Numerical Solution of Differential Equations: Problem Sheet 3

1. An alternative way to look at a one step numerical solution of the scalar equation

$$\dot{u} = f(u), \quad u(0) = u_0,$$

written as

$$U_{n+1} = U_n + \Phi(U_n, \Delta t), \quad U_0 = u_0,$$

is to consider which differential equation

$$\dot{v} = g(v, \Delta t), \quad v(0) = u_0,$$

has the discrete values as an exact solution: that is a function v that satisfies:

$$v(t_{n+1}) = U_{n+1},$$

and in particular, $v_1 = v(\Delta t) = U_1$. This can be explored by writing

$$\Phi(u, \Delta t) = \Delta t \Phi_1(u) + \Delta t^2 \Phi_2(u) + \cdots,$$

and assuming that g can be expanded:

$$g(v, \Delta t) = g_1(v) + \Delta t g_2(v) + \Delta t^2 g_3(v) + \cdots.$$

Show that consistency as $\Delta t \to 0$ requires that

$$\Phi_1(u) = f(u), \text{ and } g_1(u) = f(u).$$

Derive that

$$g_2(u) = \Phi_2(u) - \frac{1}{2} f(u) f'(u).$$

Apply this method to the differential equation:

$$\dot{u} = u^2, \ t > 0, \ u(0) = u_0.$$

Deduce that while u will have a finite time singularity, v will not provided $\Delta t > 0$ [you do not need to solve the differential equation for v for this deduction]. Use matlab to apply forward Euler to both the differential equation for u and the differential equation for v in the case $u_0 = 0.1$, $\Delta t = 0.02$ and $0 < t < 15$. Compare with the exact solution $u(t) = u_0/(1 - u_0 t)$.

2. Show that there is a value of the parameter b such that the linear multistep method defined by the formula $U_{n+3} + (2b-3)(U_{n+2} - U_{n+1}) - U_n = b\Delta t(F_{n+2} + F_{n+1})$ is fourth order accurate (with $F_n = f(t_n, U_n)$). Show further that the method is *not* zero–stable for this value of b.

3. a) Suppose $\dot{u} = f(u)$ and given data (t_{n-2}, F_{n-2}), (t_{n-1}, F_{n-1}) and (t_n, F_n) (for simplicity, assume $t_r = r\Delta t$, and let $F_r = f(U_r)$). Determine the quadratic that passes through these points, that is determine $\tilde{F}(t)$ satisfying $\tilde{F}(t_k) = U_k$, $k = n-2, \ n-1, \ n$. Integrate this quadratic over the interval (t_n, t_{n+1}) to derive the explicit Adams prediction for U_{n+1}.

b) Recalculate the quadratic so that it interpolates F_{n-1}, F_n and F_{n+1} at t_{n-1}, t_n and t_{n+1} respectively. Integrate this function over the same interval to determine the implicit Adams prediction for U_{n+1}.

c) Use matlab to implement the two step explicit scheme and apply to the equation

$$\frac{\mathrm{d}u}{\mathrm{d}t} = i * u, \ 0 \leq t \leq \pi, \text{ with } u(0) = i,$$

using uniform time steps $\Delta t = \pi/N$, $N = 10, \ 100, \ 1000$. You will need to decide how to determine U_1 and U_2 in order to start the three step scheme. Graph the value of $\log_{10} \mathcal{R}[U_N]$ versus $\log_{10}(N^{-1})$ and estimate the order of convergence to the exact value of $\mathcal{R}[u(\pi)]$. Compare the convergence rate with theoretical expectation. [And keep in mind that while our subscript varies $n = 0, 1, \cdots, N$, vector indices in matlab will vary $1, 2, \cdots, N+1$].

4. (Finals 2012, I have extended the hint at the end)

The function $u(t)$, $t \geq 0$ is determined by

$$\frac{\mathrm{d}u}{\mathrm{d}t} = f(t, u), \quad t > 0,$$

where f is a uniformly continuous function of its arguments and $u(0) = u_0$,

A linear multistep method for numerical approximation of this equation at the points $t_r = r\Delta t, \ r = 0, 1, 2, \ldots,$ with $\Delta t > 0$ is defined for integer $k > 0$ by

$$\sum_{r=0}^{k} \alpha_r U_{n+r} = \Delta t \sum_{r=0}^{k} \beta_r F_{n+r}, \ n = 0, 1, \ldots,$$

where U_n is an approximation to $u_n = u(t_n)$, $F_n = f(t_n, U_n)$, $\alpha_k \neq 0$, $\alpha_0^2 + \beta_0^2 > 0$ and $\sum_{r=0}^{k} \beta_r \neq 0$. Denote polynomials

$$\rho(z) = \sum_{r=0}^{k} \alpha_r z^r, \quad \sigma(z) = \sum_{r=0}^{k} \beta_r z^r.$$

a) [6 Marks] Define the terms: truncation error, convergence, consistency, order of convergence and zero stability for a linear multistep method.

b) [8 Marks] In the case where the roots of ρ are all simple, prove that a necessary condition for convergence is that all roots lie in $\{z \in \mathcal{C} \ : \ |z| \leq 1\}$.

c) [4 Marks] Show that consistency requires

$$\rho(1) = 0, \quad \rho'(1) = \sigma(1).$$

d) [7 Marks] Suppose the linear multistep method is of order p. By considering the function $u(t) = \mathrm{e}^t$, (so that $f(t, u) = u$) and the associated truncation error, prove that the function $\psi(z)$ defined by

$$\psi(z) = \frac{\rho(z)}{\ln z} - \sigma(z),$$

has a p-fold zero at $z = 1$

[Hint: that is, $\psi \sim (z-1)^p$ as $z \to 1$. Consider specifically $z = \mathrm{e}^{\Delta t}$ as $\Delta t \to 0$.]