Part 4: Active-set methods for linearly constrained optimization

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$$\underset{x \in \mathbb{R}^n}{\text{minimize}} \quad f(x) \;\text{ subject to }\; Ax \geq b$$

Part C course on continuoue optimization

LINEARLY CONSTRAINED MINIMIZATION

$$\underset{x \in \mathbb{R}^n}{\text{minimize}} \;\; f(x) \;\text{ subject to }\; Ax \left\{ \begin{array}{c} \geq \\ = \end{array} \right\} b$$

where the **objective function** $f : \mathbb{R}^n \longrightarrow \mathbb{R}$

- ⊙ assume that $f \in C^1$ (sometimes C^2) and Lipschitz
- ⊙ often in practice this assumption violated, but not necessary
- ⊙ important special cases:
  - ⋄ **linear programming**: $f(x) = g^T x$
  - ⋄ **quadratic programming**: $f(x) = g^T x + \frac{1}{2} x^T H x$

Concentrate here on quadratic programming

QUADRATIC PROGRAMMING

$$\textbf{QP}: \quad \underset{x \in \mathbb{R}^n}{\text{minimize}} \;\; q(x) = g^T x + \tfrac{1}{2} x^T H x \;\; \text{subject to} \;\; Ax \geq b$$

- H is n by n, real symmetric, $g \in \mathbb{R}^n$
- $A = \begin{pmatrix} a_1^T \\ \vdots \\ a_m^T \end{pmatrix}$ is m by n real, $b = \begin{pmatrix} [b]_1 \\ \vdots \\ [b]_m \end{pmatrix}$
- in general, constraints may
  - have upper bounds: $b^l \leq Ax \leq b^u$
  - include equalities: $A^e x = b^e$
  - involve simple bounds: $x^l \leq x \leq x^u$
  - include network constraints ...

PROBLEM TYPES

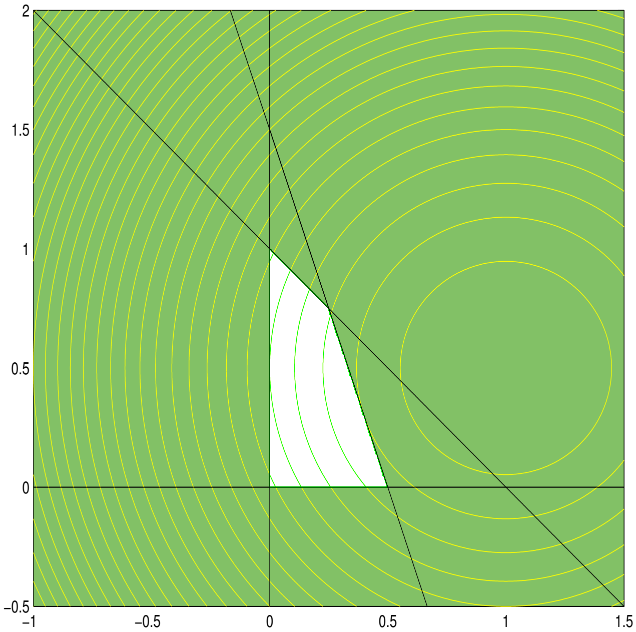
Convex problems

- H is positive semi-definite ($x^T H x \geq 0$ for all x)
- any local minimizer is global
- important special case: $H = 0 \iff$ linear programming

Strictly convex problems

- H is positive definite ($x^T H x > 0$ for all $x \neq 0$)
- unique minimizer (if any)

CONVEX EXAMPLE



Contours of objective function

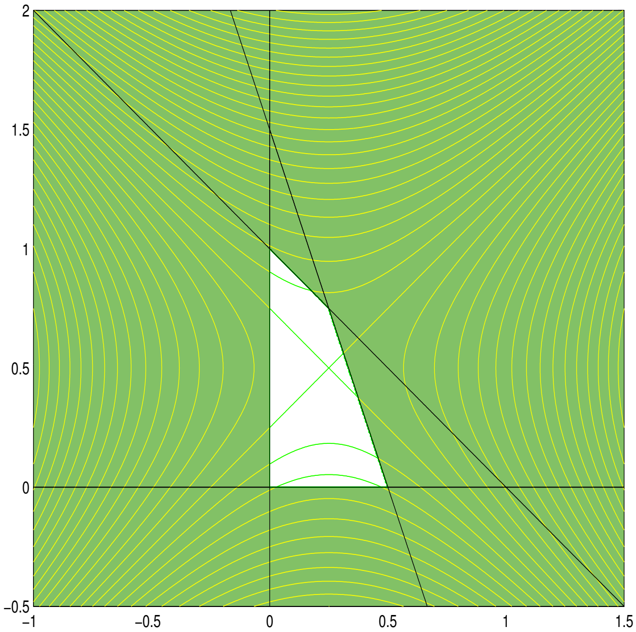
$$\min (x_1 - 1)^2 + (x_2 - 0.5)^2$$
$$\text{subject to } \; x_1 + x_2 \leq 1$$
$$3x_1 + x_2 \leq 1.5$$
$$(x_1, x_2) \geq 0$$

PROBLEM TYPES (II)

General (non-convex) problems

- H may be indefinite ($x^T H x < 0$ for some x)
- may be many local minimizers
- may have to be content with a local minimizer
- problem may be unbounded from below

NON-CONVEX EXAMPLE



Contours of objective function

$$\begin{gathered} \min -2(x_1-0.25)^2+2(x_2-0.5)^2 \\ \text{subject to } \; x_1+x_2 \leq 1 \\ 3x_1+x_2 \leq 1.5 \\ (x_1, x_2) \geq 0 \end{gathered}$$

PROBLEM TYPES (III)

Small

- values/structure of matrix data H and A irrelevant
- currently $\min(m,\ n) = O(10^2)$

Large

- values/structure of matrix data H and A important
- currently $\min(m,\ n) \geq O(10^3)$

Huge

- factorizations involving H and A are unrealistic
- currently $\min(m,\ n) \geq O(10^5)$

WHY IS QP SO IMPORTANT?

- many **applications**
  - portfolio analysis, structural analysis, VLSI design, discrete-time stabilization, optimal and fuzzy control, finite impulse response design, optimal power flow, economic dispatch ...
  - $\sim$ 500 application papers
- **prototypical** nonlinear programming problem
- **basic subproblem** in constrained optimization:

$$\begin{array}{ll} \underset{x \in \mathbb{R}^n}{\text{minimize}} & f(x) \\ \text{subject to} & c(x) \geq 0 \end{array} \quad \Longrightarrow \quad \begin{array}{ll} \underset{x \in \mathbb{R}^n}{\text{minimize}} & f + g^T x + \frac{1}{2} x^T H x \\ \text{subject to} & Ax + c \geq 0 \end{array}$$

$\Longrightarrow$ SQP methods ($\Longrightarrow$ Course Part 7)

OPTIMALITY CONDITIONS

Recall: the importance of optimality conditions is:

- to be able to recognise a solution if found by accident or design
- to guide the development of algorithms

FIRST-ORDER OPTIMALITY

$$\text{QP:} \quad \underset{x \in \mathbb{R}^n}{\text{minimize}} \;\; q(x) = g^T x + \tfrac{1}{2} x^T H x \;\; \text{subject to} \;\; Ax \geq b$$

Any point x_* that satisfies the conditions

$$\begin{array}{rl}
Ax_* \geq b & (\textbf{primal feasibility}) \\
Hx_* + g - A^T y_* = 0 \;\; \text{and} \;\; y_* \geq 0 & (\textbf{dual feasibility}) \\
[Ax_* - b]_i \cdot [y_*]_i = 0 \;\; \text{for all} \;\; i & (\textbf{complementary slackness})
\end{array}$$

for some vector of **Lagrange multipliers** y_* is a **first-order critical** (or Karush-Kuhn-Tucker) point

If $[Ax_* - b]_i = 0 \iff [y_*]_i > 0$ for all $i \implies$ the solution is **strictly complementary**

SECOND-ORDER OPTIMALITY

$$\text{QP:} \quad \underset{x \in \mathbb{R}^n}{\text{minimize}} \;\; q(x) = g^T x + \tfrac{1}{2} x^T H x \;\; \text{subject to} \;\; Ax \geq b$$

Let

$$\mathcal{N}_+ = \left\{ s \;\middle|\; \begin{array}{l} a_i^T s = 0 \;\; \text{for all} \;\; i \;\; \text{such that} \;\; a_i^T x_* = [b]_i \;\; \text{and} \;\; [y_*]_i > 0 \;\; \text{and} \\ a_i^T s \geq 0 \;\; \text{for all} \;\; i \;\; \text{such that} \;\; a_i^T x_* = [b]_i \;\; \text{and} \;\; [y_*]_i = 0 \end{array} \right\}$$

Any first-order critical point x_* for which additionally

$$s^T H s \geq 0 \;\; (\text{resp.} > 0) \;\; \text{for all} \;\; s \in \mathcal{N}_+$$

is a **second-order** (resp. **strong second-order**) critical point

Theorem 4.1: x_* is a (an isolated) local minimizer of QP $\iff$ x_* is (strong) second-order critical

WEAK SECOND-ORDER OPTIMALITY

$$\text{QP:} \quad \underset{x \in \mathbb{R}^n}{\text{minimize}} \;\; q(x) = g^T x + \tfrac{1}{2} x^T H x \;\; \text{subject to} \;\; Ax \geq b$$

Let

$$\mathcal{N} = \left\{ s \mid a_i^T s = 0 \;\; \text{for all} \;\; i \;\; \text{such that} \;\; a_i^T x_* = [b]_i \right\}$$

Any first-order critical point x_* for which additionally

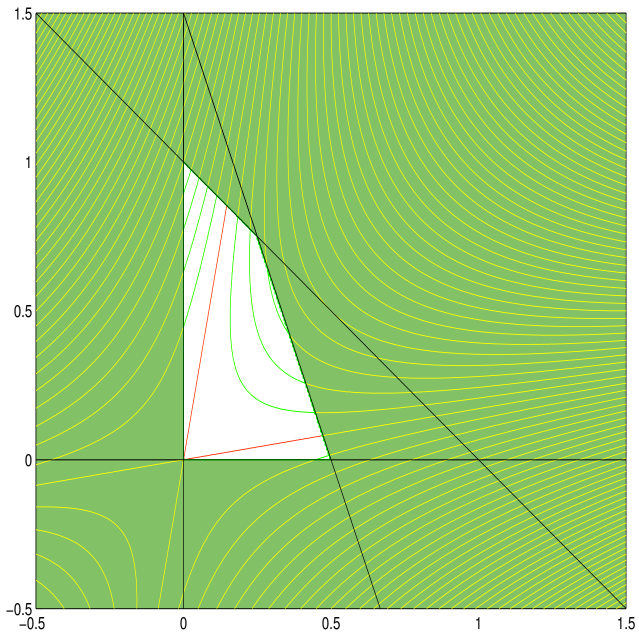
$$s^T H s \geq 0 \;\; \text{for all} \;\; s \in \mathcal{N}$$

is a **weak** second-order critical point

Note that

- a weak second-order critical point may be a maximizer!
- checking for weak second-order criticality is easy (strong is hard)

NON-CONVEX EXAMPLE



$$\begin{gathered} \min x_1^2 + x_2^2 - 6x_1x_2 \\ \text{subject to} \;\; x_1 + x_2 \leq 1 \\ 3x_1 + x_2 \leq 1.5 \\ (x_1, x_2) \geq 0 \end{gathered}$$

Contours of objective function:
note that escaping from the origin may be difficult!

[DUALITY

$$\text{QP:} \quad \underset{x \in \mathbb{R}^n}{\text{minimize}} \;\; q(x) = g^T x + \tfrac{1}{2} x^T H x \;\; \text{subject to} \;\; Ax \geq b$$

If QP is convex, any first-order critical point is a global minimizer

If H is strictly convex, the problem

$$\underset{y \in \mathbb{R}^m,\; y \geq 0}{\text{maximize}} \quad -\tfrac{1}{2} g^T H^{-1} g + (AH^{-1}g + b)^T y - \tfrac{1}{2} y^T A H^{-1} A^T y$$

is known as the **dual** of QP

- QP is the **primal**
- primal and dual have same KKT conditions
- if primal is feasible, optimal value of primal = optimal value dual
- can be generalized for simply convex case]

ALGORITHMS

Essentially two classes of methods (slight simplification)

active set methods :

primal active set methods aim for dual feasibility while maintaining primal feasibility and complementary slackness

dual active set methods aim for primal feasibility while maintaining dual feasibility and complementary slackness

interior-point methods : aim for complementary slackness while maintaining primal and dual feasibility ($\Longrightarrow$ Course Part 6)

EQUALITY CONSTRAINED QP

The basic subproblem in all of the methods we will consider is

$$\textbf{EQP}: \quad \underset{x \in \mathrm{IR}^n}{\text{minimize}} \;\; g^T x + \tfrac{1}{2} x^T H x \;\; \text{subject to} \;\; Ax = 0 \longleftarrow \boxed{\textbf{N.B.}}$$

Assume A is m by n, full-rank (preprocess if necessary)

⊙ First-order optimality (Lagrange multipliers y)

$$\begin{pmatrix} H & A^T \\ A & 0 \end{pmatrix} \begin{pmatrix} x \\ -y \end{pmatrix} = \begin{pmatrix} -g \\ 0 \end{pmatrix}$$

⊙ Second-order necessary optimality:
$s^T H s \geq 0$ for all s for which $As = 0$

⊙ Second-order sufficient optimality:
$s^T H s > 0$ for all $s \neq 0$ for which $As = 0$

EQUALITY CONSTRAINED QP (II)

$$\text{EQP}: \quad \underset{x \in \mathrm{IR}^n}{\text{minimize}} \;\; q(x) = g^T x + \tfrac{1}{2} x^T H x \;\; \text{subject to} \;\; Ax = 0$$

Four possibilities:

(i)
$$\begin{pmatrix} H & A^T \\ A & 0 \end{pmatrix} \begin{pmatrix} x \\ -y \end{pmatrix} = \begin{pmatrix} -g \\ 0 \end{pmatrix} \qquad (*)$$

and H is second-order sufficient $\Longrightarrow$ **unique** minimizer x

(ii) $(*)$ holds, H is second-order necessary, but $\exists s$ such that $Hs = 0$ and $As = 0 \Longrightarrow$ family of **weak** minimizers $x + \alpha s$ for any $\alpha \in \mathrm{IR}$

(iii) $\exists s$ for which $As = 0$, $Hs = 0$ and $g^T s < 0 \Longrightarrow$
$q(\cdot)$ unbounded along **direction of linear infinite descent** s

(iv) $\exists s$ for which $As = 0$ and $s^T H s < 0 \Longrightarrow$
$q(\cdot)$ unbounded along **direction of negative curvature** s

CLASSIFICATION OF EQP METHODS

Aim to solve

$$\begin{pmatrix} H & A^T \\ A & 0 \end{pmatrix} \begin{pmatrix} x \\ -y \end{pmatrix} = \begin{pmatrix} -g \\ 0 \end{pmatrix}$$

Three basic approaches:

full-space approach

range-space approach

null-space approach

For each of these can use

direct (factorization) method

iterative (conjugate-gradient) method

FULL-SPACE/KKT/AUGMENTED SYSTEM APPROACH

$$\begin{pmatrix} H & A^T \\ A & 0 \end{pmatrix} \begin{pmatrix} x \\ -y \end{pmatrix} = \begin{pmatrix} -g \\ 0 \end{pmatrix}$$

- **KKT matrix**

  $$K = \begin{pmatrix} H & A^T \\ A & 0 \end{pmatrix}$$

  is symmetric, indefinite $\Longrightarrow$ use Bunch-Parlett type factorization
  - $K = PLBL^TP^T$
  - P permutation, L unit lower-triangular
  - B block diagonal with 1x1 and 2x2 blocks
- LAPACK for small problems, `MA27`/`MA57` for large ones
- **Theorem 4.2**: H is second-order sufficient $\Longleftrightarrow$
  K non-singular and has precisely m negative eigenvalues

RANGE-SPACE APPROACH

$$\begin{pmatrix} H & A^T \\ A & 0 \end{pmatrix} \begin{pmatrix} x \\ -y \end{pmatrix} = \begin{pmatrix} -g \\ 0 \end{pmatrix} \qquad (*)$$

For **non-singular** H

- ⊙ eliminate x using first block of $(*) \Longrightarrow$

$$AH^{-1}A^T y = AH^{-1}g \text{ followed by } Hx = -g + A^T y$$

- ⊙ strictly convex case $\Longrightarrow$ H and $AH^{-1}A^T$ positive definite $\Longrightarrow$ Cholesky factorization
- ⊙ **Theorem 4.3**: H is second-order sufficient $\Longleftrightarrow$ H and $AH^{-1}A^T$ have same number of negative eigenvalues
- ⊙ $AH^{-1}A^T$ usually dense $\Longrightarrow$ factorization only for small m

NULL-SPACE APPROACH

$$\begin{pmatrix} H & A^T \\ A & 0 \end{pmatrix} \begin{pmatrix} x \\ -y \end{pmatrix} = \begin{pmatrix} -g \\ 0 \end{pmatrix} \qquad (*)$$

- ⊙ let n by $n-m$ S be a **basis** for null-space of $A \Longrightarrow AS = 0$
- ⊙ second block $(*) \Longrightarrow x = Sx_N$
- ⊙ premultiply first block $(*)$ by $S^T \Longrightarrow$

$$S^T HSx_S = -S^T g$$

- ⊙ **Theorem 4.4**: H is second-order sufficient $\Longleftrightarrow$ $S^T HS$ is positive definite $\Longrightarrow$ Cholesky factorization
- ⊙ $S^T HS$ usually dense $\Longrightarrow$ factorization only for small $n-m$

NULL-SPACE BASIS

Require n by $n-m$ null-space basis S for $A \implies AS = 0$

Non-orthogonal basis: let $A = (A_1 \;\; A_2)P$

- ⊙ P permutation, A_1 non-singular

$$\implies S = P^T \begin{pmatrix} -A_1^{-1}A_2 \\ I \end{pmatrix}$$

- ⊙ generally suitable for large problems. Best A_1?

Orthogonal basis: let $A = (L \;\; 0)Q$

- ⊙ L non-singular (e.g., triangular), $Q = \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix}$ orthonormal

$$\implies S = Q_2^T$$

- ⊙ more stable but … generally unsuitable for large problems

[ITERATIVE METHODS FOR SYMMETRIC LINEAR SYSTEMS

$$Bx = b$$

Best methods are based on finding solutions from the **Krylov space**

$$\mathcal{K} = \{r^0, Br^0, B(Br^0), \ldots\} \qquad\qquad (r^0 = b - Bx^0)$$

B **indefinite:** use MINRES method

B **positive definite:** use conjugate gradient method

- ⊙ usually satisfactory to find approximation rather than exact solution
- ⊙ usually try to **precondition** system, i.e., solve

$$C^{-1}Bx = C^{-1}b$$

where $C^{-1}B \approx I$]

[ITERATIVE RANGE-SPACE APPROACH

$$AH^{-1}A^T y = AH^{-1}g \text{ followed by } Hx = -g + A^T y$$

For strictly convex case $\Longrightarrow$ H and $AH^{-1}A^T$ positive definite

H^{-1} **available:** (directly or via factors),
use conjugate gradients to solve $AH^{-1}A^T y = AH^{-1}g$

- ⊙ matrix vector product $AH^{-1}A^T v = \left(A\left(H^{-1}(A^T v)\right)\right)$
- ⊙ preconditioning? Need to approximate (likely dense) $AH^{-1}A^T$

H^{-1} **not available:** use composite conjugate gradient method (Urzawa's method) iterating both on solutions to

$$AH^{-1}A^T y = AH^{-1}g \quad \text{and} \quad Hx = -g + A^T y$$

at the same time (may not converge)]

[ITERATIVE NULL-SPACE APPROACH

$$S^T H S x_N = -S^T g \text{ followed by } x = S x_N$$

- ⊙ use conjugate gradient method
  - ⋄ matrix vector product $S^T H S v_N = \left(S^T\left(H(S v_N)\right)\right)$
  - ⋄ preconditioning? Need to approximate (likely dense) $S^T H S$
  - ⋄ if we encounter s_N such that $s_N^T(S^T H S)s_N < 0 \Longrightarrow s = N s_N$ is a direction of negative curvature since $As = 0$ and $s^T H s < 0$
  - ⋄ **Advantage**: $Ax^{\text{approx}} = 0$]

[ITERATIVE FULL-SPACE APPROACH

$$\begin{pmatrix} H & A^T \\ A & 0 \end{pmatrix} \begin{pmatrix} x \\ -y \end{pmatrix} = \begin{pmatrix} -g \\ 0 \end{pmatrix}$$

- use MINRES with the preconditioner

$$\begin{pmatrix} M & 0 \\ 0 & AN^{-1}A^T \end{pmatrix}$$

  where M and $N \approx H$.

  - **Disadvantage**: $Ax^{\text{approx}} \neq 0$

- use conjugate gradients with the preconditioner

$$\begin{pmatrix} M & A^T \\ A & 0 \end{pmatrix}$$

  where $M \approx H$.

  - **Advantage**: $Ax^{\text{approx}} = 0$]

ACTIVE SET ALGORITHMS

$$\text{QP:} \quad \underset{x \in \mathbb{R}^n}{\text{minimize}} \; q(x) = g^T x + \tfrac{1}{2} x^T H x \; \text{ subject to } \; Ax \geq b$$

The **active set** $\mathcal{A}(x)$ at x is

$$\mathcal{A}(x) = \{ i \mid a_i^T x = [b]_i \}$$

If x_* solves QP, we have

$$\begin{aligned} & \arg\min q(x) \;\text{ subject to }\; Ax \geq b \\ \equiv \; & \arg\min q(x) \;\text{ subject to }\; a_i^T x = [b]_i \;\text{ for all }\; i \in \mathcal{A}(x_*) \end{aligned}$$

A **working set** $\mathcal{W}(x)$ at x is a **subset of the active set** for which the vectors $\{a_i\}$, $i \in \mathcal{W}(x)$ are linearly independent

BASICS OF ACTIVE SET ALGORITHMS

Basic idea: Pick a subset $\mathcal{W}_k$ of $\{1, \ldots, m\}$ and find

$$x_{k+1} = \arg\min q(x) \;\text{ subject to }\; a_i^T x = [b]_i \;\text{ for all }\; i \in \mathcal{W}_k$$

If x_{k+1} does not solve QP, adjust $\mathcal{W}_k$ to form $\mathcal{W}_{k+1}$ and repeat

Important issues are:

- ⊙ how do we know if x_{k+1} solves QP ?
- ⊙ if x_{k+1} does not solve QP, how do we pick the next working set $\mathcal{W}_{k+1}$?

Notation: rows of A_k are those of A indexed by $\mathcal{W}_k$
components of b_k are those of b indexed by $\mathcal{W}_k$

PRIMAL ACTIVE SET ALGORITHMS

Important feature: ensure all iterates are feasible, i.e., $Ax_k \geq b$

If $\mathcal{W}_k \subseteq \mathcal{A}(x_k)$
$\Longrightarrow$ $A_k x_k = b_k$ and $A_k x_{k+1} = b_k$
$\Longrightarrow$ $x_{k+1} = x_k + s_k$, where

$$\begin{aligned} s_k &= \arg\min \mathrm{EQP}_k \\ &= \arg \underbrace{\min q(x_k + s) \;\text{ subject to }\; A_k s = 0}_{\text{equality constrained problem}} \end{aligned}$$

Need an initial feasible point x_0

PRIMAL ACTIVE SET ALGORITHMS — ADDING CONSTRAINTS

$$s_k = \arg\min q(x_k + s) \;\text{ subject to }\; A_k s = 0$$

What if $x_k + s_k$ is not feasible?

- $\odot$ a currently inactive constraint j must become active at $x_k + \alpha_k s_k$ for some $\alpha_k < 1$ — pick the smallest such α_k
- $\odot$ move instead to $x_{k+1} = x_k + \alpha_k s_k$ and set $\mathcal{W}_{k+1} = \mathcal{W}_k + \{j\}$

PRIMAL ACTIVE SET ALGORITHMS — DELETING CONSTRAINTS

What if $x_{k+1} = x_k + s_k$ is feasible ? $\Longrightarrow$

$$x_{k+1} = \arg\min q(x) \;\text{ subject to }\; a_i^T x = [b]_i \;\text{ for all }\; i \in \mathcal{W}_k$$

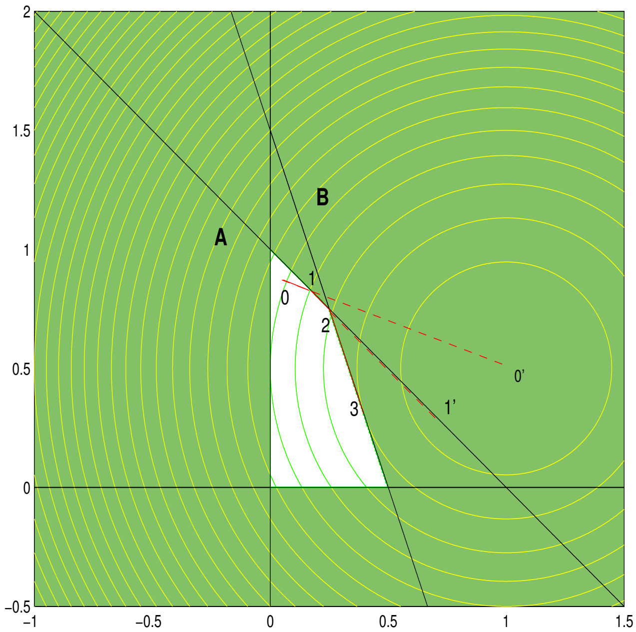
$\Longrightarrow$ $\exists$ Lagrange multipliers y_{k+1} such that

$$\begin{pmatrix} H & A_k^T \\ A_k & 0 \end{pmatrix} \begin{pmatrix} x_{k+1} \\ -y_{k+1} \end{pmatrix} = \begin{pmatrix} -g \\ b_k \end{pmatrix}$$

Three possibilities:

- $\odot$ $q(x_{k+1}) = -\infty$ (not strictly-convex case only)
- $\odot$ $y_{k+1} \geq 0 \Longrightarrow x_{k+1}$ is a first-order critical point of QP
- $\odot$ $[y_{k+1}]_i < 0$ for some $i \Longrightarrow q(x)$ may be improved by considering $\mathcal{W}_{k+1} = \mathcal{W}_k \setminus \{j\}$, where j is the i-th member of $\mathcal{W}_k$

ACTIVE-SET APPROACH



0. Starting point
0'. Unconstrained minimizer
1. Encounter constraint A
1'. Minimizer on constraint A
2. Encounter constraint B, move off constraint A
3. Minimizer on constraint B = required solution

LINEAR ALGEBRA

Need to solve a sequence of EQP$_k$s in which

$$\text{either}\quad \mathcal{W}_{k+1} = \mathcal{W}_k + \{j\} \quad\Longrightarrow\quad A_{k+1} = \begin{pmatrix} A_k \\ a_j^T \end{pmatrix}$$

$$\text{or}\quad \mathcal{W}_{k+1} = \mathcal{W}_k \setminus \{j\} \quad\Longrightarrow\quad A_k = \begin{pmatrix} A_{k+1} \\ a_j^T \end{pmatrix}$$

Since working sets change gradually, aim to **update** factorizations rather than compute afresh

RANGE-SPACE APPROACH — MATRIX UPDATES

Need factors $L_{k+1}L_{k+1}^T = A_{k+1}H^{-1}A_{k+1}^T$ given $L_kL_k^T = A_kH^{-1}A_k^T$

When $A_{k+1} = \begin{pmatrix} A_k \\ a_j^T \end{pmatrix} \Longrightarrow$

$$A_{k+1}H^{-1}A_{k+1}^T = \begin{pmatrix} A_kH^{-1}A_k^T & A_kH^{-1}a_j \\ a_j^TH^{-1}A_k^T & a_j^TH^{-1}a_j \end{pmatrix}$$

$\Longrightarrow$

$$L_{k+1} = \begin{pmatrix} L_k & 0 \\ l^T & \lambda \end{pmatrix}$$

where

$$L_kl = A_kH^{-1}a_j \ \text{ and } \ \lambda = \sqrt{a_j^TH^{-1}a_j - l^Tl}$$

Essentially reverse this to remove a constraint

NULL-SPACE APPROACH — MATRIX UPDATES

Need factors $A_{k+1} = (L_{k+1} \ \ 0)Q_{k+1}$ given

$$A_k = (L_k \quad 0)Q_k = (L_k \ \ 0)\begin{pmatrix} Q_{1\,k} \\ Q_{2\,k} \end{pmatrix}$$

To add a constraint (to remove is similar)

$$\begin{aligned} A_{k+1} &= \begin{pmatrix} A_k \\ a_j^T \end{pmatrix} = \begin{pmatrix} L_k & 0 \\ a_j^TQ_{1\,k}^T & a_j^TQ_{2\,k}^T \end{pmatrix} Q_k \\ &= \begin{pmatrix} L_k & 0 \\ a_j^TQ_{1\,k}^T & a_j^TQ_{2\,k}^T \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & U^T \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & U \end{pmatrix} Q_k \\ &= \underbrace{\left[\begin{pmatrix} L_k & 0 \\ a_j^TQ_{1\,k}^T & \sigma e_1^T \end{pmatrix}\right]}_{(L_{k+1}\ \ 0)} \underbrace{\left[\begin{pmatrix} I & 0 \\ 0 & U \end{pmatrix} Q_k\right]}_{Q_{k+1}} \end{aligned}$$

where the Householder matrix U reduces $Q_{2\,k}a_j$ to $\sigma e_1 = \begin{pmatrix} \sigma \\ 0 \end{pmatrix}$

[FULL-SPACE APPROACH — MATRIX UPDATES

$\mathcal{W}_k$ becomes $\mathcal{W}_\ell \Longrightarrow A_k = \begin{pmatrix} A_C \\ A_D \end{pmatrix}$ becomes $A_\ell = \begin{pmatrix} A_C \\ A_A \end{pmatrix}$

Solving

$$\begin{pmatrix} H & A_\ell^T \\ A_\ell & 0 \end{pmatrix} \begin{pmatrix} s_\ell \\ -y_\ell \end{pmatrix} = \begin{pmatrix} g_\ell \\ 0 \end{pmatrix} \Longrightarrow$$

$$\begin{pmatrix} H & A_k^T \\ A_k & 0 \end{pmatrix} \longleftarrow \begin{pmatrix} \boxed{\begin{matrix} H & A_C^T & A_D^T \\ A_C & 0 & 0 \\ A_D & 0 & 0 \end{matrix}} & \begin{matrix} A_A^T & 0 \\ 0 & 0 \\ 0 & I \end{matrix} \\ \begin{matrix} A_A & 0 & 0 \\ 0 & 0 & I \end{matrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \end{pmatrix} \begin{pmatrix} s_\ell \\ -y_C \\ -y_D \\ -y_A \\ u_\ell \end{pmatrix} = \begin{pmatrix} g_\ell \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix};$$

$$y_\ell = \begin{pmatrix} y_C \\ y_A \end{pmatrix}$$

. . .]

[FULL-SPACE APPROACH — MATRIX UPDATES (CONT.)

. . . can solve

$$\begin{pmatrix} H & A_k^T \\ A_k & 0 \end{pmatrix} \longleftarrow \begin{pmatrix} \boxed{\begin{matrix} H & A_C^T & A_D^T \\ A_C & 0 & 0 \\ A_D & 0 & 0 \end{matrix}} & \begin{matrix} A_A^T & 0 \\ 0 & 0 \\ 0 & I \end{matrix} \\ \begin{matrix} A_A & 0 & 0 \\ 0 & 0 & I \end{matrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \end{pmatrix} \begin{pmatrix} s_\ell \\ -y_C \\ -y_D \\ -y_A \\ u_\ell \end{pmatrix} = \begin{pmatrix} g_\ell \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

using the factors of

$$K_k = \begin{pmatrix} H & A_k^T \\ A_k & 0 \end{pmatrix}$$

and the **Schur complement**

$$S_\ell = -\begin{pmatrix} A_A & 0 & 0 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} H & A_k^T \\ A_k & 0 \end{pmatrix}^{-1} \begin{pmatrix} A_A^T & 0 \\ 0 & 0 \\ 0 & I \end{pmatrix}$$

. . .]

[SCHUR COMPLEMENT UPDATING

- Major iteration starts with factorization of

$$K_k = \begin{pmatrix} H & A_k^T \\ A_k & 0 \end{pmatrix}$$

- As $\mathcal{W}_k$ changes to $\mathcal{W}_\ell$, factorization of

$$S_\ell = -\begin{pmatrix} A_A & 0 & 0 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} H & A_k^T \\ A_k & 0 \end{pmatrix}^{-1} \begin{pmatrix} A_A^T & 0 \\ 0 & 0 \\ 0 & I \end{pmatrix}$$

  is **updated** not recomputed

- Once dim S_ℓ exceeds a given threshold, or it is cheaper to factorize/use K_ℓ than maintain/use K_k and S_ℓ, start the next major iteration]

PHASE-1

To find an initial feasible point x_0 such that $Ax_0 \geq b$

- use traditional (simplex) phase-1, or
- let $r = \min(b - Ax_{\text{guess}}, 0)$, and solve $[(x_0, \xi_0) = (x_{\text{guess}}, 1)]$

$$\underset{x \in \mathbb{R}^n,\, \xi \in \mathbb{R}}{\text{minimize}} \;\; \xi \;\; \text{subject to} \;\; Ax + \xi r \geq b \;\; \text{and} \;\; \xi \geq 0$$

Alternatively, use a single-phase method

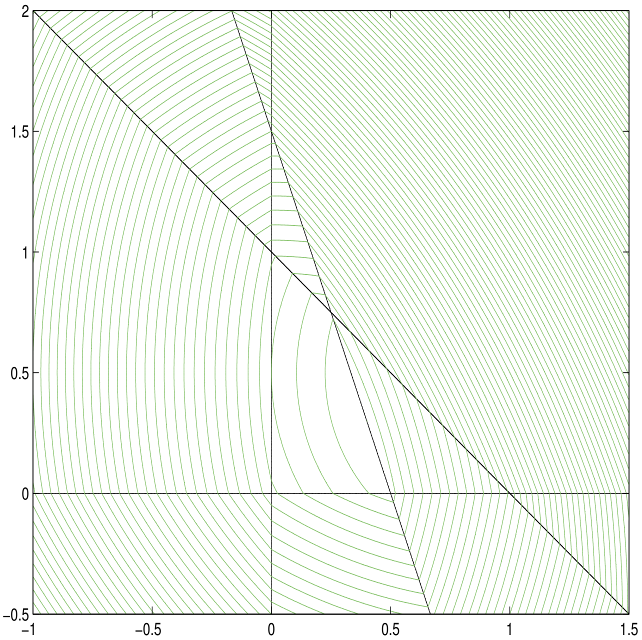
- Big-M: for some sufficiently large M

$$\underset{x \in \mathbb{R}^n,\, \xi \in \mathbb{R}}{\text{minimize}} \;\; q(x) + M\xi \;\; \text{subject to} \;\; Ax + \xi r \geq b \;\; \text{and} \;\; \xi \geq 0$$

- ℓ_1QP $(\rho > 0)$ — may be reformulated as a QP

$$\underset{x \in \mathbb{R}^n}{\text{minimize}} \;\; q(x) + \rho \| \max(b - Ax, 0) \|$$

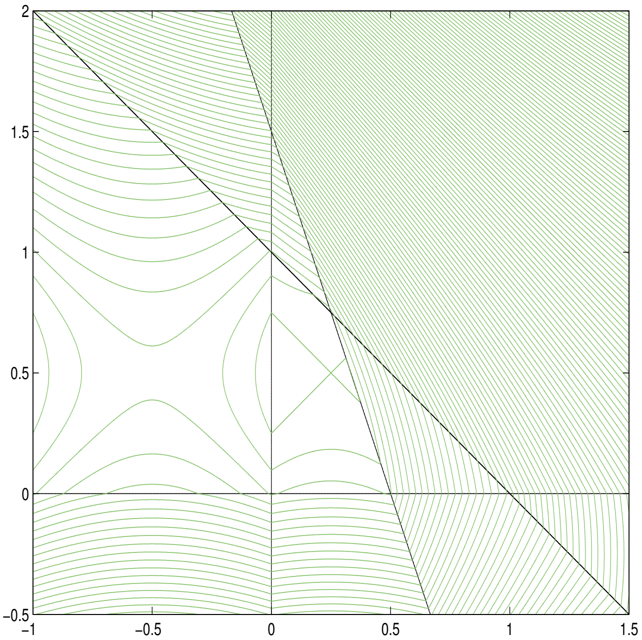
CONVEX EXAMPLE



$$\min (x_1-1)^2 + (x_2-0.5)^2$$
$$\text{subject to } \; x_1 + x_2 \le 1$$
$$3x_1 + x_2 \le 1.5$$
$$(x_1, x_2) \ge 0$$

Contours of penalty function $q(x) + \rho\|\max(b - Ax, 0)\|$ (with $\rho = 2$)

NON-CONVEX EXAMPLE



$$\min -2(x_1-0.25)^2 + 2(x_2-0.5)^2$$
$$\text{subject to } \; x_1 + x_2 \le 1$$
$$3x_1 + x_2 \le 1.5$$
$$(x_1, x_2) \ge 0$$

Contours of penalty function $q(x) + \rho\|\max(b - Ax, 0)\|$ (with $\rho = 3$)

TERMINATION, DEGENERACY & ANTI-CYCLING

So long as $\alpha_k > 0$, these methods are finite:

- finite number of steps to find an EQP with a feasible solution
- finite number of EQP with feasible solutions

If x_k is degenerate (active constraints are dependent) it is possible that $\alpha_k = 0$. If this happens infinitely often

- may make no progress (a cycle) $\Longrightarrow$ algorithm may stall

Various anti-cycling rules

- Wolfe's and lexicographic perturbations
- least-index — Bland's rule
- Fletcher's robust method

NON-CONVEXITY

- causes little extra difficulty so long as suitable factorizations are possible
- **Inertia-controlling** methods tolerate at most one negative eigenvalue in the reduced Hessian. Idea is
  1. start from working set on which problem is strictly convex (e.g., a vertex)
  2. if a negative eigenvalue appears, do not drop any further constraints until 1. is restored
  3. a direction of negative curvature is easy to obtain in 2.
- latest methods are not inertia controlling $\Longrightarrow$ more flexible

COMPLEXITY

- When the problem is convex, there are algorithms that will solve QP in a polynomial number of iterations
  - some interior-point algorithms are polynomial
  - no known polynomial active-set algorithm
- When the problem is non-convex, it is unlikely that there are polynomial algorithms
  - problem is NP complete
  - even verifying that a proposed solution is locally optimal is NP hard

NON-QUADRATIC OBJECTIVE

When $f(x)$ is **non quadratic**

- $H = H_k$ changes
- active-set subproblem

$$x_{k+1} \approx \arg\min f(x) \;\text{ subject to }\; a_i^T x = [b]_i \;\text{ for all }\; i \in \mathcal{W}_k$$

  - iteration now required but each step satisfies $A_k s = 0$ $\Longrightarrow$ linear algebra as before
  - usually solve subproblem inaccurately
    - when to stop?
    - which Lagrange multipliers in this case?
    - need to avoid zig-zagging in which working sets repeat