Numerical Solution of Differential Equations: Problem Sheet 4

1. A linear multistep method $\sum_{j=0}^{k} \alpha_j U_{n+j} = \Delta t \sum_{j=0}^{k} \beta_j f(t_{n+j}, U_{n+j})$, $n \geq 0$, for the numerical solution of the initial value problem $u' = f(t,u)$, $u(t_0) = U_0$, on the mesh $\{t_j : t_j = t_0 + j\Delta t\}$ of uniform spacing $\Delta t > 0$ is said to be *absolutely stable* for a certain Δt if, when applied to the model problem $u' = \lambda u$, $u(0) = 1$, with $\lambda < 0$, on the interval $t \in [0, \infty)$, the sequence $(|U_n|)_{n \geq k}$ decays exponentially fast; i.e., $|U_n| \leq \mathrm{C}\,\mathrm{e}^{-\mu n}$, $n \geq k$, for some positive constants C and μ.

   a) Show that a linear multistep method is absolutely stable for $\Delta t > 0$ if and only if all roots z of its *stability polynomial* $\pi(z; \overline{\Delta t}) = \rho(z) - \overline{\Delta t}\sigma(z)$, where ρ and σ are the first and second characteristic polynomial of the linear multistep method respectively and $\overline{\Delta t} = \lambda \Delta t$, belong to the open unit disk $D = \{z : |z| < 1\}$ in the complex plane.

   b) For each of the following methods find the range of $\Delta t > 0$ for which it is absolutely stable (when applied to $u' = \lambda u$, $u(0) = 1$, $\lambda < 0$, $t \in [0, \infty)$):

      b1) $U_{n+1} - U_n = \Delta t f(t_n, U_n)\,;$

      b2) $U_{n+1} - U_n = \Delta t f(t_{n+1}, U_{n+1})\,;$

      b3) $U_{n+2} - U_n = \frac{1}{3}\Delta t\,[f(t_{n+2}, U_{n+2}) + 4f(t_{n+1}, U_{n+1}) + f(t_n, U_n)]\,.$

2. Consider the θ–method

$$U_{n+1} = U_n + \Delta t\,[(1-\theta)F_n + \theta F_{n+1}]$$

   for $\theta \in [0, 1]$.

   a) Show that the method is A–stable for $\theta \in [1/2, 1]$.

   b) A method is said to be $A(\alpha)$–stable, $\alpha \in (0, \pi/2)$, if its region of absolute stability (as a set in the complex plane), contains the infinite wedge $\{\overline{\Delta t} : \pi - \alpha < \arg(\overline{\Delta t}) < \pi + \alpha\}$. Find all $\theta \in [0, 1]$ such that the θ–method is $A(\alpha)$–stable for some $\alpha \in (0, \pi/2)$.

3. Show that the second-order backward differentiation method

$$3U_{n+2} - 4U_{n+1} + U_n = 2\Delta t f(t_{n+2}, U_{n+2})$$

   is A-stable.

   In this question you will find it helpful to exploit the following result, known as *Schur's criterion* (which you are not expected to prove).

   Consider the polynomial $\phi(z) = c_k z^k + \ldots + c_1 z + c_0$, $c_k \neq 0$, $c_0 \neq 0$, with complex coefficients. The polynomial ϕ is said to be a *Schur polynomial* if each of its roots z_j satisfies $|z_j| < 1$, $j = 1, \ldots, k$.

   Given the polynomial $\phi(z)$, as above, consider the polynomial

$$\hat{\phi}(z) = \bar{c}_0 z^k + \bar{c}_1 z^{k-1} + \ldots + \bar{c}_{k-1} z + \bar{c}_k\ ,$$

where $\bar{c}_j$ denotes the complex conjugate of c_j, $j = 1, \ldots, k$. Further, define

$$\phi_1(z) = \frac{1}{z}\left[\hat{\phi}(0)\phi(z) - \phi(0)\hat{\phi}(z)\right] .$$

Clearly ϕ_1 has degree $\leq k-1$.

Then the polynomial ϕ is a Schur polynomial if and only if $|\hat{\phi}(0)| > |\phi(0)|$ and ϕ_1 is a Schur polynomial.

4. a) Determine the order of the linear multistep method

$$25U_{n+4} - 48U_{n+3} + 36U_{n+2} - 16U_{n+1} + 3U_n = 12\Delta t F_{n+4}.$$

Use matlab to help find the roots of $\rho(z) = 0$ and hence show that the method is zero stable. [If you want to try to work on the roots algebraically, you can assume that there are two real roots and a pair of complex conjugate roots for $\rho(z) = 0$ but you are still likely to need matlab or a calculator.]

b) Show that the scheme in (a) is absolutely stable as $\mathrm{Re}[\lambda\Delta t] \to -\infty$.

c) One way which can sometimes be used to determine the region of absolute stability is having written the polynomials in (b) as

$$\rho(z) = h\sigma(z),$$

where $h = \lambda\Delta t$, to consider the locus of points in the h-plane where $|z| = 1$. Write $z = \exp(\imath s)$, $-\pi \leq s \leq \pi$, $h = \rho(\mathrm{e}^{is})/\sigma(\mathrm{e}^{is}) = x(s) + \imath y(s)$ and work out explicit expressions for $x(s)$ and $y(s)$. By calculating where $\frac{\mathrm{d}x}{\mathrm{d}s} = 0$, show that the scheme is not A-stable but that it is $A(\alpha)$ stable and estimate an upper bound for α. Use matlab to plot the locus and confirm that your analytic estimate is in agreement with the calculated plot.

5. (Finals 2010) Consider the linear multistep method

$$U_{n+3} + aU_{n+1} + bU_n = \Delta t f(t_{n+2}, U_{n+2}), \quad n = 0, 1, \ldots \quad (*)$$

for the numerical solution of the initial-value problem

$$u'(t) = f(t, u(t)), \quad u(0) = u_0,$$

on the uniform mesh $\{t_n = n\Delta t \ : \ n = 0, 1, 2, \ldots\}$ of spacing $\Delta t > 0$, where U_0, U_1,U_2 are given real numbers, a and b are real parameters, to be chosen, and prime $'$ denotes the derivative with respect to t.

(a) Show that there exists a unique choice of parameters a and b such that the linear multistep method (*) is consistent. What is the order of accuracy of the method for this choice of a and b?

(b) What does it mean to say that a linear multistep method is *zero-stable*?

By applying the linear multistep method (*) with a and b as in part (a) to the initial value problem $u'(t) = 0$, $u(0) = 1$, show that there exist starting values U_1 and U_2, such that the sequence $(U_n)_{n=0}^{\infty}$, generated by the linear multistep method from the starting values $U_0 = 1$,U_1, U_2, satisfies the inequality

$$|U_n| \geq \left(\frac{3}{2}\right)^n \quad \text{for all } n \geq 0. \qquad (**)$$

Deduce from (**) that if a linear multistep method of the form (*) is consistent, then it is not zero stable.

(c) What does it mean to say that a linear multistep method is *convergent*?

Show that there exist no values of a and b such that the linear multistep method (*) is convergent.

[*Any theorem that you refer to must be stated carefully.*]