NSDE 1: LECTURE 11

TYRONE REES*

The numerical solution of parabolic problems

For the rest of the course we're going to look at problems of the form:

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(a(x,t)\frac{\partial u}{\partial x}\right) + f(x,t),$$

with $u(x,0) = u_0(x)$ and appropriate boundary condititions.

In practice, we're going to concentrate on the model problem:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}.$$

Before we look at how to solve this numerically, let's explore what the analytic solution looks like. To do this, we're going to use the Fourier transform:

$$\hat{u}(\xi) = F[u](\xi) = \int_{-\infty}^{\infty} u(x)e^{-ix\xi}dx$$

Applying this to the PDE, we get:

$$\int_{-\infty}^{\infty} \frac{\partial u}{\partial t}(x,t)e^{-ix\xi}dx = \int_{-\infty}^{\infty} \frac{\partial^2 u}{\partial t^2}(x,t)e^{-ix\xi}dx$$
$$\frac{d}{dt}\underbrace{\int_{-\infty}^{\infty} u(x,t)e^{-ix\xi}dx}_{\hat{u}(\xi,t)} = (i\xi)^2 \underbrace{\int_{-\infty}^{\infty} u(x,t)e^{-ix\xi}dx}_{\hat{u}(\xi,t)}$$

where we've integrated by parts twice on the RHS and ignored boundary terms at $\pm\infty$. So

$$\frac{d\hat{u}}{dt} = -\xi^2\hat{u}.$$

This has solutions

$$\hat{u}(\xi,t) = A(\xi)e^{-\xi^2 t} = \hat{u}(\xi,0)e^{-\xi^2 t}.$$

*Rutherford Appleton Laboratory, Chilton, Didcot, UK, tyrone.rees@stfc.ac.uk

Now we just need to apply the inverse Fourier transform:

$$\begin{aligned}
u(x,t) &= F^{-1}\left(e^{-\xi^2 t}\hat{u}_0\right) \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{u}_0(\xi)e^{-\xi^2 t}e^{i\xi x}d\xi \\
&= \text{... messy calculation ...} \\
&= \frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty} e^{-(x-y)^2/(4t)}u_0(y)dy
\end{aligned}$$

`AnalyticSolution.m`

This behaviour can be explained by **Parseval's identity:**

$$\|u\|^2_{L_2(-\infty,\infty)} = \frac{1}{2\pi}\|\hat{u}\|^2_{L_2(-\infty,\infty)},$$

where

$$\|u\|_{L_2(\infty,\infty)} = \left(\int_{-\infty}^{\infty} |u(x)|^2 dx\right)^{1/2}$$

If we apply this to our function:

$$\begin{aligned}
\|u(\cdot,t)\|^2_{L_2(-\infty,\infty)} &= \frac{1}{2\pi}\|\hat{u}(\cdot,t)\|^2_{L_2(-\infty,\infty)} \\
&= \frac{1}{2\pi}\|e^{-\xi^2 t}\hat{u}_0(\cdot)\|^2_{L_2(-\infty,\infty)} \\
&\leq \frac{1}{2\pi}\max_{\xi}|e^{-\xi^2 t}|\|\hat{u}_0\|_2^2 \\
&\leq \frac{1}{2\pi}\|\hat{u}_0\|_2^2 \\
&= \|u_0\|_2^2 \qquad \text{[by Parseval's identity]}
\end{aligned}$$

This is an important property of the solution, that we must make sure is satisfied by any numerical approximation.

Proof of Parseval's identity

Let $v(x)$ and $w(x)$ be two functions.

$$\begin{aligned}
\int_{-\infty}^{\infty} \hat{w}(x)v(x)dx &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} w(\xi)e^{-i\xi x}d\xi v(x)dx \\
&= \int_{-\infty}^{\infty} w(\xi)\int_{-\infty}^{\infty} e^{-i\xi x}v(x)dxd\xi \\
&= \int_{-\infty}^{\infty} w(\xi)\hat{v}(\xi)d\xi
\end{aligned}$$

Now, if we let $w(\xi) = \overline{\hat{v}(\xi)}$, then

$$\hat{w}(\xi) = \int_{-\infty}^{\infty} \overline{\hat{v}(x)} e^{-i\xi x} dx = \overline{\int_{-\infty}^{\infty} \hat{v}(x) e^{i\xi x} dx} = 2\pi\bar{v}$$

Where we've used the fact that $v = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{v}e^{i\xi x}dx$ (by definition of the inverse Fourier transform). Therefore

$$2\pi \int_{-\infty}^{\infty} \bar{v}(x)v(x)dx = \int_{-\infty}^{\infty} \overline{\hat{v}(\xi)}\hat{v}(\xi)ds$$
$$\Rightarrow \|v\|_2^2 = \frac{1}{2\pi}\|\hat{v}\|_2^2$$

Discretization Suppose $x \in [0, X]$ and $t \in [0, T]$, where $T > 0$ is a given final time, and 0, X are the left and right boundaries.

Construct a finite-difference grid:

$$\Delta x = x_{\text{end}}/N \text{ in the } x\text{-direction}$$
$$\Delta t = T/M \text{ in the } t\text{-direction}$$

so that

$$x_j = j\Delta x, \text{ and } t_m = m\Delta t.$$

Since

$$\frac{\partial u}{\partial t}(x_j, t_m) = \lim_{\Delta t \to 0} \frac{u(x_j, t_m + \Delta t) - u(x_j, t_m)}{\Delta t}$$

and

$$\frac{\partial^2 u}{\partial x^2}(x_j, t_m) = \lim_{\Delta x \to 0} \frac{u(x_j + \Delta x, t_m) - 2u(x_j, t_m) + u(x_j - \Delta x, t_m)}{(\Delta x)^2}$$

We can approximate $u(x_j, t_m) \approx U_j^m$, so that

$$\frac{\partial u}{\partial t}(x_j, t_m) = \frac{U_j^{m+1} - U_j^m}{\Delta t}$$

and

$$\frac{\partial^2 u}{\partial x^2}(x_j, t_m) = \frac{U_{j-1}^m - 2U_j^m + U_{j+1}^m}{(\Delta x)^2}.$$

This scheme can be illustrated by the stencil:

Putting it together, suppose we want to approximate the PDE

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

for $x \in [0, X]$ and $t \in [0, T]$, where $u(\mathbf{x}, 0) = u_0(\mathbf{x})$, and $u(0, t) = u_a(t)$, $u(X, t) = u_b(t)$. Then we get that

$$U_j^0 = u_0(x_j), \qquad j = 0, \pm 1, \pm 2, \cdots$$

and then we can update U at the next time step by

$$\begin{aligned} U_0^{m+1} = u_a(t_{m+1}),\ \ U_{N+1}^{m+1} = u_b(t_{m+1}) \\ \frac{U_j^{m+1} - U_j^m}{\Delta t} = \frac{U_{j-1}^m - 2U_j^m + U_{j-1}^m}{(\Delta t)^2}. \end{aligned}$$

The final relation can also be written as

$$U_j^{m+1} = U_j^m + \mu(U_{j-1}^m - 2U_j^m + U_{j+1}^m),$$

where $\mu = \frac{\Delta t}{\Delta x^2}$.