NSDE 1: LECTURE 10

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Absolute stability

Zero-stability tells us that a method will converge if we take h small enough, but tells us nothing about what the solution will look like for a fixed h.

To motivate us, consider the model problem

$$u' = \lambda u,$$

which has exact solution $u = e^{\lambda t}$. If $Re(\lambda) < 0$, then the solutions decay as t increases.

`euler_stability(0.01)`, `euler_stability(0.05)`, `euler_stability(0.1)`, `euler_stability(0.11)`, `euler_stability(0.15)`

Suppose we apply Euler's method:

$$U_{n+1} = U_n + h\lambda U_n = (1 + h\lambda)U_n.$$

if $|1 + h\lambda| > 1$, the solution will grow at each time step. Note that only the combination $\bar{h} = h\lambda$ matters. The method is stable when $|1 + \bar{h}| < 1$. In the complex plane this is equivalent to the unit circle, centered on -1.

To make this more general, consider again the model problem above. Then a linear multistep method for this problem looks like:

$$\sum_{j=0}^{k}(\alpha_j - \lambda h \beta_j)U_{n+j} = 0.$$

The difference equation

$$\sum_{j=0}^{k}(\alpha_j - \bar{h}\beta_j)U_{n+j} = 0$$

has general solution

$$U_n = \sum_{s} p_s(n) z_s^n,$$

where z_s is a zero of the *stability polynomial*

$$\pi(z, \bar{h}) = \rho(h) - \bar{h}\sigma(z) = \sum_{j=0}^{k}(\alpha_j - \bar{h}\beta_j)z^j$$

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Since $\lim_{t\to\infty} u(t) = 0$, we want $\lim_{t\to\infty} U_n = 0$, and so we need $|z_s| < 1$ for all $s = 1, 2, \ldots, k$.

A linear multistep method is called *absolutely stable* in an open set $\mathcal{R}_A$ of the complex plane if, for all $\bar{h} \in \mathcal{R}_A$, all roots $z_s = z_s(\bar{h}), s = 1, \ldots k$ of $\pi(z, \bar{h})$ satisfy $|z_s| < 1$.

The set $\mathcal{R}_A$ is called the *region of absolute stability.*

Example: implicit Euler

$$U_{n+1} - U_n = hf(t_{n+1}, U_{n+1})$$

$$\rho(z) = z - 1, \qquad \sigma(z) = z$$

$$\pi(z, \bar{h}) = z - 1 - \bar{h}z = 0$$
$$\iff z = \frac{1}{1 - \bar{h}}$$

$|z| < 1$ when $|1 - \bar{h}| > 1$. Therefore $\mathcal{R}_A = \{\bar{h} \in \mathbb{C} \ : \ |1 - \bar{h}| > 1\}$ This is the outside of the unit circle, centered at 1, in the complex plane. Note that as this contains the whole of the left half plane, this method is stable for any h (since we know that $Re(\lambda) < 0$).

Definition A linear multistep method is said to be A-stable it its region of absolute stability, $\mathcal{R}_A$, contains the whole of the open left-hand complex half plane, $Re(\bar{h}) < 0$.

The implicit Euler method is A-stable.

Dahlquist Barrier Theorem

- No explicit linear multistep method is A-stable
- The order of an A-stable implicit linear multistep method is ≤ 2.
- The second-order A-stable linear multistep method with the smallest error constant is the trapezium rule method.

Stiffness

When deal with systems of differential equations, it is common that the methods we have studied may not work well with some systems where different parts of the solution evolve on different time scales.

Consider

$$u'' + (1 + a)u' + au = 0$$

which we can write as

$$\begin{aligned} u' &= v \\ v' &= -(1 + a)v - au \end{aligned}$$

or, in matrix form

$$\frac{d}{dt}\begin{bmatrix} u \\ v \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -a & -(1+a) \end{bmatrix}}_{A} \begin{bmatrix} u \\ v \end{bmatrix}$$

which has solutions $u = c_1 e^{-t} + c_2 e^{-at}$.

If $a \gg 1$, then u and v will have different scales, $O(1)$ and $O(a^{-1})$.

Note that the eigenpairs of A such that $A\mathbf{v}_i = \lambda_i \mathbf{v}_i$ are given by

$$\lambda_i = -1, \mathbf{v}_1 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}) \text{ and } \lambda_2 = -a, \mathbf{v}_2 = \begin{bmatrix} -1/\sqrt{1+a^2} \\ 1/\sqrt{1+a^2} \end{bmatrix}.$$

Therefore, letting $V = [\mathbf{v}_1, \mathbf{v}_2]$ and $\Lambda = diag(\lambda_1, \lambda_2)$, we can write $A = V\Lambda V^{-1}$

$$\begin{aligned} \mathbf{u}' &= A\mathbf{u} \\ \Rightarrow \mathbf{u}' &= V\Lambda V^{-1}\mathbf{u} \\ \Rightarrow V^{-1}\mathbf{u}' &= \Lambda V^{-1}\mathbf{u} \end{aligned}$$

Making the change of variables

$$\begin{bmatrix} p \\ q \end{bmatrix} = V^{-1} \begin{bmatrix} u \\ v \end{bmatrix}$$

therefore gives the decoupled system of equations

$$\begin{aligned} \frac{dp}{dt} &= -p \\ \frac{dq}{dt} &= -aq. \end{aligned}$$

Treating these separately would lead to very different requirements for stability: the equation for p could take much shorter time steps and still be stable. However, when we solve the systems together, for stability we need to take the time step required for stability of the bottom equation.

Such a system is called *Stiff.*

Another example of a stiff system is Van-der-pol's oscillator:

$$u'' + \mu(u^2 - 1)u' + u = 0$$

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vdp_stiff(0.16), vdp_stiff(0.165), vdp_stiff(0.17)
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