NSDE 1: LECTURE 12

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Recap To solve

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

for $x \in [0, X]$ and $t \in [0, T]$, where $u(x, 0) = u_0(x)$, and $u(0, t) = u_a(t)$, $u(X, t) = u_b(t)$, set up a grid

$$x_j = j\Delta x, \text{ and } t_m = m\Delta t$$

where

$$\Delta x = X/N \text{ in the } x\text{-direction}$$
$$\Delta t = T/M \text{ in the } t\text{-direction.}$$

We approximate the derivatives by $u(x_j, t_m) \approx U_j^m$, so that the derivatives are approximately

$$\frac{\partial u}{\partial t}(x_j, t_m) \approx \frac{U_j^{m+1} - U_j^m}{\Delta t}$$

and

$$\frac{\partial^2 u}{\partial x^2}(x_j, t_m) \approx \frac{U_{j-1}^m - 2U_j^m + U_{j+1}^m}{(\Delta x)^2}.$$

This gives

$$U_0^{m+1} = u_a(t_{m+1}), \ \ U_{N+1}^{m+1} = u_b(t_{m+1})$$
$$U_j^{m+1} = U_j^m + \mu(U_{j-1}^m - 2U_j^m + U_{j+1}^m)$$

where $\mu = \frac{\Delta t}{\Delta x^2}$.

Stability

If U is a function defined on the infinite grid $x_j = j\Delta x$, $j = 0, \pm 1, \pm 2, \dots$, the semidiscrete Fourier transform of U is defined as

$$\hat{U}(k) = \Delta x \sum_{j=-\infty}^{\infty} U_j \, e^{-ikx_j}, \qquad k \in [-\pi/\Delta x, \pi/\Delta x]$$

Now, since $\hat{U}(k)$ is a continuous function, we can take the regular inverse Fourier transform to obtain

$$U_j = \frac{1}{2\pi} \int_{-\pi/\Delta x}^{\pi/\Delta x} \hat{U}(k) \, e^{ikj\Delta x} \, dk.$$

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It can be shown that there's an associated Parseval's Identity:

$$\|U\|_{\ell_2}^2 = \frac{1}{2\pi}\|\hat{U}\|_{L_2}^2,$$

where

$$\|\hat{U}\|_{L_2} = \left(\int_{-\pi/\Delta x}^{\pi/\Delta x} |\hat{U}(k)|^2 dk\right)^{1/2}$$

and

$$\|U\|_{\ell_2} = \left(\Delta x \sum_{j=-\infty}^{\infty} |U_j|^2\right)^{1/2}$$

[see problem sheet]

Definition: We say that a finite difference scheme for the unsteady heat equation is *(practically) stable* in the ℓ_2 norm if

$$\|U^m\|_{\ell_2} \leq \|U^0\|_{\ell_2},\ m = 1, \dots M,$$

where $U^m = \{U_j^m\}$.

Now, recall the Euler scheme:

$$U_j^{m+1} = U_j^m + \mu(U_{j-1}^m - 2U_j^m + U_{j+1}^m)$$

and inserting the inverse Fourier transform we get

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi/\Delta x}^{\pi/\Delta x} e^{ikj\Delta x}\hat{U}^{m+1}(k)dk =& \frac{1}{2\pi}\int_{-\pi/\Delta x}^{\pi/\Delta x} e^{ikj\Delta x}\hat{U}^m(k)dk + \\ & \mu\left(\frac{1}{2\pi}\int_{-\pi/\Delta x}^{\pi/\Delta x} e^{ikj-1\Delta x}\hat{U}^m(k)dk\right. \\ & + 2\frac{1}{2\pi}\int_{-\pi/\Delta x}^{\pi/\Delta x} e^{ikj\Delta x}\hat{U}^m(k)dk \\ & \left. + \frac{1}{2\pi}\int_{-\pi/\Delta x}^{\pi/\Delta x} e^{ikj+1\Delta x}\hat{U}^m(k)dk\right)\end{aligned}$$

Rearranging, we get

$$\frac{1}{2\pi}\int_{-\pi/\Delta x}^{\pi/\Delta x} e^{ikj\Delta x}\left(\hat{U}^{m+1}(k) - \hat{U}^m(k) - \mu\left(e^{ik\Delta x} - 2 + e^{-ik\Delta x}\right)\hat{U}^m(k)\right)dk = 0,$$

which, in turn means that

$$\hat{U}^{m+1}(k) = \hat{U}^m(k) + \mu(e^{ik\Delta x} - 2 + e^{ik\Delta x})\hat{U}^m(k)$$

for all wave numbers $k \in [-\pi/\Delta x, \pi/\Delta x]$. Therefore

$$\hat{U}^{m+1}(k) = \lambda(k)\hat{U}^m(k)$$

where $\lambda(k) = 1 + \mu(e^{ik\Delta x} - 2 + e^{ik\Delta x})\hat{U}^m(k)$ Now,

$$\begin{aligned}
\|U^{m+1}\|_{\ell_2} &= \frac{1}{\sqrt{2\pi}}\|\hat{U}^{m+1}\|_{L_2} \qquad \text{Parseval} \\
&= \frac{1}{\sqrt{2\pi}}\|\lambda\hat{U}^m\|_{L_2} \\
&\leq \frac{1}{\sqrt{2\pi}} \max_k |\lambda(k)| \|\hat{U}^m\|_{L_2} \\
&= \max_k |\lambda(k)| \|U^m\|_{\ell_2} \qquad \text{Parseval}
\end{aligned}$$

Since we want that

$$\|U^{m+1}\|_{\ell_2} \leq \|U^m\|_{\ell_2}, \qquad m = 0, 1, 2, \ldots, M-1$$

we need that

$$\max_k |\lambda(k)| \leq 1$$

$$\max_k |1 + \mu(e^{ik\Delta x} - 2 + e^{ik\Delta x})| \leq 1$$

Now, since $e^{ik\Delta x} = \cos k\Delta x + i \sin k\Delta x$, and $e^{-ik\Delta x} = \cos k\Delta x - i \sin k\Delta x$, we can write this as

$$\max_k |1 + 2\mu(\cos k\Delta x - 1)| \leq 1$$

$$\max_k |1 - 4\mu \sin^2\left(\frac{k\Delta x}{2}\right)| \leq 1$$

Now, the condition that

$$-1 \leq 1 - 4\mu \sin^2\left(\frac{k\Delta x}{2}\right) \leq 1 \qquad \forall k \in [-\pi/\Delta x, \pi/\Delta x]$$

holds if and only if

$$\mu = \frac{\Delta t}{(\Delta x)^2} \leq \frac{1}{2}.$$

We've proved the following theorem:

Theorem Suppose that U_j^m is the solution of

$$\frac{U_j^{m+1} - U_j^m}{\Delta t} = \frac{U_{j+1}^m - 2U_j^m + U_{j-1}^m}{(\Delta x)^2}, \ j = 1, 2, \ldots$$

where $U_j^0 = u_0(x_j)$ and $\mu = \frac{\Delta t}{(\Delta x)^2} \le \frac{1}{2}$. Then

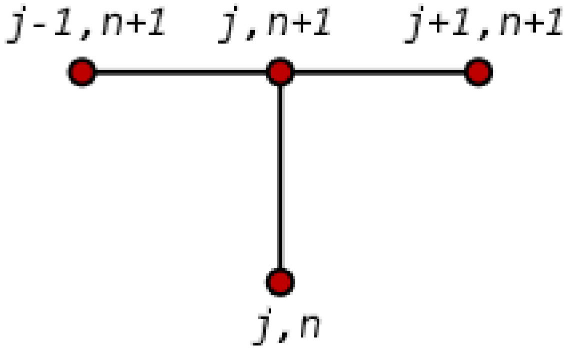
$$\|U^m\|_{\ell_2} \le \|U^0\|_{\ell_2},\ m = 1, 2, \ldots M.$$

We say that the explicit Euler scheme is *conditionally stable.*

The implicit Euler scheme Consider instead the scheme

$$\frac{U_j^{m+1} - U_j^m}{\Delta t} = \frac{U_{j+1}^{m+1} - 2U_j^{m+1} + U_{j-1}^{m+1}}{(\Delta x)^2},\ j = 1, 2, \ldots$$

where $U_j^0 = u_0(x_j)$



Now we have that

$$U_j^{m+1} - \mu(U_{j+1}^{m+1} - 2U_j^{m+1} + U_{j-1}^{m+1}) = U_j^m,$$

$U_j^0 = u_0(x_j)$, where again $\mu = \Delta t/(\Delta x)^2)$.

Using an identical argument to that used for Explicit Euler, we find the amplification factor is here

$$\lambda(k) = \frac{1}{1 + 4\mu \sin^2\left(\frac{k\Delta x}{2}\right)}.$$

In contrast to earlier, this satisfies $|\lambda(k)| \le 1$ for all values of μ.

Theorem Suppose that U_j^m is the solution of

$$\frac{U_j^{m+1} - U_j^m}{\Delta t} = \frac{U_{j+1}^{m+1} - 2U_j^{m+1} + U_{j-1}^{m+1}}{(\Delta x)^2},\ j = 1, 2, \ldots$$

where $U_j^0 = u_0(x_j)$ and $\mu = \frac{\Delta t}{(\Delta x)^2} \le \frac{1}{2}$. Then

$$\|U^m\|_{\ell_2} \le \|U^0\|_{\ell_2},\ m = 1, 2, \ldots M.$$

We say that the explicit Euler scheme is *unconditionally stable.*

Fourier modes The same results can be reached using a simplier argument, by inserting the *Fourier mode* into the scheme, i.e. setting

$U_j^m = [\lambda(k)]^m e^{ikj\Delta x}$. For example, substituting this into the explicit Euler scheme:

$$U_j^{m+1} = U_j^m + \mu \left(U_{j+1}^m - 2U_j^m + U_{j-1}^m \right)$$

gives

$$\lambda(k) = 1 + \mu(e^{ik\Delta x} - 2 + e^{-ik\Delta x}),$$

and hence the result follows as before.