Numerical Solution of Differential Equations I: Problem Sheet 6

1. Consider a θ-method for the numerical solution of the initial value problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}\,, \qquad x \in \mathbb{R}\,, \quad t > 0\,,$$

$$u(x,0) = u_0(x)\,, \qquad x \in \mathbb{R}.$$

Suppose that the parameter θ has been chosen according to the formula

$$\theta = \frac{1}{2} + \frac{h^2}{12\Delta t}\,.$$

Show that the resulting scheme is unconditionally stable in the ℓ_2 norm and has a truncation error which is $\mathcal{O}((\Delta t)^2 + h^2)$, provided that derivatives of u of sufficiently high order exist and are bounded functions of x and t, $(x,t) \in \mathbb{R} \times [0,\infty)$.

2. The diffusion equation $u_t = u_{xx}$, $-\infty < x < \infty$, subject to the initial condition $u(x,0) = u_0(x)$, $-\infty < x < \infty$, is approximated by the finite difference scheme (*Crandall's scheme*):

$$U_j^{n+1} - \frac{1}{2}(\nu - \zeta)(U_{j+1}^{n+1} - 2U_j^{n+1} + U_{j-1}^{n+1}) = U_j^n + \frac{1}{2}(\nu + \zeta)(U_{j+1}^n - 2U_j^n + U_{j-1}^n)$$

with $U_j^0 = u_0(x_j)$, where $\Delta t > 0$, $h > 0$, $\nu = \Delta t/h^2$ and ζ is a fixed constant. Show that if ν is fixed real number, then the truncation error, T_j^n, obeys

$$T_j^n = \begin{cases} \mathcal{O}(h^2) & \text{if } \zeta \neq 1/6 \\ \mathcal{O}(h^4) & \text{if } \zeta = 1/6\,. \end{cases}$$

3. Heat transfer in a finite-length, straight, cylindrical body of varying area where the temperature is assumed to vary only with axial distance and time, can be reduced to a non-dimensional problem for temperature $u(x,t)$ given by

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(A(x)\frac{\partial u}{\partial x}\right), \quad 0 < x < 1,$$

$$u(0,t) = u_L(t), \quad u(1,t) = u_R(t), \quad u(x,0) = u_0(x),$$

where $A(x)$ is a non-dimensional cross-sectional area of the cylinder.

Determine an explicit Euler discretisation using central differences for the two cases

(a) where $A'(x)$ is known and the equation written

$$\frac{\partial u}{\partial t} = A(x)\frac{\partial^2 u}{\partial x^2} + A'(x)\frac{\partial u}{\partial x},$$

(b) where $A'(x)$ is either not known or too complicated to evaluate, either analytically or numerically.

(c) Explain how you would use a fictitious point to treat the boundary condition at $x = 0$ if it were changed to $u_x(0, t) = v(t)$, where v is a given function.

d) Write a matlab routine using Explicit Euler to calculate values for the case where there are Dirichlet boundary conditions $u(0, t) = \sin \pi t$ and $u(1, t) = 0$ and initial condition $u(x, 0) = 0$ for $0 \leq t \leq 4$. Use your code to determine a value for $u(0.5, 3)$ using $N = 21, 51, 101, 201$ space mesh points. Note that as this is an explicit scheme, you will need to think about stability limits and if the time step is very small, you may not want to attempt to store all previous computed time steps. Compute solutions for $a(x) = 2 - x,\ 0 \leq x \leq 1$.

4. The function $u(x, t)$, $x \in \mathbb{R}$, $t \geq 0$ satisfies $u(x, 0) = u_0(x)$ and $u_t = u_{xx}$, $t > 0$.

A numerical solution is found using an explicit Euler method with central space differences on a mesh $h, \Delta t$ in the x, t directions respectively such that $U_r^n \approx u_r^n = u(rh, n\Delta t)$. Suppose that $N = 1/\Delta t$ and $M = 1/h$ are integers.

Suppose the initial value function is

$$u_0(x) = A \cos(k_1 x) + B \cos(k_2 x).$$

Determine the numerical solution U_r^n, the exact solution $u(x, t)$ and the restriction of the continuous solution to the mesh to give u_r^n. For the case $B = 0$, determine the error $e_0^N = u_0^N - U_0^N$ at $t = 1$. Investigate the asymptotic behaviour of this error, e_0^N, as $\Delta t \to 0$ and as $h \to 0$.

You may find it useful to use that, for some constant $0 < a \leq N$,

$$(1 - \frac{a}{N})^N = \exp\Big(-N \sum_{r=1}^{\infty} \frac{a^r}{rN^r}\Big).$$

5. (2005 Finals)

Consider the initial value problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad -\infty < x < \infty, \quad 0 < t \leq T,$$

$$u(x, 0) = u_0(x), \qquad -\infty < x < \infty,$$

where T is a fixed real number, and u_0 is a real-valued, bounded and continuous function of $x \in (-\infty, \infty)$.

(a) (5 marks) Formulate the θ scheme for the numerical solution of this initial value problem on a mesh of uniform spacing $h > 0$ and $\Delta t = T/M$ in the x and t co-ordinate directions, respectively, where M is a positive integer. You should state the scheme so that $\theta = 0$ corresponds to the explicit (forward) Euler scheme.

(b) (10 marks) Let U_r^n denote the θ-scheme approximation to $u(rh, n\Delta t)$, $0 \leq n \leq M$, $r \in \mathbb{Z}$, where $\mathbb{Z}$ denotes the set of all integers. Suppose that $\|U^0\|_{\ell_\infty} = \max_{r \in \mathbb{Z}} |U_r^0|$ is finite. Show that if $\theta \in [0, 1]$, then

$$\|U^n\|_{\ell_\infty} \leq \|U^0\|_{\ell_\infty}$$

for all m, $1 \le m \le M$, provided that $A(\theta)\Delta t \le h^2$, where $A(\theta)$ is a constant, depending on the choice of θ, which you should determine. Deduce that the implicit (backward) Euler scheme is unconditionally stable in the $\|\cdot\|_{\ell_\infty}$ norm.

(c) (10 marks) Let U_r^n denote the θ-scheme approximation to $u(rh, n\Delta t)$, $0 \le n \le M$, $r \in \mathbb{Z}$, where $\mathbb{Z}$ denotes the set of all integers. Suppose that

$$\|U^0\|_{\ell_2} = \Big(h \sum_{r\in\mathbb{Z}} |U_r^0|^2\Big)^{1/2}$$

is finite. Show that if $\theta \in [\frac{1}{2}, 1]$, then

$$\|U^n\|_{\ell_2} \le \|U^0\|_{\ell_2}$$

for all m, $1 \le m \le M$, for any Δt and h.

Now suppose that $\theta \in [0, \frac{1}{2})$. Show that $\|U^n\|_{\ell_2} \le \|U^0\|_{\ell_2}$ if, and only if, $B(\theta)\Delta t \le h^2$, where $B(\theta)$ is a constant, depending on the choice of θ, which you should determine.