NSDE 1: LECTURE 9

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$$u' = f(t, u), \qquad u(t_0) = u_0.$$

The general form of a linear multistep method is

$$\sum_{j=0}^{k} \alpha_j U_{n+j} = h \sum_{j=0}^{k} \beta_j f(t_{n+j}, U_{n+j})$$

The linear multistep method is said to be *convergent* if, for all initial value problems $u' = f(t, u)$, $u(t_0) = u_0$ (which satisfies the assumptions of Picard's theorem),

$$\lim_{h\to 0,\ nh=t-t_0} U_n = u(t)$$

for all $t \in [t_0, T_M]$ and for all solutions $\{U_n\}_{n=0}^{N}$ with consistent starting condition, i.e., with starting values

$$U_s = \eta_s(h)$$

for which $\lim_{h\to 0} \eta_s(h) = U_0$, $s = 0, 1, \ldots, k-1$.

Convergence is a vital property of a numerical method. It tells us that, if we take smaller and smaller time steps, we will get better and better approximations to the true solution.

Theorem Zero stability is a necessary condition for convergence.

Proof Since the scheme converges for all f, choose $f = 0$. In particular, let $u(0) = 0$ with solution $u = 0$.

Then

$$\sum_{j=0}^{k} \alpha_j U_{n+j} = 0 \tag{0.1}$$

As the method is convergent, $U_n \to 0$ as $h \to 0$, $nh \to t$ for consistent starting values $U_0, \ldots, U_{k-1}$.

Now, let $z = re^{i\theta}$ be a root of $\rho(z) = 0$.

Choose the starting data $U_m = hr^m cos(m\theta)$, which is consistent with the initial condition. Note that U_m satisfies (0.1), since by writing

$$U_m = Re(hr^m e^{im\theta}) = hRe(z^m),$$

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then

$$\sum_{j=0}^{k} \alpha_j U_{n+j} = h \sum_{j=0}^{k} Re(z^{m+k}) = h \sum_{j=0} Re(z^n \rho(z)) = 0,$$

since z was a root of $\rho(z) = 0$.

Therefore, the values $U_n = hr^n \cos(n\theta)$ are solutions of the discrete method starting from a consistent set of initial values.

- Suppose $\theta \neq 0, \pi$. then

  $$U_n^2 - U_{n+1}U_{n-1} = h^2 r^{2n} \left[\cos^2(n\theta) - \cos((n+1)\theta)\cos((n-1)\theta)\right]$$

  so

  $$U_n^2 - U_{n+1}U_{n-1} = h^2 r^{2n} \sin^2 \theta$$

  The LHS tends to zero as $n \to \infty$, $nh \to t$, as all values converge to zero. The right hand side must therefore also tend to zero. However, this is only possibly if $r \leq 1$.
- if $\theta = 0$ or π, then since

  $$U_n = hr^n \qquad (\theta = 0)$$

  or

  $$U_n = hr^n(-1)^n \qquad (\theta = \pi)$$

  we again have that since $U_n \to 0$, $r \leq 1$.

Now, if z is a double root of $\rho(z)$, $U_n = hnr^n \cos(n\theta)$ also satisfies (0.1), and so $r < 1$. Therefore all roots on the unit circle must be simple.

Hence, if the scheme is convergent, the roots of the first characteristic polynomial satisfy root condition, and hence the scheme is zero-stable.

Theorem

Consistency is a necessary condition for convergence.

Proof

Recall that a linear multistep method is consistent if and only if

$$\sum_{j=0}^{k} \alpha_j = 0, and \sum_{j=0}^{k} j\alpha_j = \sum_{j=0}^{k} \beta_j.$$

Assume that a linear multi-step method is convergent for all functions f.

- Consider the function $f = 0$ with initial value $u(0) = 1$, so the solution is $u = 1$. Now $U_0 = U_1 = \cdots U_{k-1}$ is a consistent set of initial data, and convergence gives $U_n \to 1$ as $nh \to t$, $h \to 0$. The method is

$$\sum_{j=0}^{k} U_{n+j} = 0,$$

and since, in the limit, $U_{n+s} \to 1$ for $s = 1, \dots k$, we have $\sum_{j=0}^{k} \alpha_k = 0$, as required.

- Next consider $f = 1$ with initial value $u(0) = 0$, so that $u' = 1$. Here the solution is $u(t) = t$, and so $U_n \to t$ as $nh \to t$, $h \to 0$. The method is

$$\sum_{j=0}^{k} \alpha_j U_{n+j} = h \sum_{j=0}^{k} \beta_j.$$

Convergence tells us that $U_{n+r} = (n + r)h$, so

$$\sum_{j=0}^{k} \alpha_j (n + j)h = h \sum_{j=0}^{k} \beta_j$$

i.e.

$$\sum_{j=0}^{k} \alpha_j n + \sum_{j=1}^{k} j\alpha_j = h \sum_{j=0}^{k} \beta_j$$

since the first term is zero by part (i), we must have

$$\sum_{j=1}^{k} j\alpha_j = h \sum_{j=0}^{k} \beta_j$$

Putting these two results together, convergence implies the scheme will be consistent.

Dahlquist Theorem

For a linear multi-step method that is consistent with the ODE $u' = f(t, u)$, where f obeys a Lipschitz condition, and starting with consistent initial data, zero-stability is necessary and sufficient for convergence.

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Dahlquist's theorem tells us that if a linear multistep method is not zero-stable, then it's global error cannot be made arbirarily small by taking h sufficiently small. In fact, if the root condition is violated, then no matter how good the initial data is, there exists a solution that will grow by an arbitrarily large number.