NSDE 1: LECTURE 8

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$$u' = f(t, u), \qquad u(t_0) = u_0.$$

The general form of a linear multistep method is

$$\sum_{j=0}^{k} \alpha_j U_{n+j} = h \sum_{j=0}^{k} \beta_j f(t_{n+j}, U_{n+j})$$

A method is zero-stable if there exists a constant K such that, for any two sequences $U_0, U_1, \ldots, U_{k-1}$, and $\hat{U}_0, \hat{U}_1, \ldots, \hat{U}_{k-1}$,

$$|U_n - \hat{U}_n| \leq K \max\{|U_0 - \hat{U}_0|, |U_1 - \hat{U}_1|, \ldots, |U_{k-1} - \hat{U}_{k-1}|\}$$

for $t_n \leq T_M$ and as $h \to 0$.

The first characteristic polynomial is given by

$$\rho(z) = \sum_{j=0}^{k} \alpha_j z^j$$

The second characteristic polynomial is given by

$$\sigma(z) = \sum_{j=0}^{k} \beta_j z^j$$

Theorem (Root condition)

A linear multistep method is zero stable for any ODE

$$u' = f(t, u)$$

where f obeys a Lipschitz condition if and only if all zeros of its first characteristic polynomial lie inside the closed unit disk, with any that lie on the unit circle being simple.

Lemma

Consider the kth order homogeneous linear recurrence relation

$$\alpha_k y_{n+k} + \cdots + \alpha_1 y_{n+1} + \alpha_0 y_n = 0, \qquad n = 0, 1, 2, \ldots$$

with $\alpha_k \neq 0$, $\alpha_0 \neq 0$, $\alpha_j \in \mathbb{R}$, $j = 1, \ldots, k$, and the corresponding characteristic polynomial:

$$\rho(z) = \alpha_k z^k + \cdots + \alpha_1 z + \alpha_0.$$

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Let z_r, $1 \leq r \leq l$, $l \leq k$, be distinct roots of the polynomial ρ, and let $m_r \geq 1$ denote the multiplicity of z_r, with $m_1 + \cdots + m_l = k$.

If a sequence (y_n) of complex numbers statisfies the recurrence relation above, then

$$y_n = \sum_{r=1}^{l} p_r(n) z_r^n \qquad \forall n \geq 0,$$

where $p_r(\cdot)$ is a polynomial in n of degree $m_r - 1$, $1 \leq r \leq l$. In particular, if all roots are simple, (i.e. $m_r = 1$, $1 \leq r \leq k$), then the p_r are constants.

(See, e.g, Suli and Mayers (Lemma 12.1) for a sketch of the proof)

Proof of Root condition (necessity)

We want to prove that zero-stability implies the root condition. Suppose that

$$\sum_{j=0}^{k} \alpha_j U_{n+j} = h \sum_{j=0}^{k} \beta_j f(t_{n+j}, U_{n+j})$$

is zero stable. Then applying the method to the ODE $u' = 0, u(0) = 0$ gives

$$\alpha_k U_{n+k} + \alpha_{k-1} U_{n+k-1} + \cdots + \alpha_1 U_{n+1} + \alpha_o U_n = 0$$

This is a difference equation, and from the lemma, it's general solution is of the form

$$U_n = \sum_s p_s(n) z_s^n,$$

where z_s is a zero of

$$\rho(z) = \sum_{j=0}^{k} \alpha_j z^j$$

of multiplicity $m_s \geq 1$ and p_s is a polynomial of degree $m_s - 1$.

If $|z_s| > 1$ for some s then there are starting values such that the solution grows like $|z_s|^n$.

If $|z_s| = 1$ and z_s has multiplicity $m_s > 1$, then there are starting values such that the solution grows like $n^{m_s - 1}$. In either case, there are solutions which grow unbounded as $n \to \infty$, $h \to 0$, nh fixed.

Consider starting data $U_0, U_1, \ldots, U_{k-1}$ that gives such an unbounded solution, and the starting data

$$\hat{U}_0, \hat{U}_1 = \cdots = \hat{U}_{k-1} = 0,$$

which gives $\hat{U}_n = 0$ for all $n \geq 0$.

Therefore, if a method violates the root condition, it cannot be zero stable.

The proof of the converse is technical, and outside the scope of this course.

Example

Simpson rule method:

$$
\begin{aligned}
U_{n+2} - U_n &= \frac{h}{3}(f_{n+2} + 4f_{n+1} + f_n) \\
\rho(z) &= z^2 - 1 \\
z &= \pm 1
\end{aligned}
$$

(where $f_n = f(t_n, U_n)$).

Simple roots on the unit circle, so the method is zero-stable.

Example

Adams-Bashforth method

$$U_{n+4} - U_{n+3} = \frac{h}{24}(-9f_n + 37f_{n+1} - 59f_{n+2} + 55f_{n+3})$$

$$
\begin{aligned}
\rho(z) &= z^4 - z^3 = z^3(z-1) \\
z_1 &= z_2 = z_3 = 0, z_4 = 1
\end{aligned}
$$

Example A 3-step 6-th order accurate method:

$$11U_{n+3} + 27U_{n+2} - 27U_{n+1} - 11U_n = 3h(f_{n+3} + 9f_{n+2} + 9f_{n+1} + f_n)$$

$$\rho(z) = 11z^3 + 27z^2 - 27z - 11$$

$$z_1 = 1, z_2 \approx -0.3189, z_3 \approx -3.1356$$

since $|z_3| > 1$, the method is not zero stable.

Convergence

The linear multistep method is said to be *convergent* if, for all initial value problems $u' = f(t, u),\ u(t_0) = u_0$ (which satisfies the assumptions of Picard's theorem),

$$\lim_{h \to 0,\ nh = t - t_0} U_n = u(t)$$

for all $t \in [t_0, T_M]$ and for all solutions $\{U_n\}_{n=0}^N$ with consistent starting condition, i.e., with starting values

$$U_s = \eta_s(h)$$

for which $\lim_{h \to 0} \eta_s(h) = U_0$, $s = 0, 1, \ldots, k-1$.

Convergence is a vital property of a numerical method. It tells us that, if we take smaller and smaller time steps, we will get better and better approximations to the true solution.