NSDE 1: LECTURE 14

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Recap

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

for $x \in [0, X]$ and $t \in [0, T]$ where $u(x, 0) = u_0(x)$, and with mixed boundary conditions $\frac{\partial u}{\partial x}(0, t) = u_a(t)$, $u(X, t) = u_b(t)$ Set up a grid:

$$x_j = j\Delta x, j = 0, \ldots, N+1 \text{ and } t_m = m\Delta t, m = 1, \ldots, M.$$

$$U_j^{m+1} - \theta\mu(U_{j+1}^{m+1} - 2U_j^{m+1} + U_{j-1}^{m+1}) = U_j^m + (1-\theta)\mu(U_{j+1}^m - 2U_j^m + U_{j-1}^m)$$

for $j = 1, \ldots N$.

Neumann boundary conditions Suppose that we have a Neumann boundary condition at x_0, so $\frac{\partial u}{\partial x}(x_0, t) = u_a(t)$, with a Dirichlet b.c. at x_{N+1}. Now U_0^m is also unknown, so what do we do?

The solution is to introduce a *ficticious point*, U_{-1}^m, and set

$$\frac{U_1^m - U_{-1}^m}{2\Delta x} = u_a(t_m)$$

(We must use central differences, otherwise the order of accuracy of the boundary condition would be less than that of the PDE). Then $U_{-1}^m = U_1^m - 2\Delta x u_a(t_m)$. We can therefore substitute this into our scheme, giving

$$(I - \theta\mu K^N)U^{m+1} = (I + (1-\theta)\mu K^N)U^m + \mathbf{f}^N$$

where

$$K^N = \begin{bmatrix} -2 & 2 & 0 & \cdots & 0 \\ 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & \\ & & \ddots & \ddots & \ddots \\ & & 0 & 1 & -2 \end{bmatrix}, \qquad \mathbf{f} = \begin{bmatrix} -2(1-\theta)\mu\Delta x u_a(t_m) - 2\theta\mu\Delta x u_a(t_{m+1}) \\ 0 \\ \vdots \\ 0 \\ (1-\theta)\mu u_b(t_m) + \theta\mu u_b(t_{m+1}) \end{bmatrix}$$

Truncation error

Suppose we solve the heat equation with Dirichlet boundary conditions:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad x \in [a, b], \ t \in [0, T]$$
$$u(x, 0) = u_0(x)$$
$$u(a, t) = u_a(t), u(b, t) = u_b(t)$$

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using Explicit Euler in time and central differences in space

$$\frac{U_j^{m+1} - U_j^m}{\Delta t} = \frac{U_{j-1}^m - 2U_j^m + U_{j-1}^m}{\Delta x^2}$$

$$U_j^0 = u_0(x_j), j = 1, \dots, N$$

$$U_0^{m+1} = u_a(t_{m+1}), U_{N+1}^{m+1} = u_b(t_{m+1}), m = 1, \dots, M$$

We define the truncation error of the scheme as

$$T_j^m = \frac{u_j^{m+1} - u_j^m}{\Delta t} - \frac{u_{j-1}^m - 2u_j^m + u_{j-1}^m}{\Delta x^2}$$

where $u_j^m = u(x_j, t_m)$.

Then, expanding using Taylor series, we get

$$u_j^{m+1} = \left[u + \Delta t u_t + \frac{\Delta t^2}{2} u_{tt} + \frac{\Delta t^3}{6} u_{ttt} + \cdots \right]_j^m$$

and so the time derivative gives

$$\begin{aligned}\frac{u_j^{m+1} - u_j^m}{\Delta t} &= \frac{\left[u + \Delta t u_t + \frac{\Delta t^2}{2} u_{tt} + \frac{\Delta t^3}{6} u_{ttt} + \cdots - u\right]_j^m}{\Delta t} \\ &= u_t + \frac{\Delta t}{2} u_{tt} + \cdots\end{aligned}$$

Now, for the central differences approximation:

$$\begin{aligned}u_{j+1}^m &= \left[u + \Delta x u_x + \frac{\Delta x^2}{2} u_{xx} + \frac{\Delta x^3}{6} u_{xxx} + \frac{\Delta x^4}{24} u_{xxxx} + \cdots \right]_j^m \\ -2u_j^m &= -2u_j^m \\ u_{j-1}^m &= \left[u - \Delta x u_x + \frac{\Delta x^2}{2} u_{xx} - \frac{\Delta x^3}{6} u_{xxx} + \frac{\Delta x^4}{24} u_{xxxx} + \cdots \right]_j^m\end{aligned}$$

and so we get

$$\begin{aligned}\frac{u_{j+1}^m - 2u_j^m + u_{j-1}^m}{\Delta x^2} &= \frac{\Delta x^2 u_{xx} + \frac{\Delta x^4}{12} u_{xxxx} + \cdots}{\Delta x^2} \\ &= u_{xx} + \frac{\Delta x^2}{12} u_{xxxx} + \cdots\end{aligned}$$

Therefore, putting it together:

$$T_j^m = u_t - u_{xx} + \frac{\Delta t}{2} u_{tt} - \frac{\Delta x^2}{12} u_{xxxx} + \cdots = O(\Delta t + (\Delta x)^2)$$

For other schemes, choose the point that you expand about accordingly to minimize the algebra!

- Explicit Euler: u_j^m – $O(\Delta t + \Delta x^2)$
- Implicit Euler: u_j^{m+1} – $O(\Delta t + \Delta x^2)$
- Crank-Nicolson: $u_j^{m+1/2}$ – $O(\Delta t^2 + \Delta x^2)$

Error analysis of the Explicit Euler scheme Let us define the *global error* as

$$e_j^m = u(x_j, t_m) - U_j^m.$$

Note that, for a Dirichlet problem,

$$e_0^{m+1} = 0, e_{N+1}^{m+1} = 0, e_j^0 = 0, j = 1, \ldots, N$$

Then, we have

$$\begin{aligned} U_j^{m+1} &= U_j^m + \mu(U_{j-1}^m - 2U_j^m + U_{j+1}^m) \\ u_j^{m+1} &= u_j^m + \mu(u_{j-1}^m - 2u_j^m + u_{j+1}^m) + \Delta t T_j^m \end{aligned}$$

and so

$$\begin{aligned} e_j^{m+1} &= e_j^m + \mu(e_{j-1}^m - 2e_j^m + e_{j+1}^m) + \Delta t T_j^m \\ &= (1-2\mu)e_j^m + \mu e_{j-1}^m + \mu e_{j+1}^m + \Delta t T_j^m \end{aligned}$$

Let $E^m = \max_{0 \le j \le N+1} |e_j^m|$ and $T^m = \max_{1 \le j \le N} |T_j^m|$. As long as $(1 - 2\mu) \ge 0$ (recall that this scheme is absolutely stable if $\mu \le 1/2$) we can write

$$\begin{aligned} E^{m+1} &\le (1-2\mu)E^m + \mu E^m + \mu E^m + \Delta t T^m \\ &= E^m + \Delta t T^m \end{aligned}$$

Therefore, since $E^0 = 0$,

$$\begin{aligned} E^m &\le \Delta t \sum_{i=0}^{m-1} T^i \\ &\le m\Delta t \max_{0 \le i \le m-1} T^i \\ &\le T \max_{0 \le m \le M} \max_{1 \le j \le N} |T_j^m| \end{aligned}$$

Therefore, since the Euler scheme has truncation error

$$T_j^m = O(\Delta x^2 + \Delta t)$$

we have that

$$\max_{0 \leq m \leq M} \max_{1 \leq j \leq N} |u(x_j, t_m) - U_j^m| \leq Const.(\Delta x^2 + \Delta t)$$