Numerical Solution of Differential Equations I: Problem Sheet 1

1. Let the real function $u(t)$, defined for $t \in [0, \infty)$ satisfy the differential equation
$$u' = \frac{\mathrm{d}u}{\mathrm{d}t} = f(t, u), \quad t > 0,$$
with initial condition $u(0)$ given.

   Verify that the following functions satisfy a Lipschitz condition with respect to u, uniformly in t, for $0 \le t \le \infty,\ u \in \mathbb{R}$:

   a) $f(t, u) = 2u(1 + t)^{-4}$;
   b) $f(t, u) = \mathrm{e}^{-(1+t)^2} \tan^{-1} u$;
   c) $f(t, u) = 2u(1 + u^2)^{-1}(1 + \mathrm{e}^{-|t|})$

   and for each case determine a bound for the truncation error when using Euler's method to approximate $u(t)$ at equally spaced points $t_n = n\Delta t$, where $\Delta t > 0$ and $n = 0, 1, 2, \ldots$. In (a) you may assume that the solution u is bounded with $|u| \le u_{max}$ for some positive $u_{max} \in \mathbb{R}$.

2. Suppose that m is a fixed positive integer. Show that the initial value problem
$$u' = u^{2m/(2m+1)} , \qquad u(0) = 0 ,$$
has infinitely many continuously differentiable solutions. Why does this not contradict Picard's Theorem?

3. Van der Pol's equation
$$u'' - \varepsilon(1 - u^2)u' + u = 0$$
subject to the initial conditions $u(0) = a_1$ and $u'(0) = a_2$, where a_1 and a_2 are given real numbers, and $\varepsilon > 0$ a parameter, models electrical circuits connected with electronic oscillators. Rewrite the equation as a coupled system of two first–order differential equations with appropriate initial conditions. Formulate Euler's method for this system, when $\varepsilon = 1$, $a_1 = 1/2$ and $a_2 = 1/2$, on the interval $[0, T]$, for some $T > 0$ using n points with uniform spacing $\Delta t = 1/(n - 1)$. Evaluate algebraically the Euler approximation to $u(t)$ and $u'(t)$ at the point $t = \Delta t$.

   Use matlab to calculate the solution using Euler's method and graph the results for $T = 20$ and $n = 101,\ 1001,\ 10001$ for $\varepsilon = 1,\ \varepsilon = 5$.

4. Consider the initial value problem
$$u' = \log\log(4 + u^2) , \qquad t \in [0, 1] , \qquad u(0) = 1 ,$$
and the sequence $(U_n)_{n=0}^{N}$, $N \ge 1$, generated by the explicit Euler method
$$\frac{U_{n+1} - U_n}{\Delta t} = \log\log(4 + U_n^2) , \qquad n = 0, \ldots, N - 1 , \qquad U_0 = 1 ,$$
using the time points $t_n = n\Delta t$, $n = 0, \ldots, N$, with spacing $\Delta t = 1/N$. Here log denotes the logarithm with base e.

a) Let T_n denote the truncation error of Euler's method at $t = t_n$ for this initial value problem. Show that $|T_n| \leq \Delta t/(4\mathrm{e})$.

b) Verify that

$$|u_{n+1} - U_{n+1}| \leq (1 + \Delta t L)|u_n - U_n| + \Delta t|T_n| \,, \qquad n = 0, \ldots, N-1 \,,$$

where $L = 1/(2\log 4)$.

c) Let $e_n = u_n - U_n$. Prove by induction that

$$|e_n| \leq (1 + \Delta t L)^n |e_0| + [(1 + \Delta t L)^n - 1]\frac{T}{L}.$$

d) Find a positive integer N_0, as small as possible, such that

$$\max_{0 \leq n \leq N} |u_n - U_n| \leq 10^{-4}$$

whenever $N \geq N_0$.

5. [2005 Finals]

Consider the initial value problem $u' = f(u)$, $u(0) = 1$, where $f(u) = \tan^{-1}(1 + u^2)$. [You may assume that this problem has a unique solution $t \to u(t)$, defined for all $t \in \mathcal{R}$ and that the functions u' and u'' are defined and continuous for all $t \in \mathcal{R}$.]

a) [8 marks] Show that $|u''| \leq \frac{\pi}{4}$ for all $t \in \mathcal{R}$. Show further that the function f satisfies the following Lipschitz condition:

$$|f(u) - f(v)| \leq \frac{1}{2}|u - v| \quad \forall u, v \in \mathcal{R}.$$

b) [8 marks] The implicit Euler approximation U_n to $u_n = u(t_n)$ on the mesh $\{t_n \ : \ t_n = n\Delta t, \ n = 0, 1, \ldots\}$ of uniform spacing $\Delta t \in (0, 1]$, is obtained from the formula

$$\frac{U_n - U_{n-1}}{\Delta t} = f(U_n). \quad n = 1, 2, 3, \ldots, \quad U_0 = 1.$$

Let $g(u) = u - \Delta t f(u)$. Show that the function g is strictly monotonic increasing and $\lim_{u \to \pm\infty} g(u) = \pm\infty$. By rewriting Euler's method as $g(U_n) = U_{n-1}$, deduce that, given $U_{n-1} \in \mathcal{R}$, the Euler approximation U_n is uniquely defined in $\mathcal{R}$

c) [9 marks] Show that the truncation error T_n of the implicit Euler method applied to the initial value problem under consideration satisfies

$$|T_n| \leq \frac{\pi}{8}\Delta t, \quad n = 1, 2, 3, \ldots.$$

Show further that

$$|u_n - U_n| \leq \frac{2}{2 - \Delta t}|u_{n-1} - U_{n-1}| + \frac{2\Delta t}{2 - \Delta t}|T_n|, \quad n = 1, 2, \ldots,$$

and deduce that

$$|u_n - U_n| \leq \frac{\pi}{4}\left[\left(1 + \frac{\Delta t}{2 - \Delta t}\right)^n - 1\right]\Delta t, \quad n = 1, 2, \ldots.$$

Show that if $\Delta t \leq [25\pi(\mathrm{e} - 1)]^{-1}$, then U_n approximates u_n to within 10^{-2} for all $t_n \in [0, 1]$. [You may use without proof that $(1 + \frac{\Delta t}{2 - \Delta t})^n \leq \mathrm{e}^{t_n}$ for all $\Delta t \in (0, 1]$ and all $n \geq 0$.]