Finding a Positive Semidefinite Interval for a Parametric Matrix*

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ABSTRACT

Let C and E be symmetric (n,n)-matrices such that C is positive semidefinite and E is of rank one or two. This paper is concerned with finding real numbers $\underline{t} \leqslant 0$ and $\bar{t} \geqslant 0$ such that $C(t) = C + tE$ is positive semidefinite if and only if $t \in [\underline{t}, \bar{t}]$. Explicit expressions for $\underline{t}$ and $\bar{t}$ are derived, and a method for computing $\underline{t}$ and $\bar{t}$ is presented along with preliminary numerical experience.

1. INTRODUCTION

Let C and E be given symmetric (n,n)-matrices such that C is positive semidefinite and E is of rank one or two. This paper is concerned with finding real numbers $\underline{t} \leqslant 0$ and $\bar{t} \geqslant 0$ so that the parametric matrix

$$C(t) = C + tE$$

is *positive semidefinite* if and only if $t \in [\underline{t}, \bar{t}]$. It is assumed that

$$E = uu' + \lambda vv',$$

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where u and v are linearly independent n-vectors and $\lambda = 0$, 1, or -1. (Note: In general, a symmetric rank one or two matrix can be written as $\pm(uu' + \lambda vv')$. However, if $C + tE$ is positive semidefinite if and only if $t \in [\underline{t}, \bar{t}]$, then $C + t(-E)$ is positive semidefinite if and only if $t \in [-\bar{t}, -\underline{t}]$. Therefore, the assumption that $E = uu' + \lambda vv'$ causes no loss of generality.)

This problem arises in connection with the parametric Hessian quadratic programming problem [1]

$$\text{minimize} \quad \left\{c'x + \tfrac{1}{2}x'C(t)x \,|\, a_i'x \leqslant b_i,\ i = 1,\ldots,m\right\},$$

where $c, a_1, \ldots, a_m$ are n-vectors and $b_1, \ldots, b_m$ are scalars. The solution of the problem has applications in structural design and portfolio analysis.

Previous results have been given in association with quasi-Newton methods for the unconstrained minimization of functionals [2]. Such methods are concerned with choosing t and E such that if C is *positive definite* then $C(t)$ is also *positive definite*.

Section 2 contains background material and preliminary results. These results will be used in Section 3 to derive explicit expressions for $\underline{t}$ and $\bar{t}$. A method for computing $\underline{t}$ and $\bar{t}$ is given in Section 4, along with the results of limited numerical testing.

2. BACKGROUND AND PRELIMINARY RESULTS

This section presents various results concerning the matrix $C(t)$ and its eigenvalues. Lemma 2.1 is a variation of a result given by Pearson [3], Lemma 2.2 is due to Wilkinson [4], and Lemma 2.3 can be found in Noble and Daniel [5]. The proofs are omitted.

LEMMA 2.1. *If C is nonsingular, then $C(t)$ is nonsingular if and only if $\beta(t) \neq 0$, where*

$$\beta(t) = 1 + (u'x + \lambda v'y)t + \lambda\left[(u'x)(v'y) - (u'y)^2\right]t^2,$$

and where $x = C^{-1}u$ and $y = C^{-1}v$. Also, $\det(C(t)) = \det(C)\beta(t)$.

LEMMA 2.2. *Suppose that C has eigenvalues*

$$0 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_n,$$

and consider $\hat{C}(t) = C + t(uu')$. *If* $t \leqslant 0$ *is arbitrary but fixed, then* $\hat{C}(t)$ *has eigenvalues* $\lambda_i(t)$, $i = 1, \dots, n$, *such that*

$$\lambda_1(t) \leqslant \lambda_1 \leqslant \lambda_2(t) \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_n(t) \leqslant \lambda_n.$$

LEMMA 2.3. *Suppose that* C *has rank* $r \leqslant n$. *There exists an* (n, r)*-matrix* Q_1 *and an* $(n, n-r)$*-matrix* Q_2 *such that* $Q = [Q_1, Q_2]$ *is an orthogonal* (n, n)*-matrix satisfying*

$$Q'CQ = \begin{bmatrix} Q_1' \\ Q_2' \end{bmatrix} C [Q_1, Q_2] = \begin{bmatrix} C_1 & 0 \\ 0 & 0 \end{bmatrix}, \tag{1}$$

where C_1 *is a positive definite diagonal* (r, r)*-matrix whose diagonal elements are the nonzero eigenvalues of* C.

It follows from Lemma 2.3 that

$$Q'C(t)Q = \begin{bmatrix} C_1 + t(u_1u_1' + \lambda v_1 v_1') & t(u_1u_2' + \lambda v_1 v_2') \\ t(u_2u_1' + \lambda v_2 v_1') & t(u_2u_2' + \lambda v_2 v_2') \end{bmatrix}, \tag{2}$$

where $u_1 = Q_1'u$, $u_2 = Q_2'u$, $v_1 = Q_1'v$, and $v_2 = Q_2'v$. It is noted that if $u \in R(C)$, where $R(C)$ denotes the range space of C, then $u_2 = 0$. Similarly, $v \in R(C)$ implies $v_2 = 0$.

The next lemma will be used in Section 3 to express results obtained using (2) in terms of C, u, and v rather than C_1, u_1, and v_1.

LEMMA 2.4. *Let* C, Q, *and* C_1 *be as in Equation* (1), *and let* u *and* v *be any* n*-vectors in* $R(C)$. *Define* $u_1 = Q_1'u$ *and* $v_1 = Q_1'v$. *If* x^* *is the unique solution to* $C_1 x^* = u_1$, *then* $x^* = x_1$ *and*

$$v_1'x_1 = v'x,$$

where x *is any solution to* $Cx = u$ *and* $x_1 = Q_1'x$.

Proof. Let x be any solution to $Cx = u$. It follows that $Q'C(QQ')x = Q'u$, which implies that $C_1 x_1 = u_1$. Since C_1 is nonsingular, then $x^* = x_1$. Now,

$$v'x = v'QQ'x = v_1'x_1 + v_2'x_2 = v_1'x_1,$$

since $u, v \in R(C)$ implies $u_2 = v_2 = 0$. ∎

The expressions for $\underline{t}$ and $\bar{t}$ are now derived.

3. THE POSITIVE SEMIDEFINITE INTERVAL

The expressions for $\underline{t}$ and $\bar{t}$ are derived for six distinct cases. The cases are determined according to the choice for E and the relationship between C, u, and v.

Case 1: $E = uu' + \lambda vv'$; $u, v \in R(C)$

From Equation (2), with $u_2 = v_2 = 0$ [since $u, v \in R(C)$], it follows that $C(t)$ is positive semidefinite if and only if

$$C_1(t) = C_1 + t(u_1u_1' + \lambda v_1v_1')$$

is positive semidefinite. Let the eigenvalues of $C_1(t)$ be represented by $\lambda_1(t), \dots, \lambda_r(t)$ where $\lambda_i(0) > 0$, $i = 1, \dots, r$. It follows from Lemma 2.1 that

$$\det(C_1(t)) = \det(C_1)\beta_1(t) = \prod_{i=1}^{r} \lambda_i(t), \tag{3}$$

where

$$\beta_1(t) = 1 + (u_1'x_1 + \lambda v_1'y_1)t + \lambda\left[u_1'x_1v_1'y_1 - (u_1'y_1)^2\right]t^2,$$

and where $x_1 = C_1^{-1}u_1$ and $y_1 = C_1^{-1}v_1$. If $\lambda \neq 0$, the Cauchy-Schwarz inequality [6] implies that $\beta_1(t)$ has two distinct roots. Using Lemma 2.4, the roots can be written as

$$r_1 = 2\Big/\left\{ -u'x - \lambda v'y - \sqrt{(u'x - \lambda v'y)^2 + 4\lambda(u'y)^2} \right\},$$

$$r_2 = 2\Big/\left\{ -u'x - \lambda v'y + \sqrt{(u'x - \lambda v'y)^2 + 4\lambda(u'y)^2} \right\}.$$

It follows from (3) that $\lambda_i(t) = 0$ for some i if and only if $t = r_1$ or $t = r_2$. This, along with the continuity of the $\lambda_i(t)$ and the fact that $\lambda_i(0) > 0$ for $i = 1, \dots, r$, is used to determine $\underline{t}$ and $\bar{t}$ from r_1 and r_2.

If $\lambda = 1$, then $r_2 < r_1 < 0$. Define $\underline{t} = r_1$ and $\bar{t} = +\infty$. For $t \geqslant r_1$ it follows that $\lambda_i(t) \geqslant 0$, $i = 1, \dots, r$, and hence $C(t)$ is positive semidefinite if $t \in [\underline{t}, \bar{t}]$. For $r_2 < t < r_1$ we have $\beta_1(t) < 0$, so that there is some i with $\lambda_i(t) < 0$. Lemma 2.2 then implies that $\lambda_i(t) < 0$ for some i whenever $t < \underline{t}$. Thus, $C(t)$ is positive semidefinite only if $t \in [\underline{t}, \bar{t}]$.

If $\lambda = 0$, then $\beta_1(t)$ has the single root $r_1 = -1/u'x$. As in the case for $\lambda = 1$, set $\underline{t} = r_1$ and $\bar{t} = +\infty$.

If $\lambda = -1$, then $r_1 < 0 < r_2$. Define $\underline{t} = r_1$ and $\bar{t} = r_2$. In a manner analogous to that for $\lambda = 1$, it can be shown that $C(t)$ is positive semidefinite if and only if $t \in [\underline{t}, \bar{t}]$.

Case 2: $E = uu' + \lambda vv'$; $\lambda = 0, 1$; $u \notin R(C)$ or $v \notin R(C)$

First, it is noted that

$$x'C(t)x = x'Cx + t\left[(x'u)^2 + \lambda(x'v)^2\right],$$

which is nonnegative for all x and for all $t \geqslant 0$. Thus, $\bar{t} = +\infty$.

Now $\underline{t}$ is to be determined. Suppose that $u \in R(C)$ and $v \notin R(C)$. Since $v \notin R(C)$, there exists an n-vector x satisfying $Cx = 0$ and $v'x = 1$. Since $u \in R(C)$, then $u'x = 0$. Therefore, $x'C(t)x = t$, which implies that C is not positive semidefinite if $t < 0$. Hence $\underline{t} = 0$. Analogously, if $u \notin R(C)$ and $v \in R(C)$, then $\underline{t} = 0$.

Now suppose that both $u, v \notin R(C)$. Either $v \in R(C|u)$ or $v \notin R(C|u)$, where $(C|u)$ is the matrix formed by appending u to C. Suppose $v \in R(C|u)$; then there exists an n-vector s and a scalar α satisfying $Cs + \alpha u = v$. Since $u \notin R(C)$, there also exists a vector x with $Cx = 0$ and $u'x = 1$. Consequently, $v'x = \alpha$, which yields $x'C(t)x = (1 + \lambda\alpha^2)t$. Clearly, this implies that $\underline{t} = 0$. Now suppose that $v \notin R(C|u)$, so that there exists an n-vector x such that $Cx = 0$, $u'x = 0$, and $v'x = 1$. Therefore, $x'C(t)x = \lambda t$, which also gives $\underline{t} = 0$.

In conclusion, $C(t)$ is positive semidefinite if and only if $t \in [\underline{t}, \bar{t}]$, where $\underline{t} = 0$ and $\bar{t} = +\infty$. Note that these results hold if $\lambda = 0$ and $u \notin R(C)$.

Case 3: $E = uu' - vv'$; $u \notin R(C)$ and $v \in R(C)$

Since $\lambda = -1$, $u \notin R(C)$, and $v \in R(C)$, Equation (2) reduces to

$$Q'C(t)Q = \begin{bmatrix} C_1(t) & tu_1u_2' \\ tu_2u_1' & tu_2u_2' \end{bmatrix}. \tag{4}$$

Since $u_2 \neq 0$, there exists [7] an $(n-r, n-r)$-matrix $(Q_2^*)'$ such that $u_2'Q_2^* = [\beta, 0]$, where $\beta = \pm\sqrt{u_2'u_2}$. Define the orthogonal (n, n)-matrix Q^* by

$$Q^* = \begin{bmatrix} I & 0 \\ 0 & Q_2^* \end{bmatrix}.$$

It then follows from (4) that

$$(Q^*)'Q'C(t)QQ^* = \begin{bmatrix} C_2(t) & 0 \\ 0 & 0 \end{bmatrix}, \tag{5}$$

where $C_2(t)$ is the $(r+1, r+1)$-matrix given by

$$C_2(t) = \begin{bmatrix} C_1(t) & t\beta u_1 \\ t\beta u_1' & t\beta^2 \end{bmatrix}.$$

Now (5) implies that $C(t)$ is positive semidefinite if and only if $C_2(t)$ is positive semidefinite. The numbers $\underline{t}$ and $\bar{t}$ will be derived by examining the determinant of $C_2(t)$.

First, take the matrix-vector product of the first r columns of $C_2(t)$ with

$$w(t) = -t\beta [C_1(t)]^{-1} u_1,$$

and add it to the last column of $C_2(t)$. Since the determinant is invariant under this operation, it follows that

$$\det(C_2(t)) = \det \begin{bmatrix} C_1(t) & 0 \\ t\beta u_1' & t\beta^2\{1 - tu_1'[C_1(t)]^{-1}u_1\} \end{bmatrix}$$

$$= t\beta^2\{1 - tu_1'[C_1(t)]^{-1}u_1\} \det(C_1(t)). \tag{6}$$

It follows from Lemmas 2.1 and 2.4, [1, p. 6], and (6) that

$$\det(C_2(t)) = t\beta^2 \det(C_1)(1 - tv'y), \tag{7}$$

where y is any solution to $Cy = v$ and $v'y > 0$. From Equation (7) it is seen that $\det(C_2(t)) < 0$ whenever $t < 0$ or $t > 1/v'y$. This implies that $C(t)$ is positive semidefinite only if $t \in [\underline{t}, \bar{t}]$, where $\underline{t} = 0$ and $\bar{t} = 1/v'y$. It remains to show that $C(t)$ is positive semidefinite if $t \in [\underline{t}, \bar{t}]$. From (7), $\det(C_2(t)) = 0$ if and only if $t = \underline{t}$ or $t = \bar{t}$. The continuity of the eigenvalues of $C_2(t)$ and the fact that $C_2(0)$ has nonnegative eigenvalues implies that $C_2(t)$ has nonnegative eigenvalues whenever $t \in [\underline{t}, \bar{t}]$. Therefore, $C(t)$ is positive semidefinite if and only if $t \in [\underline{t}, \bar{t}]$, where $\underline{t} = 0$ and $\bar{t} = 1/v'y$.

Case 4: $E = uu' - vv'$; $u \in R(C)$, $v \notin R(C)$

This case is analogous to case 3. It can be shown that $\underline{t} = -1/u'x$ and $\bar{t} = 0$, where x is any solution to $Cx = u$.

Case 5: $E = uu' - vv'$; $u, v \notin R(C)$; $v \notin R(C|u)$

Since $v \notin R(C|u)$, there exists an n-vector x such that $Cx = 0$, $u'x = 0$, and $v'x = 1$. Therefore, $x'C(t)x = -t$, which implies that $\bar{t} = 0$. Since $v \notin R(C|u)$, then $u \notin R(C|v)$, which can be used to establish that $\underline{t} = 0$. Hence, C is positive semidefinite if and only if $t = 0$.

Case 6: $E = uu' - vv'$; $u, v \notin R(C)$; $v \in R(C|u)$

Since $u, v \notin R(C)$ and $v \in R(C|u)$, there exists a nonunique vector x and a unique scalar α such that $Cx + \alpha u = v$. Premultiply this equation with the Q_2' of (1) to get $v_2 = \alpha u_2$. Proceeding as in case 3, a matrix $C_2(t)$ is found that is positive semidefinite if and only if $C(t)$ is positive semidefinite. Also, it can be shown that

$$\det(C_2(t)) = t\beta^2 \det(C_1)\left[(1-\alpha^2) - t(v-\alpha u)'x\right], \tag{8}$$

where $(v - \alpha u)'x > 0$. Equation (8) is used to determine $[\underline{t}, \bar{t}]$. There are three possibilities. If $1 - \alpha^2 = 0$ then $\det(C_2(t)) < 0$ for all $t \neq 0$, which implies that $\underline{t} = \bar{t} = 0$. If $1 - \alpha^2 < 0$, it can be shown, in a manner analogous to case 3, that $\underline{t} = (1-\alpha^2)/(v-\alpha u)'x$ and $\bar{t} = 0$. Similarly, if $1 - \alpha^2 > 0$ then $\underline{t} = 0$ and $\bar{t} = (1-\alpha^2)/(v - \alpha u)'x$.

Expressions for $\underline{t}$ and $\bar{t}$ have been derived for all possible cases. The results are summarized in Table 1. The next section shows how $\underline{t}$ and $\bar{t}$ may be computed.

4. COMPUTATION OF THE INTERVAL

This section presents a method for the computation of $\underline{t}$ and $\bar{t}$. It also presents results of some limited numerical experience.

From Table 1 it is clear that to compute $\underline{t}$ and $\bar{t}$, it is only necessary to either find a solution, or show that no solution exists, to each of $Cx = u$, $Cy = v$, and $Cx + \alpha u = v$. The obvious complication is that, when C is singular, the numerical rank (and hence the range space) of C may be hard to determine [8]. There appear to be two approaches to tackling this problem. The first is to try to ensure that the rank of C is correctly identified. This

TABLE 1

THE INTERVAL ENDPOINTS

E	$Cx = u$?	$Cy = v$?	$Cx + \alpha u = v$?	Interval end points
$uu' + vv'$[a]	Yes	Yes		$\bar{t} = +\infty$,
				$\underline{t} = 2/\left\{-u'x - v'y - \sqrt{(u'x - v'y)^2 + 4(u'y)^2}\right\}$
	Otherwise			$\underline{t} = 0,\ \bar{t} = +\infty$
$uu' - vv'$	Yes	Yes		$\bar{t} = 2/\left\{-u'x + v'y + \sqrt{(u'x + v'y)^2 - 4(v'x)^2}\right\}$,
				$\underline{t} = 2/\left\{-u'x + v'y - \sqrt{(u'x + v'y)^2 - 4(v'x)^2}\right\}$
	Yes	No		$\underline{t} = -1/u'x,\ \bar{t} = 0$
	No	Yes		$\underline{t} = 0,\ \bar{t} = 1/v'y$
	No	No	No	$\underline{t} = 0,\ \bar{t} = 0$
	No	No	Yes	$1 - \alpha^2 = 0 \Rightarrow \underline{t} = 0,\ \bar{t} = 0$
				$1 - \alpha^2 > 0 \Rightarrow \underline{t} = 0,\ \bar{t} = (1 - \alpha^2)/(v - \alpha u)'x$
				$1 - \alpha^2 < 0 \Rightarrow \underline{t} = (1 - \alpha^2)/(v - \alpha u)'x,\ \bar{t} = 0$

[a] Includes $v = 0$.

approach is typified by methods which compute a spectral (singular value) decomposition of C [8, pp. 289–290]. Unfortunately, such methods tend to be expensive. The other approach is to use a less expensive factorization (such as a Cholesky factorization with symmetric pivoting) and hope that the rank is correctly identified. Although this latter approach is theoretically risky, it has proved satisfactory in practice (cf. using the QR factorization for rank deficient least squares problems [8, pp. 162–167]). This section makes use of a Cholesky factorization with symmetric pivoting.

Suppose that C has rank $r \leqslant n$. There exists [9] a nonunique permutation matrix P and a triangular matrix R (unique for a given P) such that $P'CP = R'R$, where

$$R = \begin{bmatrix} R_{11} & R_{12} \\ 0 & 0 \end{bmatrix},$$

and where R_{11} is an upper triangular (r, r)-matrix and R_{12} is an $(r, n - r)$-matrix.

The equation $Cx = u$ is considered first. The results for $Cy = v$ will be analogous. Let $x_p = P'x$ and $u_p = P'u$, so that solving $Cx = u$ is equivalent to solving $R'Rx_p = u_p$. Now set $y = Rx_p$ and solve

$$\begin{bmatrix} R'_{11} & 0 \\ R'_{12} & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} u_{p1} \\ u_{p2} \end{bmatrix},$$

where $y' = [y_1', y_2']$ and $u_p' = [u_{p1}', u_{p2}']$. Clearly, y_1 is uniquely determined by $R_{11}'y_1 = u_{p1}$, y_2 is undetermined, and $Cx = u$ has no solution if $R_{12}'y_1 = u_{p2}$. Suppose that $R_{12}'y_1 = u_{p2}$, and solve

$$\begin{bmatrix} R_{11} & R_{12} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_{p1} \\ x_{p2} \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix},$$

where $x_p' = [x_{p1}', x_{p2}']$. This implies that $y_2 = 0$ and that, for an arbitrary x_{p2}, x_{p1} is the unique solution to

$$R_{11}x_{p1} = y_1 - R_{12}x_{p2}. \tag{9}$$

Thus, if $R_{12}'y_1 = u_{p2}$, then $Cx = u$ has the nonunique solution $x = Px_p$, where x_{p2} is arbitrary and x_{p1} is determined by (9).

The equation $Cx + \alpha u = v$, where $u, v \notin R(C)$, is now considered. Define $v_p = P'v$ and consider the equivalent equation $R'Rx_p = v_p - \alpha u_p$. Set $y = Rx_p$, and consider

$$\begin{bmatrix} R_{11}' & 0 \\ R_{12}' & 0 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} v_{p1} \\ v_{p2} \end{bmatrix} - \alpha\begin{bmatrix} u_{p1} \\ u_{p2} \end{bmatrix}.$$

Set $y_1 = y_{1v} - \alpha y_{1u}$, where y_{1v} and y_{1u} are uniquely determined by $R_{11}'y_{1v} = v_{p1}$ and $R_{11}'y_{1u} = u_{p1}$, respectively. Clearly, $Cx + \alpha u = v$ has a solution if and only if $R_{12}'y_1 = v_{p2} - \alpha u_{p2}$ has a solution α. The latter equation is equivalent to $R_{12}'y_{1v} - v_{p2} = \alpha(R_{12}'y_{1u} - u_{p2})$, from which α can be determined. If α exists, then $Cx + \alpha u = v$ has the solution

$$\alpha = \frac{\left(R_{12}'y_{1v} - v_{p2}\right)_i}{\left(R_{12}'y_{1u} - u_{p2}\right)_i}, \qquad 1 \leqslant i \leqslant n - r,$$

where the subscript i denotes the ith component of the vectors, and

$$x = Px_p,$$

where $x_p' = [x_{p1}', x_{p2}']$, x_{p2} is arbitrary, and x_{p1} is the unique solution to

$$R_{11}x_{p1} = y_{1v} - \alpha y_{1u} - R_{12}x_{p2}.$$

TABLE 2
NUMERICAL RESULTS

Example	n	$\underline{t}$	$\bar{t}$	TL	TU
1	3	−1	3	−0.9999995	−2.999995
2	2	−9999.99999	∞	−10000.0039	∞
3	2	−0.00001	∞	−0.00001	∞
4	2	−0.00001	∞	−0.0000099	∞
5	3	−0.00007071	∞	−0.00007071	∞
6	5	−0.3498	1.0	−0.3498	1.0
7	5	−1.3498	0.0	−1.3498	0.0

In summary, given the Cholesky factorization $P'CP = R'R$ of the matrix C, it is possible to select the appropriate expressions for $\underline{t}$ and $\bar{t}$ from Table 1, and then to evaluate $\underline{t}$ and $\bar{t}$.

The above method for the computation of $\underline{t}$ and $\bar{t}$ has been implemented in the double precision FORTRAN subprogram DPSINT [10]. This subprogram uses the LINPACK [9] and BLAS [11] subprograms to perform matrix factorizations, solve linear equations, and calculate inner products.

Numerical experience with the DPSINT code is limited. There are three phases to the testing. In the first phase, DPSINT was used to solve thirteen examples in which C is a diagonal (5,5)-matrix with 0's and 1's along the diagonal. The examples were chosen to reflect the different possibilities in the choices of E and in the relationship between C, u, and v.

In the second phase, DPSINT solved seven examples obtained from [12]. The results are summarized in Table 2, where TL and TU are the computed values of $\underline{t}$ and $\bar{t}$, respectively. Example 7 is interesting in that the matrix C equals $C(\bar{t})$ from example 6.

In the third phase of testing, three unconstrained minimization problems where solved using the BFGS [2, p. 74] method. At each iteration of the BFGS method, the approximate second derivative matrix B is updated to B^*, via a rank two update formula of the form $B^* = B + uu' - vv'$, for appropriate vectors u and v. The testing involved finding the positive semidefinite interval for the parametric matrix $B^*(t) = B + t(uu' - vv')$. The optimization algorithm generated 84 examples for DPSINT. As expected, $1.0 \in (\underline{t}, \bar{t})$, which reflects the hereditary positive definite property of the BFGS update formula.

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