Numerical Solution of Differential Equations I: Problem Sheet 2

1. Consider the Runge–Kutta method

$$\frac{U_{n+1} - U_n}{\Delta t} = (c_1\, k_1 + c_2\, k_2),$$

where

$$k_1 = f(t_n, U_n),$$

and

$$k_2 = f(t_n + b_{2,1}\Delta t, U_n + b_{2,1}\Delta t k_1),$$

and where c_1, c_2, $b_{2,1}$ are real parameters.

   a) Show that there is a choice of these parameters such that the truncation error of the method,

$$T_n = \frac{u_{n+1} - u_n}{\Delta t} - [c_1 f(t_n, u_n) + c_2 f(t_n + b_{2,1}\Delta t, u_n + b_{2,1}\Delta t f(t_n, u_n))],$$

   is order 2 as $\Delta t \to 0$.

   b) Suppose that a second-order method of the above form is applied to the initial value problem $u' = -\lambda u$, $u(0) = 1$, where λ is a positive real number. Show that the sequence $\{U_n\}_{n\geq 0}$ is bounded if and only if $\Delta t \leq \frac{2}{\lambda}$.

   c) Use the analytic solution to $u' = -\lambda u$, $u(0) = 1$, namely $u(t) = \exp(-\lambda t)$, to show further that, for such λ,

$$|u_n - U_n| \leq \frac{1}{6}\lambda^3 \Delta t^2 t_n\ , \quad n \geq 0\ .$$

2. For the system $\mathbf{u}' = \mathbf{f}(\mathbf{u})$, Improved Euler is

$$\mathbf{U}_{n+1} \quad = \quad \mathbf{U}_n + \frac{1}{2}\Delta t[\mathbf{f}(\mathbf{U}_n) + \mathbf{f}(\mathbf{U}_n + \Delta t \mathbf{f}(\mathbf{U}_n))].$$

In the case of a spring system (or Van der Pol when $\alpha = 0$),

$$\mathbf{u} = \begin{pmatrix} u \\ v \end{pmatrix}, \quad \mathbf{f} = \begin{pmatrix} v \\ -u \end{pmatrix}.$$

Show that in this system, Improved Euler iterates satisfy

$$U_{n+1}^2 + V_{n+1}^2 = (1 + \frac{1}{4}\Delta t^4)(U_n^2 + V_n^2). \qquad (*)$$

By modifying your matlab routine to calculate solutions to the van der Pol equation from the last sheet, implement Improved Euler for this harmonic oscillator. Estimate analytically how many steps are needed for the numerical solution starting from $u_0 = 1,\ v_0 = 0$ to increase beyond a circle of radius 2 in the u, v phase plane for cases $\Delta t = 0.1,\ 0.05$. Use a `while` construction in matlab together with a loop counter rather than a fixed `do` construction to compare the number of matlab iterations with the theoretical result.

3. For the spring system in the previous question, show that the hybrid system

$$\begin{aligned} V_{n+1} &= V_n - \Delta t\ U_n, \\ U_{n+1} &= U_n + \Delta t\ V_{n+1}, \end{aligned}$$

does not preserve the Hamiltonian $H(u,v) = \frac{1}{2}(u^2+v^2)$ but that it does preserve the modified Hamiltonian

$$\hat{H}(u,v) = \frac{1}{2}(u^2+v^2) - \frac{1}{2}\Delta t uv.$$

Determine the eigenvalues and eigenvectors of the matrix

$$\begin{pmatrix} 1 & -\frac{\Delta t}{2} \\ -\frac{\Delta t}{2} & 1 \end{pmatrix}.$$

Use these to explain the trajectory of the numerical solution which passes through $(1,0)$ and to predict the variation in H which should occur for the numerical solution when Δt is small. Sketch the trajectories (u,v) and (U_n, V_n) which pass through $(1,0)$ and relate the sketch to the eigenvalues and eigenvectors just determined.

Modify your matlab routine from the previous question to implement this alternative scheme. For the same cases as before, $\Delta t = 0.1$ and $\Delta t = 0.05$, starting from $u_0 = 1,\ v_0 = 0$, calculate $H_{max} = \frac{1}{2}\max(U_n^2 + V_n^2)$ and $H_{min} = \frac{1}{2}\min(U_n^2 + V_n^2)$ for a few periods, for example $t \leq 30$. Compare with the range predicted for the maximum and minimum values of H.

4. The choice of which equation for initial update in the scheme in the previous question was arbitrary. Show that this alternative scheme for the same differential system:

$$\begin{aligned} U_{n+1} &= U_n + \Delta t\ V_n, \\ V_{n+1} &= V_n - \Delta t\ U_{n+1}, \end{aligned}$$

also preserves area in the (u,v) plane (you need only consider the area of a simple rectangle in (u,v) space), and preserves a modified Hamiltonian, which you should determine (you may assume the modified Hamiltonian is a homogeneous second order polynomial in u, v).

5. (Finals 2009) Consider the initial value problem $u' = f(t,u)$, $u(0) = u_0$, on the non empty closed interval $[0, X]$ of the real line, where f is a smooth function of its arguments and u_0 is a given real number. On a uniform mesh

$$\{t_n \ :\ t_n = n\Delta t, n = 0, \ldots, N\}$$

of spacing $\Delta t = X/N$, $N \geq 1$, the solution to the in initial value problem is approximated at the mesh points $(t_n)_{n=1}^N$ by the sequence $(U_n)_{n=1}^N$, defined by the one step method

$$U_{n+1} = U_n + \frac{\Delta t}{2}[f(t_n, U_n) + f(t_{n+1}, U_n + \Delta t f(t_n, U_n))],\ n = 0, \ldots N-1,$$

where $U_0 = u_0$. Let $u_n = u(t_n)$, $n = 1, \ldots, N$.

a) [3+6 marks] Define the Truncation error $T_n,\ n = 1, \ldots, N-1$, of this method. Show that

$$T_n = \mathcal{O}(\Delta t^2) \quad \text{as} \quad \Delta t \to 0.$$

b) [8 marks] Suppose that there exists a positive constant L such that

$$|f(t,u) - f(t,v)| \leq L|u - v|$$

for all $t \geq 0$ and all $u, v \in \mathcal{R}$. Show that there exists a positive constant $K = K(L, X)$ such that

$$\max_{1 \leq n \leq N} |u_n - U_n| \leq K \max_{0 \leq n \leq N-1} |T_n|.$$

Hence deduce that the method is second-order convergent.

c) [4+4 marks] Suppose that the method is applied to the initial-value problem $u' = \lambda u$, $u(0) = 1$, where $\lambda < 0$ is a fixed constant. The solution $u(t) = \mathrm{e}^{\lambda t}$ to this initial-value problem is strictly monotonic decreasing. Find the set of all $\Delta t > 0$ such that the corresponding sequence of numerical approximations $(U_n)_{n=n}^{N}$ with $U_0 = 1$, is strictly monotonic decreasing.

Discuss the practical implications of the resulting restriction on the mesh-size Δt when λ is negative and $|\lambda| >> 1$.