NSDE 1: LECTURE 16

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Let $\Omega = (a, b) \times (c, d)$. Consider the 2D heat equation:

$$\frac{\partial u}{\partial t} = \nabla^2 u \qquad (x, y) \in \Omega,\ t \in (0, T]$$

subject to the initial condition:

$$u(x, y, 0) = u_0(x, y), \qquad (x, y) \in \bar{\Omega}$$

and the Dirichlet boundary condition:

$$u|_{\partial\Omega} = B(x, y, t), \qquad t \in [0, T],\ (x, y) \in \partial\Omega,$$

where $\partial\Omega$ is the boundary of Ω.

Define a grid:

$$\Delta x = (b - a)/(N_x + 1),\ \Delta y = (d - c)/(N_y + 1),\ \Delta t = T/M,$$

and set

$$\begin{aligned} x_i &= a + i\Delta x,\ i = 0, \dots, N_x + 1 \\ y_j &= c + j\Delta y,\ j = 0, \dots, N_y + 1 \\ t_m &= m\Delta t,\ m = 0, \dots, M. \end{aligned}$$

Let us define

$$\begin{aligned} \delta_x^2 U_{i,j} &= U_{i+1,j} - 2U_{i,j} + U_{i-1,j} \\ \delta_y^2 U_{i,j} &= U_{i,j+1} - 2U_{i,j} + U_{i,j-1} \end{aligned}$$

Then the 2D $\theta-$scheme is given by

$$\frac{U_{i,j}^{m+1} - U_{i,j}^m}{\Delta t} = (1 - \theta)\left(\frac{\delta_x^2 U_{i,j}^m}{\Delta x^2} + \frac{\delta_y^2 U_{i,j}^m}{\Delta y^2}\right) + \theta\left(\frac{\delta_x^2 U_{i,j}^{m+1}}{\Delta x^2} + \frac{\delta_y^2 U_{i,j}^{m+1}}{\Delta y^2}\right)$$

$$\begin{aligned} U_{i,j}^0 &= u_0,\ i = 0, \dots, N_x + 1,\ j = 0, \dots, N_y + 1 \\ U_{i,j}^{m+1} &= B(x_i, y_j, t_{m+1}),\ (x_i, y_j) \in \partial\Omega,\ m = 0, \dots, M - 1. \end{aligned}$$

We usually order the vector of unknowns in what's called a Lexicographic ordering:

$$U^m = [U_{1,1}^m, \dots, U_{N_x,1}, U_{1,2}, \dots, U_{N_x,2}, \dots, U_{N_x,Ny}]^T$$

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If we do this, then the matrix to be solved in an implicit method has 5 diagonals (yet not all next to each other).

Stability

As before, we analyze stability by doing a Fourier analysis, or by inserting the Fourier mode, in this case

$$U_{i,j}^m = [\lambda(k_x, k_y)]^m \, e^{i(k_x x_i + k_y y_j)}$$

This gives

$$\begin{aligned}\lambda - 1 =& - 4(1-\theta)\left[\mu_x \sin^2\left(\frac{k_x \Delta x}{2}\right) + \mu_y \sin^2\left(\frac{k_y \Delta y}{2}\right)\right] \\ & - 4\theta\lambda\left[\mu_x \sin^2\left(\frac{k_x \Delta x}{2}\right) + \mu_y \sin^2\left(\frac{k_y \Delta y}{2}\right)\right]\end{aligned}$$

where

$$\mu_x = \frac{\Delta t}{\Delta x^2}, \qquad \mu_y = \frac{\Delta t}{\Delta y^2}$$

and so

$$\lambda = \frac{1 - 4(1-\theta)\left[\mu_x \sin^2\left(\frac{k_x \Delta x}{2}\right) + \mu_y \sin^2\left(\frac{k_y \Delta y}{2}\right)\right]}{1 + 4\theta\left[\mu_x \sin^2\left(\frac{k_x \Delta x}{2}\right) + \mu_y \sin^2\left(\frac{k_y \Delta y}{2}\right)\right]}$$

For practical stability in ℓ_2, we require that

$$|\lambda(k_x, k_y)| \leq 1 \;\forall\; (k_x, k_y) \in \left[-\frac{\pi}{\Delta x}, \frac{\pi}{\Delta x},\right] \times \left[-\frac{\pi}{\Delta y}, \frac{\pi}{\Delta y}\right]$$

which demands

$$-1 \leq \frac{1 - 4(1-\theta)\left[\mu_x + \mu_y\right]}{1 + 4\theta\left[\mu_x + \mu_y\right]} \leq 1,$$

and so

$$2(1-2\theta)(\mu_x + \mu_y) \leq 1$$

$$\begin{aligned}\text{Implicit Euler}(\theta = 1) \qquad & \text{unconditionally stable} \\ \text{Crank-Nicolson}(\theta = 1/2) \qquad & \text{unconditionally stable} \\ \text{Explicit Euler}(\theta = 0) \qquad & \text{conditionally stable :}\end{aligned}$$

$$\mu_x + \mu_y = \Delta t\left(\frac{1}{\Delta x^2} + \frac{1}{\Delta y^2}\right) \leq \frac{1}{2}$$

Discrete maximum principle

We can write the $\theta-$scheme as

$$\begin{aligned}(1+2\theta(\mu_x+\mu_y))U_{i,j}^{m+1} =&(1-2(1-\theta)(\mu_x+\mu_y))U_{i,j}^{m} \\ &+(1-\theta)\mu_x(U_{i+1,j}^{m}+U_{i-1,j}^{m}) \\ &+(1-\theta)\mu_y(U_{i,j+1}^{m}+U_{i,j-1}^{m}) \\ &+\theta\mu_x(U_{i+1,j}^{m+1}+U_{i-1,j}^{m+1}) \\ &+\theta\mu_y(U_{i,j+1}^{m+1}+U_{i,j-1}^{m+1}) \qquad (*)\end{aligned}$$

If $(\mu_x+\mu_y)(1-\theta)\leq 1/2$, then the $\theta-$scheme obeys a Discrete maximum principle, so that

$$U_{min}\leq U_{i,j}^m\leq U_{max},$$

where

$$U_{min}=\min\left\{\min\left\{U_{i,j}^0\right\},\min\left\{U_{i,j}^m\right\}_{(x_i,y_j)\in\partial\Omega}\right\}$$

and

$$U_{max}=\max\left\{\max\left\{U_{i,j}^0\right\},\max\left\{U_{i,j}^m\right\}_{(x_i,y_j)\in\partial\Omega}\right\}$$

Proof: exactly similarly to the 1D proof.

Summary For

$$(\mu_x+\mu_y)(1-\theta)\leq 1/2$$

The $\theta-$scheme obeys the discrete maximum principle. This is more demanding than the ℓ_2-stability condition:

$$(\mu_x+\mu_y)(1-2\theta)\leq 1/2, \quad 0\leq\theta\leq 1/2$$

Error analysis

We define the truncation error

$$T_{i,j}^m=\frac{u_{i,j}^{m+1}-u_{i,j}^m}{\Delta t}-(1-\theta)\left(\frac{\delta_x^2u_{i,j}^m}{\Delta x^2}+\frac{\delta_y^2u_{i,j}^m}{\Delta y^2}\right)+\theta\left(\frac{\delta_x^2u_{i,j}^{m+1}}{\Delta x^2}+\frac{\delta_y^2u_{i,j}^{m+1}}{\Delta y^2}\right),$$

where $u_{i,j}^m=u(x_i,y_j,t_m)$. After applying some Taylor expansions we get

$$T_{i,j}^m=\begin{cases}O(\Delta x^2+\Delta y^2+\Delta t^2), & \theta=1/2\\ O(\Delta x^2+\Delta y^2+\Delta t), & \theta\neq 1/2\end{cases}$$

We now look at the global error. First, note that we can rearrange the truncation error formula to give

$$\begin{aligned}(1+2\theta(\mu_x+\mu_y))u_{i,j}^{m+1} =&(1-2(1-\theta)(\mu_x+\mu_y))u_{i,j}^m\\ &+(1-\theta)\mu_x(u_{i+1,j}^m+u_{i-1,j}^m)\\ &+(1-\theta)\mu_y(u_{i,j+1}^m+u_{i,j-1}^m)\\ &+\theta\mu_x(u_{i+1,j}^{m+1}+u_{i-1,j}^{m+1})\\ &+\theta\mu_y(u_{i,j+1}^{m+1}+u_{i,j-1}^{m+1})\\ &+\Delta t T_{i,j}^m. \qquad (**)\end{aligned}$$

Defining the global error

$$e_{i,j}^m = U_{i,j}^m - u(x_i, y_j, t_m)$$

then $e_{i,j}^0 = 0$ and $e_{i,j}^m = 0$ for $(x_i, y_j) \in \partial\Omega$, and, subtracting (*) from (**), we get

$$\begin{aligned}(1+2\theta(\mu_x+\mu_y))e_{i,j}^{m+1} =&(1-2(1-\theta)(\mu_x+\mu_y))e_{i,j}^m\\ &+(1-\theta)\mu_x(e_{i+1,j}^m+e_{i-1,j}^m)\\ &+(1-\theta)\mu_y(e_{i,j+1}^m+e_{i,j-1}^m)\\ &+\theta\mu_x(e_{i+1,j}^{m+1}+e_{i-1,j}^{m+1})\\ &+\theta\mu_y(e_{i,j+1}^{m+1}+e_{i,j-1}^{m+1})\\ &+\Delta t T_{i,j}^m\end{aligned}$$

Then if $E^m = \max_{i,j} |e_{i,j}^m|$ and $T^m = \max_{i,j} |T_{i,j}^m|$, and we assume that

$$1 - 2(1-\theta)(\mu_x + \mu_y) \geq 0,$$

(Discrete Maximum Principle) then we have

$$(1 + 2\theta(\mu_x + \mu_y))E^{m+1} \leq 2\theta(\mu_x + \mu_y)E^{m+1} + E^m + \Delta t T^m$$

and hence

$$E^{m+1} \leq E^m + \Delta t T^m.$$

As in the 1D case, as $E^0 = 0$,

$$E^m \leq T \max_m \max_{i,j} |T_{i,j}^m|$$

(where T is the maximum time), and so

$$\max_m \max_{i,j} |u(x_i, y_j, t_m) - U_{i,j}^m| \leq T \max_m \max_{i,j} |T_{i,j}^m|$$