NSDE 1: LECTURE 13

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Recap To solve

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

for $x \in (-\infty, \infty)$ and $t \in [0, T]$, where $u(x, 0) = u_0(x)$, and $u \to 0$ as $\to \pm\infty$ Set up a grid:

$$x_j = j\Delta x, \text{ and } t_m = m\Delta t.$$

We've considered two schemes: Explicit Euler in time with central differences in space:

$$U_j^{m+1} = U_j^m + \mu(U_{j-1}^m - 2U_j^m + U_{j+1}^m)$$

and Implicit Euler in time with central differences in space:

$$U_j^{m+1} - \mu(U_{j+1}^{m+1} - 2U_j^{m+1} + U_{j-1}^{m+1}) = U_j^m,$$

where $\mu = \Delta t/(\Delta x)^2$.

We analyzed stability using the semidiscrete Fourier transform:

$$\hat{U}(k) = \Delta x \sum_{j=-\infty}^{\infty} U_j \, e^{-ikx_j}, \qquad k \in [-\pi/\Delta x, \pi/\Delta x],$$

with inverse:

$$U_j = \frac{1}{2\pi} \int_{-\pi/\Delta x}^{\pi/\Delta x} \hat{U}(k) \, e^{ikj\Delta x} \, dk$$

`aliasing.m`

Fourier modes The same stability results can be reached using a simplier argument, by inserting the *Fourier mode* into the scheme, i.e. setting $U_j^m = [\lambda(k)]^m e^{ikj\Delta x}$. You should be able to do either method.

Consider another scheme: the $\theta-$scheme:

$$\frac{U_j^{m+1} - U_j^m}{\Delta t} = (1-\theta)\frac{U_{j+1}^m - 2U_j^m + U_{j-1}^m}{(\Delta x)^2} + \theta\frac{U_{j+1}^{m+1} - 2U_j^{m+1} + U_{j-1}^{m+1}}{(\Delta x)^2}$$

where $\theta \in [0, 1]$ is a parameter:

- $\theta = 0$: Explicit Euler scheme
- $\theta = 1$: Implicit Euler scheme

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- $\theta = 1/2$: Crank-Nicolson scheme

To analyse stability, note

$$U_j^{m+1} - U_j^m = (1-\theta)\mu(U_{j-1}^m - 2U_j^m + U_{j+1}^m) + \mu\theta(U_{j-1}^{m+1} - 2U_j^{m+1} + U_{j+1}^{m+1})$$

inserting the Fourier mode:

$$\begin{aligned}[\lambda(k)]^{m+1}e^{ikj\Delta x} - [\lambda(k)]^m e^{ikj\Delta x} &= (1-\theta)\mu([\lambda(k)]^m e^{ik(j-1)\Delta x} - 2[\lambda(k)]^m e^{ikj\Delta x} + [\lambda(k)]^m e^{ik(j+1)\Delta x}) \\ &+ (1-\theta)\mu([\lambda(k)]^{m+1} e^{ik(j-1)\Delta x} - 2[\lambda(k)]^{m+1} e^{ikj\Delta x} + [\lambda(k)]^{m+1} e^{ik(j+1)\Delta x})\end{aligned}$$

Dividing through by $[\lambda(k)]^m e^{ikx_j}$ gives:

$$\lambda(k) - 1 = (1-\theta)\mu(e^{-ik\Delta x} - 2 + e^{ik\Delta x}) + \theta\mu\lambda(k)(e^{-ik\Delta x} - 2 + e^{ik\Delta x}).$$

Therefore, by the same arguments as last time:

$$\lambda(k) - 1 = -4(1-\theta)\mu\sin^2\left(\frac{k\Delta x}{2}\right) - 4\theta\mu\lambda(k)\sin^2\left(\frac{k\Delta x}{2}\right)$$

and so

$$\lambda(k) = \frac{1 - 4(1-\theta)\mu\sin^2\left(\frac{k\Delta x}{2}\right)}{1 + 4\theta\mu\sin^2\left(\frac{k\Delta x}{2}\right)}$$

For practical stability, we require that

$$|\lambda(k)| \leq 1 \quad \forall\ k \in \left[\frac{-\pi}{\Delta x}, \frac{\pi}{\Delta x}\right]$$

i.e.

$$-1 \leq \frac{1 - 4(1-\theta)\mu p}{1 + 4\theta\mu p} \leq 1,$$

where we've written $p = \sin^2\left(\frac{k\Delta x}{2}\right)$, which takes values between 0 and 1. We can write this as

$$-1 \leq 1 - \frac{4\mu p}{1 + 4\theta\mu p} \leq 1$$

This is clearly less than one, and monotonic decreasing, so we just have to check what happens at $p = 1$. We get that the inequality is always satisfied of $\theta \leq 1/2$, and otherwise is satisfied if

$$\mu \leq \frac{1}{2(1-2\theta)}.$$

Therefore:

- For $\theta \in [1/2, 1]$: unconditionally stable

- For $\theta \in [0, 1/2[$: stable if $\mu \leq \frac{1}{2(1-2\theta)}$

Boundary conditions

In practice, we'll be solving on a bounded domain, and so we'll need boundary conditions: As well as an initial condition $u(x, 0) = u_0(x)$, we need, e.g.,

- Dirichlet boundary conditions: $u(x_0, t) = u_a(t)$, $u(x_{N+1}, t) = u_b(t)$
- Neumann boundary conditions: $\frac{\partial u}{\partial t}(x_0, t) = u_a(t)$, $\frac{\partial u}{\partial t}(x_{N+1}, t) = u_b(t)$
- Mixed boundary conditions: $\frac{\partial u}{\partial t}(x_0, t) = u_a(t)$, $u(x_{N+1}, t) = u_b(t)$

How do we solve the system?

We can write the θ-scheme as

$$U_j^{m+1} - \theta\mu(U_{j+1}^{m+1} - 2U_j^{m+1} + U_{j-1}^{m+1}) = U_j^m + (1-\theta)\mu(U_{j+1}^m - 2U_j^m + U_{j-1}^m)$$

for $j = 1, \dots N$. In the Dirichlet case, we know $U_j^0 = u_0(x_j)$ for all j, and also $U_0^m = u_a(t_m)$, $U_{N+1}^m = u_b(t_m)$, so we can simply plug these in where needed. This can be be written in matrix form:

$$U^{m+1} - \theta\mu \begin{bmatrix} -2 & 1 & 0 & \cdots & 0 \\ 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & \\ & & \ddots & \ddots & \ddots \\ & & 0 & 1 & -2 \end{bmatrix} U^{m+1} = U^m +$$

$$(1-\theta)\mu \begin{bmatrix} -2 & 1 & 0 & \cdots & 0 \\ 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & \\ & & \ddots & \ddots & \ddots \\ & & 0 & 1 & -2 \end{bmatrix} U^m + (1-\theta)\mu \begin{bmatrix} u_a(t_m) \\ 0 \\ \vdots \\ 0 \\ u_b(t_m) \end{bmatrix} + \theta\mu \begin{bmatrix} u_a(t_{m+1}) \\ 0 \\ \vdots \\ 0 \\ u_b(t_{m+1}) \end{bmatrix},$$

or, if we write

$$K = \begin{bmatrix} -2 & 1 & 0 & \cdots & 0 \\ 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & \\ & & \ddots & \ddots & \ddots \\ & & 0 & 1 & -2 \end{bmatrix}, \qquad \mathbf{f} = \begin{bmatrix} (1-\theta)\mu u_a(t_m) + \theta\mu u_a(t_{m+1}) \\ 0 \\ \vdots \\ 0 \\ (1-\theta)\mu u_b(t_m) + \theta\mu u_b(t_{m+1}) \end{bmatrix}$$

then

$$(I - \theta\mu K)U^{m+1} = (I + (1-\theta)\mu K)U^m + \mathbf{f}$$

At each step, a tridiagonal matrix must be solved (by, e.g., the Thomas algorithm – see problem sheet 5).