \infty \end{cases}$$

is a solution for all c *–* **an infinite number of solutions!**.

This spells trouble for an algorithm designed to find an approximation to the solution – which one would it pick?!

One way to guarantee that we don't have this problem is to ensure that the problem is smooth enough. Let us, for simplicity, consider the 1D problem,

$$u' = f(t,u), \qquad u(t_0) = u_0.$$

This is what we mean by 'smooth enough'. Let (t_0, u_0) be the point of the initial data. We need the problem to satisfy two conditions:

- We can draw 'butterfly wings' from (t_0, u_0) the function f stays within this region.
- We can draw a box around the solution in some region, and the solution leaves the box from the edge (not the top).

If these conditions hold, then we can be sure that a solution exists, and that it is unique.

We state these conditions more rigorously:

THEOREM 1.2. **Picard's theorem**

Let $R = \{z \in \mathbb{R}^2 : |t - t_0| \le h, |u - u_0| \le k\}$

- *Suppose that* $f(\cdot,\cdot)$ *is a continuous function in* $U \supset R$ *with*

$$M = \max_{(t,u)\in R} |f(t,u)|$$

- *Suppose that* $\exists L > 0$ *s.t.* $\forall (t,u_1),(t,u_2) \in R$,

$$|f(t,u_1) - f(t,u_2)| \le L|u_1 - u_2|$$

  Lipschitz condition

- *Suppose that* $Mh \le k$

Then there exists a unique continuously differentiable function $t \to u(t)$ *satisfying the IVP:*

$$u' = f(t,u), \qquad u(t_0) = u_0$$

for all $t \in [t_0 - h, t_0 + h]$.