NSDE 1: LECTURE 15

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Another concept of Stability

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

for $x \in (-\infty, \infty)$ and $t \in [0, T]$, where $u(x, 0) = u_0(x)$, and $u(x, t) \to 0$ as $x \to \pm\infty$.

We shall say that a finite difference scheme for this problem is *von Neumann*-stable in the ℓ_2 norm if there exists a positive constant $C = C(T)$ such that

$$\|U^m\|_{\ell_2} \leq C\|U^0\|_{\ell_2}, \qquad m = 1, \ldots, M = T/\Delta t,$$

where, as usual

$$\|U^m\|_{\ell_2} = \left(\Delta x \sum_{j=-\infty}^{\infty} |U_j^m|^2\right)^{1/2}.$$

Note that practical stability $\Rightarrow$ von Neumann stability.

Lemma

If

$$\hat{U}^{m+1}(k) = \lambda(k)\hat{U}^m(k)$$

and

$$|\lambda(k)| \leq 1 + C\Delta t \qquad \forall k \in [-\pi/\Delta x, \pi/\Delta x],$$

then the scheme is von Neumann-stable.

Proof

By Parseval's identity for the semidiscrete Fourier Transform:

$$\begin{aligned}
\|U^{m+1}\|_{\ell_2} &= \frac{1}{\sqrt{2\pi}}\|\hat{U}^{m+1}\|_{L_2} \\
&= \frac{1}{\sqrt{2\pi}}\|\lambda(k)\hat{U}^m\|_{L_2} \\
&\leq \frac{1}{\sqrt{2\pi}} \max_k |\lambda(k)| \|\hat{U}^m\|_{L_2} \\
&= \max_k |\lambda(k)| \|U^m\|_{\ell_2}
\end{aligned}$$

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and hence

$$\|U^{m+1}\|_{\ell_2} \leq (1 + C\Delta t)\|U^m\|_{\ell_2}, \qquad m = 0, 1, \dots, M-1.$$

Applying this result repeatedly, we get that

$$\begin{aligned}\|U^m\|_{\ell_2} &\leq (1 + C\Delta t)^m \|U^0\|_{\ell_2}, \qquad m = 0, 1, \dots, M-1. \\ &\leq (1 + C\Delta t)^M \|U^0\|_{\ell_2} \qquad [\text{as } 1 + C\Delta t > 1, M > m] \\ &\leq e^{C\Delta t M} \|U^0\|_{\ell_2} \qquad [\text{as } 1 + x \leq e^x, x > 0] \\ &= e^{CT} \|U^0\|_{\ell_2} \qquad [\text{as } M = T/\Delta t]\end{aligned}$$

□

Note that, as is apparent from the proof, von-Neumann stability isn't helpful over long time periods. However, using it we can state an important theorem:

Theorem: The Lax-equivalence theorem For a consistent difference approximation to a well posed linear evolutionary problem, stability as $\Delta t \to 0$ is necessary and sufficient for convergence.

(proof omitted)

This is the PDE equivalent of Dahlquist's theorem for multistep methods for ODEs.

The discrete Maximum principle

Consider the Dirchlet problem. Mathematically, the maximum of the solution must lie on the boundary (either initially, or at a space boundary). It is important that our numerical scheme also satisfies this property.

Proposition: the discrete Maximum principle The θ-method with $0 \leq \theta \leq 1$ and $\mu(1-\theta) \leq 1/2$ gives approximations U_j^m satisfying:

$$U_{min} \leq U_j^m \leq U_{max},$$

where

$$U_{min} = \min \left\{ \min_{0 \leq m \leq M} \{U_0^m\}, \min_{0 \leq j \leq N+1} \{U_j^0\}, \min_{0 \leq m \leq T} \{U_{N+1}^m\} \right\}$$

and

$$U_{max} = \max \left\{ \max_{0 \leq m \leq M} \{U_0^m\}, \max_{0 \leq j \leq N+1} \{U_j^0\}, \max_{0 \leq m \leq T} \{U_{N+1}^m\}. \right\}$$

Proof We can rewrite the $\theta-$scheme as

$$\begin{aligned}(1 + 2\theta\mu)U_j^{m+1} =& \theta\mu(U_{j+1}^{m+1} + U_{j-1}^{m+1}) \\ &+ (1-\theta)\mu(U_{j+1}^m + U_{j-1}^m) \\ &+ [1 - 2(1-\theta)\mu]U_j^m.\end{aligned}$$

By hypothesis:

$$\theta\mu \geq 0 \quad (1-\theta)\mu \geq 0 \quad 1-2(1-\theta)\mu \geq 0,$$

with the last of these being from the assumption that $\mu(1-\theta) \leq 1/2$.

Now, suppose that U attains it's maximum at an *internal* grid point U_j^{m+1}, so $1 \leq j \leq N$, $0 \leq m \leq M-1$ (if not, the proof is complete). Define

$$U^\star = \max\{U_{j+1}^{m+1}, U_{j-1}^{m+1}, U_{j+1}^m, U_{j-1}^m, U_j^m\}.$$

Then

$$\begin{aligned}(1+2\theta\mu)U_j^{m+1} &\leq \theta\mu U^\star + 2(1-\theta)\mu U^\star + [1-2(1-\theta)\mu]U^\star \\ &= (1+2\theta\mu)U^\star\end{aligned}$$

and, so

$$U_j^{m+1} \leq U^\star$$

However, since U_j^{m+1} is assumed to be the maximum value overall

$$U^\star \leq U_j^{m+1},$$

and, therefore,

$$U_j^{m+1} = U^\star.$$

The maximum is therefore also attained at the points neighbouring (x_j, t_{m+1}). The same argument applies to these neighbouring points, and can be continued until the boundary is reached.

Therefore, the maximum is attained at a boundary point.

□

Note that the θ-scheme obeys the discrete maximum principle for

$$\mu(1-\theta) \leq 1/2.$$

This is more demanding than the stability condition:

$$\mu(1-2\theta) \leq 1/2 \qquad \text{for } 0 \leq \theta \leq 1/2.$$

So, e.g., Crank-Nicolson is unconditionally stable, yet it only obeys the maximum principle for

$$\mu = \frac{\Delta t}{\Delta x^2} \leq 1.$$