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# Weakly Singular Discrete Gronwall Inequalities

*Es werden Verallgemeinerungen der klassischen Gronwallschen Ungleichung für den Fall angegeben, daß der Kern der zugeordneten Integralgleichung schwach singulär ist. Es wird sowohl die stetige als auch die diskrete Version angegeben. Die stetige wurde mit einbezogen, weil sie eine analoge Behandlung der diskreten anregt. Die Arbeit wurde durch Konvergenzuntersuchungen von Diskretisierungsmethoden für Volterrasche Integral- und Integro-Differentialgleichungen motiviert. Sämtliche Ergebnisse werden in einer Form gegeben, die sehr brauchbar für Spezialisten in Numerischer Analysis ist.*

*Generalizations of the classical Gronwall inequality when the kernel of the associated integral equation is weakly singular are presented. The continuous and discrete versions are both given; the former is included since it suggests the latter by analogy. This work is motivated by convergence studies of discretization methods for Volterra integral and integro-differential equations. The results are all given in a form designed to be of most use to numerical analysts.*

Представляются обобщения классического неравенства Гронуалла в случае что ядро соотнесенного интегрального уравнения является слабо-сингулярным. Даются и непрерывный и дискретный варианты; непрерывный вариант включается, потому что он предлагает аналогичную обработку. Эта работа побуждается исследованиями по сходимости относительно методов дискретизации для интегральных и интегро-интегральных уравнений типа Вольтерра. Все результаты описываются в таком виде что они являются очень полезными для специалистов нумерического анализа.

## 1. Introduction

In 1919 GRONWALL [12] introduced the following result:

Lemma 1.1: *Let the function x be continuous and non-negative on the interval* $[0, T]$. *If*

$$x(t) \leqq a + b \int_0^t x(s) \, ds , \qquad 0 \leqq t \leqq T , \tag{1.1}$$

*where a, b are positive constants, then*

$$x(t) \leqq a \, e^{bt} , \qquad 0 \leqq t \leqq T . \tag{1.2}$$

This lemma, which provides a bound on the solution of (1.1) in terms of the solution of the related integra equation

$$y(t) = a + b \int_0^t y(s) \, ds , \qquad 0 \leqq t \leqq T , \tag{1.3}$$

is one of the basic tools in the theory of differential equations. It has been extended and used considerably in various contexts, and Gronwall inequalities has now become a generic term for the many variants of this lemma. A reasonably comprehensive account of Gronwall inequalities is given by BEESACK [3].

In the Picard-Cauchy type of iteration for establishing the existence and uniqueness of solutions of differential and integral equations lemma 1.1 and its variants play a significant role. This is demonstrated by, for example, WALTER [29]. Inequalities of the type (1.1) are also encountered frequently in the perturbation and stability theory of ordinary differential equations, for instance, see BELLMAN [1].

Recurrent inequalities involving sequences of real numbers, which may be regarded as discrete Gronwall inequalities, have been extensively applied in the analysis of finite difference equations. In numerical analysis the following discrete analogue of lemma 1.1 is widely used.

Lemma 1.2: *If* $x_i$, $i = 0, 1, \dots, N$, *is a sequence of non-negative real numbers satisfying*

$$x_0 \leqq \delta , \qquad x_i \leqq \delta + Mh \sum_{j=0}^{i-1} x_j , \qquad 1 \leqq i \leqq N , \tag{1.4}$$

*where* $\delta > 0$ *and* $M > 0$ *is bounded independently of h* ($Nh \leqq T$) *then*

$$x_i \leqq \delta \exp (Mih) , \qquad 0 \leqq i \leqq N . \tag{1.5}$$

The main value of this lemma is that it can be used to demonstrate convergence of the solution of some discretization to that of the corresponding operator equation. This necessarily requires that the difference between the discrete solution and its associated operator equation, that is, the sequence $\{x_i\}_{i=0}^N$, be uniformly bounded with respect to $N$ and $h$ where $N$ and $h$ are such that $Nh$ remains constant as $N \to \infty$ and $h \to 0$. LINZ [20] and HOLYHEAD, MCKEE and TAYLOR [17], for example, consider linear first kind Volterra integral equations, and HENRICI [13] provides an elementary introduction to the application of this result to ordinary differential equations.

More recently discrete generalizations of Gronwall's inequality have been discussed by several authors, notably PACHPATTE [23] and POPENDA and WERBOWSKI [24], and since the completion of this manuscript the attention of the authors has been brought to a paper by BEESACK [4].

The purpose of this paper is to derive generalized discrete Gronwall inequalities in a form which may be directly applied by numerical analysts when proving convergence of product integration methods for weakly singular Volterra integral and integro-differential equations. It is anticipated that in a numerical scheme as the stepsize $h$ is decreased to zero the discretization will in some sense tend to the underlying integral equation. It therefore seems natural to expect that any analysis for the integral equation will have a parallel in the discrete problem. Following this view a continuous generalization of Gronwall's inequality is first presented and then discrete manipulative steps, analogous to those employed to derive the continuous inequality, are used to obtain the main result of this paper, a generalized discrete Gronwall inequality.

Illustrative examples of the application of these generalized Gronwall inequalities are given; further examples of the use of these inequalities in deriving convergence results for Volterra type equations may be found in BRUNNER [5], CAMERON and McKEE [6], DIXON and McKEE [9], TE RIELE [25] and SCOTT [26].

## 2. A linear generalization of Gronwall's inequality

The following standard result from functional analysis will be required.

Lemma 2.1: *Let $X$ be a Banach space with norm $\|\cdot\|_X$, and let $L(X)$ denote the Banach space of linear operators from $X$ into itself.*

*If $K \in L(X)$ is such that $\|K^r\| < 1$ for some $r \in N$ then $\sum_{n=0}^{\infty} K^n$ converges and is the inverse of $(I - K)$.*

The above lemma may be used to obtain the following elementary existence result for linear second kind Volterra integral equations.

Theorem 2.1: *For each $t \in \Omega := [0, T]$ let the Volterra kernel $k(t, s)$ be integrable over $(0, t)$ as a function of $s$. Suppose that for some $\mu \in N$ the $\mu$th iterated kernel $k^{(\mu)}(t, s)$ is continuous on $D := \{(t, s): 0 \leqq s \leqq t \leqq T\}$.*

*Then for each $\psi \in C(\Omega)$ the integral equation*

$$y = \psi + Ky, \tag{2.1}$$

*where $(Ky)(t) = \int_0^t k(t, s)\, y(s)\, ds$, $t \in \Omega$, has a unique solution $y \in C(\Omega)$ given by*

$$y = \sum_{n=0}^{\infty} K^n \psi . \tag{2.2}$$

*Equivalently*

$$y = \sum_{n=1}^{\infty} \psi^n \tag{2.3}$$

*where the sequence $\{\psi^n\}_{n=1}^{\infty}$ is defined by $\psi^1 = \psi$, $\psi^n = K\psi^{n-1}$, $n \geqq 2$.*

*Furthermore, there exists a constant $C$, independent of $\psi$, such that*

$$\|y\|_\infty \leqq C\|\psi\|_\infty . \tag{2.4}$$

Note that the series at (2.2) is the resolvent series and that at (2.3) is the Neumann series.

The theorem is proved by showing that $\|K^r\| < 1$ for all $r$ sufficiently large and then employing lemma 2.1.

Lemma 2.2: *Let the Volterra kernel $k(t, s)$ satisfy*

(CI) *$k(t, s)$ is non-negative;*

(CII) *$k(t, s)$ is integrable over $(0, t)$ as a function of $s$ for each $t \in \Omega$;*

(CIII) *there exists $\mu \in N$ such that the $\mu$th iterated kernel $k^{\mu}(t, s)$ is continuous in $t$, $s$.*

*Define $F \in L(C(\Omega))$ by*

$$Fy = y - Ky \tag{2.5}$$

*where $(Ky)(t) = \int_0^t k(t, s)\, y(s)\, ds$, $t \in \Omega$. Then $Fy \geqq 0$ implies $y \geqq 0$.*

The proof follows that of theorem 5 of BEESACK [2].

The following linear generalization of Gronwall's inequality is a direct consequence of lemma 2.2.

Theorem 2.2: *Let the Volterra kernel $k(t, s)$ satisfy CI–CIII. For any $\psi \in C(\Omega)$ if $x \in C(\Omega)$ satisfies the integral inequality*

$$x \leqq \psi + Kx \tag{2.6}$$

*where $(Kx)(t) = \int_0^t k(t, s)\, x(s)\, ds$, $t \in \Omega$, then*

$$x \leqq y \tag{2.7}$$

*where $y$ is the unique $C(\Omega)$ solution of the integral equation*

$$y = \psi + Ky . \tag{2.8}$$

*Furthermore, if $x \geqq 0$ and $\psi \geqq 0$, there exists $C$, independent of $\psi$, such that*

$$\|x\|_\infty \leqq C\|\psi\|_\infty . \tag{2.9}$$

The bound (2.7) is the best possible result since equality in (2.6) implies equality in (2.7). For the case $\mu = 1$ the inequality (2.7) was first given by CHU and METCALF [8].

The results stated in this section are essentially well-known ([2], [3], [15], [16], [27], [28]). They have been included since it is felt that a numerical analyst may not be particularly familiar with them and, more importantly, to emphasise the strong similarities between the continuous results given here and the analogous discrete results which will be derived in section 4.

## 3. A weakly singular integro-differential equation arising from the diffusion of discrete particles in a turbulent fluid

Consider the weakly singular integro-differential equation

$$y'(t) = f(t, y(t)) + c \int_0^t \frac{y'(s)}{(t - s)^{\alpha}}\, ds + q(t) , \qquad t \in \Omega , \tag{3.1}$$

with $y(0)$ given, $c$ a constant and $0 \leqq \alpha < 1$. It is assumed that $f(t, y)$ and $q(t)$ are continuous on $\Omega \times \mathbb{R}$ and $\Omega$ respectively, and that $f(t, y)$ is (uniformly) Lipschitz continuous in $y$. When $\alpha = \frac{1}{2}$ (3.1) models the motion of a particle in a turbulent fluid; further details are given in MCKEE [21].

Equation (3.1) will be used to illustrate the results of the previous section, and in section 6 convergence of a numerical method for solving (3.1) will be proved using a discrete analogue of theorem 2.2.

Integrating (3.1) over $(0, t)$, interchanging the order of integration and integrating by parts yields

$$y(t) = \left(1 - \frac{ct^{1-\alpha}}{1-\alpha}\right) y(0) + \int_0^t q(s)\,\mathrm{d}s + \int_0^t \left(f(s, y(s)) + \frac{cy(s)}{(t-s)^\alpha}\right) \mathrm{d}s\,. \tag{3.2}$$

If $y(0)$ is subject to a perturbation then the change in solution $x(t) = |\delta y(t)|$ will satisfy an integral inequality of the form

$$x(t) \leqq \psi(t) + M \int_0^t \frac{x(s)}{(t-s)^\alpha}\,\mathrm{d}s\,, \qquad t \in \Omega\,.$$

Investigating the effect of the perturbation on the solution motivates the following result.

Theorem 3.1: *Let $x(t)$ be continuous and non-negative on $[0, T]$. If*

$$x(t) \leqq \psi(t) + M \int_0^t \frac{x(s)}{(t-s)^\alpha}\,\mathrm{d}s\,, \qquad 0 \leqq t \leqq T\,, \tag{3.3}$$

*where $0 \leqq \alpha < 1$, $\psi(t)$ is a non-negative, monotonic increasing continuous function on $[0, T]$ and $M$ is a positive constant, then*

$$x(t) \leqq \psi(t)\, E_{1-\alpha}\big(M\Gamma(1-\alpha)\, t^{1-\alpha}\big)\,, \qquad 0 \leqq t \leqq T\,, \tag{3.4}$$

*where $E_{1-\alpha}(z)$ is the Mittag-Leffler function defined for all $\alpha < 1$ by*

$$E_{1-\alpha}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n(1-\alpha)+1)}\,.$$

The Mittag-Leffler function, which is a generalization of the exponential function to which it reduces when $\alpha = 0$, has been discussed in the literature and references may be found in ERDÈLYI [10] (see also FRIEDMAN [11] and KERSHAW [18]).

This result follows by showing that the solution of the related integral equation

$$y(t) = \psi(t) + M \int_0^t \frac{y(s)}{(t-s)^\alpha}\,\mathrm{d}s\,, \qquad 0 \leqq t \leqq T\,, \tag{3.5}$$

satisfies

$$y(t) = \psi(t)\, E_{1-\alpha}\big(M\Gamma(1-\alpha)\, t^{1-\alpha}\big)\,,$$

and invoking theorem 2.2 with $\mu = \varrho + 1$ where $\varrho \in \mathbb{N}$ is chosen such that $(\varrho - 1)/\varrho < \alpha \leqq \varrho/(\varrho + 1)$.

If $\psi(t) \equiv \psi$, $0 \leqq t \leqq T$, (3.4) is the best possible result. For a more general function $\psi(t)$ (which is not necessarily monotonic) the best possible result is given by

$$x(t) \leqq \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t \psi(s)\, E_{1-\alpha}\big(M\Gamma(1-\alpha)\,(t-s)^{1-\alpha}\big)\,\mathrm{d}s\,, \qquad 0 \leqq t \leqq T\,, \tag{3.6}$$

where the right hand side of the inequality (3.6) is the exact solution of the integral equation (3.5).

## 4. A discrete existence result

Discrete manipulative steps, analogous to those used in the continuous problem, are employed to derive a discrete version of theorem 2.1 and, in the next section, of theorem 2.2.

A discrete version of the space $C(\Omega)$ is required. Let $h_0$, $T$ be given with $0 < h_0 \leqq T$ and $T/h_0 = N_0$, a positive integer. Define $J := \{h : h = T/N, N \in \mathbb{N}, N \geqq N_0\}$ and for $h \in J$ set $\Omega^h = \{0, 1, \dots, N\}$. Define the discrete space

$$C(\Omega^h) := \{y^h : y^h = (y_0, y_1, \dots, y_N)^{\mathrm{T}}\,,\ y_i \in \Re\,,\ 0 \leqq i \leqq N\}$$

with the maximum norm

$$\|y^h\|_\infty = \max_{0 \leqq i \leqq N} |y_i|\,.$$

The idea of discrete Volterra kernels and discrete iterated kernels will be required (cf. DIXON and MCKEE [9] and SCOTT [26]).

Definition 4.1: Let the discrete function $k_{ij}$ defined on the integers $i$, $j$, $0 \leqq i, j \leqq N$, satisfy $k_{ij} = 0$ for $j \geqq i$. The *discrete iterated kernels* $k_{ij}^{(n)}$, $n = 1, 2, \dots$, of the discrete Volterra kernel $k_{ij}$ are defined to be

$$k_{ij}^{(1)} = k_{ij}; \qquad k_{ij}^{(n)} = h \sum_{l=j+1}^{i-1} k_{il} k_{lj}^{(n-1)}, \qquad n \geqq 2 .$$

(Here and elsewhere it is assumed that $\sum_{j \in \emptyset} D_j = 0$ and $\prod_{j \in \emptyset} D_j = 1$ if $\emptyset$ is the empty set; thus $k_{ii-1}^{(n)} = 0$ is assumed for $\geqq 2$.)

A discrete version of theorem 2.1 is now presented.

Theorem 4.1: *Let $k_{ij}$ be a discrete Volterra kernel and for each $i \in \Omega^h$ let $h \sum_{j=0}^{i-1} k_{ij}$ be bounded independently of $h$. Suppose that for some $\mu \in \mathfrak{N}$ the $\mu$th discrete iterated kernel $k_{ij}^{(\mu)}$ is bounded in $i$, $j$, independently of $h$.*

*Then for each $h \in J$ and $\psi^h \in C(\Omega^h)$ the discrete operator equation*

$$y^h = \psi^h + K^h y^h , \tag{4.1}$$

*where $(K^h y^h)_i = h \sum_{j=0}^{i-1} k_{ij} y_j$, $i \in \Omega^h$, has a unique solution $y^h \in C(\Omega^h)$ given by*

$$y^h = \sum_{n=0}^{\infty} (K^h)^n \psi^h . \tag{4.2}$$

*Equivalently*

$$y^h = \sum_{n=1}^{\infty} (\psi^h)^n , \tag{4.3}$$

*where the sequence $\{(\psi^h)^n\}_{n=1}^{\infty}$ is defined by $(\psi^h)^1 = \psi^h$; $(\psi^h)^n = K^h(\psi^h)^{n-1}$, $n \geqq 2$.*

*Furthermore, there exists a constant $C$, independent of $\psi^h$ and $h$, such that*

$$\|y^h\|_\infty \leqq C \|\psi^h\|_\infty . \tag{4.4}$$

The series at (4.2) is the analogue of the resolvent series while that at (4.3) is the analogue of the Neumann series.

Proof: By repeated substitutions for $y^h$ in the right hand side of (4.1) it is straightforward to show that

$$y^h = \hat{\varphi}^h + \hat{K}^h y^h$$

where $\hat{\psi}^h = \sum_{n=0}^{\mu-1} (K^h)^n \psi^h \in C(\Omega^h)$ and $(\hat{K}^h y^h)_i = h \sum_{j=0}^{i-1} k_{ij}^{(\mu)} y_j$, $i \in \Omega^h$. For $n \geqq 1$

$$|k_{ij}^{(n\mu)}| \leqq \|k^{h(\mu)}\|_\infty^n \frac{(h(i-j-1))^{n-1}}{(n-1)!}$$

where $\|k^{h(\mu)}\|_\infty := \max_{i,j} |k_{ij}^{(\mu)}| < M$, with $M$ independent of $h$. This is trivially true for $n = 1$; assume inductively that it is true for some $n \geqq 1$.

It can be demonstrated that for any integers $p$, $q$

$$k_{ij}^{(p+q)} = h \sum_{l=j+1}^{i-1} k_{il}^{(p)} k_{lj}^{(q)} .$$

Therefore

$$|k_{ij}^{((n+1)\mu)}| = \left|h \sum_{l=j+1}^{i-1} k_{il}^{(n\mu)} k_{lj}^{(\mu)}\right| \leqq \max_{i,j} |k_{ij}^{(\mu)}| \, \|k^{h(\mu)}\|_\infty^n \frac{h^n}{(n-1)!} \sum_{l=j+1}^{i-1} (i-l-1)^{n-1} .$$

Observe that

$$\sum_{l=j+1}^{i-1} (i-l-1)^{n-1} \leqq \sum_{l=j+1}^{i-1} \int_l^{l+1} (i-s)^{n-1} \, ds = \frac{(i-j-1)^n}{n} .$$

and thus

$$|k_{ij}^{((n+1)\mu)}| \leqq \|k^{h(\mu)}\|_\infty^{n+1} \frac{(h(i-j-1))^n}{n!}$$

which completes the induction.

Since $((\hat{K}^h)^n y^h)_i = h \sum_{j=0}^{i-1} k_{ij}^{(n\mu)} y_j$, $i \in \Omega^h$, it follows that

$$|((\hat{K}^h)^n y^h)_i| \leqq \frac{h^n \|k^{h(\mu)}\|_\infty \|y^h\|_\infty}{(n-1)!} \sum_{j=0}^{i-1} (i-j-1)^{n-1} .$$

Bounding the summation by the integral $\int_0^i (i-s)^{n-1} \, ds$ and using $ih \leqq Nh \leqq T$ then gives

$$\|(\hat{K}^h)^n\| = \|(K^h)^{n\mu}\| \leqq \|k^{h(\mu)}\|_\infty^n \frac{T^n}{n!} .$$

Therefore for $r$ sufficiently large $\|(\hat{K}^h)^r\| < 1$ and applying lemma 2.1 with $X = C(\Omega^h)$ yields

$$y^h = \sum_{n=0}^{\infty} (\hat{K}^h)^n \hat{\psi}^h . \tag{4.5}$$

Substituting for $\hat{K}^h$, $\hat{\psi}^h$ yields the discrete resolvent solution

$$y^h = \sum_{n=0}^{\infty} (K^h)^n \psi^h .$$

The equivalent discrete Neumann series (4.3) can be derived by using induction to prove that for $n \geqq 1$

$(\psi^h)^n = (K^h)^{n-1} \psi^h$.

To obtain (4.4), taking norms at (4.5)

$$\|y^h\|_\infty \leqq \sum_{n=0}^{\infty} \|(\hat{K}^h)^n\| \, \|\hat{\psi}^h\|_\infty \leqq C_1 \|\hat{\psi}^h\|_\infty$$

where $C_1 = \sum_{n=0}^{\infty} \frac{(MT)^n}{n!} = \exp(MT)$ is bounded independently of $h$. But

$$\|\hat{\psi}^h\|_\infty \leqq \sum_{n=0}^{\mu-1} \|K^h\|^n \, \|\psi^h\|_\infty ,$$

and $\|K^h\|$ is bounded independently of $h$, since $h \sum_{j=0}^{i-1} k_{ij}$ is bounded independently of $h$ for each $i \in \Omega^h$. The required result follows.

Observe that (4.1) is merely a strictly lower triangular system of equations so that (4.2) is equivalent to the finite sum

$$y^h = \sum_{n=0}^{N} (K^h)^n \psi^h \tag{4.2'}$$

and (4.3) is equivalent to

$$y^h = \sum_{n=1}^{N+1} (\psi^h)^n \tag{4.3'}$$

since $(K^h)^{N+1} \equiv 0$. This apparently suggests that the above analysis, which showed that under the hypothesis of the theorem $\sum_{n=0}^{\infty} (K^h)^n$ and $\sum_{n=1}^{\infty} (\psi^h)^n$ converged, was unnecessary. However discrete Gronwall inequalities are to be derived to be employed in convergence analysis for numerical schemes; the parameter $h$ will then represent the steplength and to prove convergence it will be necessary to let $h \to 0$ while $N \to \infty$ with $Nh$ fixed. It will therefore be more useful when considering convergence to use (4.2) and (4.3) in place of (4.2′) and (4.3′), respectively; the assumptions $h \sum_{j=0}^{i-1} k_{ij} \leqq M$ and $\|k^{h(\mu)}\|_\infty \leqq M'$, with $M$, $M'$ independent of $h$, will then ensure that the constant $C$ in (4.4) is independent of $h$ and $N$.

## 5. A generalized discrete Gronwall inequality

The following lemma and theorem are analogous to lemma 2.2 and theorem 2.2.

Lemma 5.1: *Let the discrete Volterra kernel $k_{ij}$ satisfy*

(DI) *$k_{ij}$ is non-negative;*

(DII) *$h \sum_{j=0}^{i-1} k_{ij}$ is bounded independently of $h$ for each $i \in \Omega^h$;*

(DIII) *there exists $\mu \in \mathbb{N}$ such that the $\mu$th discrete iterated kernel $k_{ij}^{(\mu)}$ is bounded in $i$, $j$, independently of $h$.*

*Define $F^h \in L(C(\Omega^h))$ by*

$$F^h y^h = y^h - K^h y^h \tag{5.1}$$

*where $(K^h y^h)_i = h \sum_{j=0}^{i-1} k_{ij} y_j$, $i \in \Omega^h$. Then $F^h y^h \geqq 0$ implies $y^h \geqq 0$.*

Theorem 5.1: *Let the discrete Volterra kernel $k_{ij}$ satisfy* DI—DIII. *For $h \in J$ and any $\psi^h \in C(\Omega^h)$, if $x^h \in C(\Omega^h)$ satisfies the discrete integral inequality*

$$x^h \leqq \psi^h + K^h x^h , \tag{5.2}$$

*where $(K^h x^h)_i = h \sum_{j=0}^{i-1} k_{ij} x_j$, $i \in \Omega^h$, then*

$$x^h \leqq y^h \tag{5.3}$$

*where $y^h$ is the unique $C(\Omega^h)$ solution of the discrete integral equation*

$$y^h = \psi^h + K^h y^h . \tag{5.4}$$

*Furthermore, if $x^h \geqq 0$ and $\psi^h \geqq 0$, there exists $C$, independent of $\psi^h$ and $h$, such that*

$$\|x^h\|_\infty \leqq C \|\psi^h\|_\infty . \tag{5.5}$$

The importance of $C$ being independent of $h$ is that when the above result is employed to prove convergence of some discrete algorithm the bound (5.5) will imply the order of convergence depends only upon the order of the consistency error $\psi^h$.

## 6. A discrete Abel's singularity

In this section the results of section 5 are used to derive a particular discrete Gronwall inequality which may be employed to prove convergence of product integration methods for solving Volterra integral equations with weakly singular kernels.

Lemma 6.1: *Let* $0 < \alpha < 1$, $\gamma < 1$, $\beta \geqq 1$. *Then for* $0 \leqq j \leqq i - 1$

$$\sum_{k=j+1}^{i-1} \frac{k^{\beta-1}}{(i^\beta - k^\beta)^\alpha (k^\beta - j^\beta)^\gamma} < \frac{1}{\beta} \frac{B(1-\gamma, 1-\alpha)}{(i^\beta - j^\beta)^{\alpha+\gamma-1}},$$

*where* $B(a, b)$ *is the Beta function defined for* $\operatorname{Re}(a) > 0$, $\operatorname{Re}(b) > 0$ *by*

$$B(a, b) = \int_0^1 \frac{dw}{w^{1-a}(1-w)^{1-b}}.$$

Proof: The summation is compared with the integral

$$\int_j^i \frac{s^{\beta-1}}{(i^\beta - s^\beta)^\alpha (s^\beta - j^\beta)^\gamma} ds$$

treated as an area under a curve. Let

$$f(s) = \frac{s^{\beta-1}}{(i^\beta - s^\beta)^\alpha (s^\beta - j^\beta)^\gamma}.$$

If $0 < \alpha < 1$, $\gamma \leqq 0$ and $\beta \geqq 1$, $f$ is the product of three nondecreasing functions, so that

$$\sum_{k=j+1}^{i-1} \frac{k^{\beta-1}}{(i^\beta - k^\beta)^\alpha (k^\beta - j^\beta)^\gamma} < \sum_{k=j+1}^{i-1} \int_k^{k+1} \frac{s^{\beta-1}}{(i^\beta - s^\beta)^\alpha (s^\beta - j^\beta)^\gamma} ds$$

and the required bound follows by evaluating the integral using the change of variable $s^\beta = j^\beta + (i^\beta - j^\beta) w$.

Suppose $0 < \alpha < 1$, $\gamma > 0$ and $\beta \geqq 1$. The derivative of $f$ vanishes when

$$s^{2\beta}(1 - \beta - \alpha\beta + \gamma\beta) + s^\beta((i^\beta - j^\beta)(\beta - 1) - \alpha\beta j^\beta - \gamma\beta j^\beta) - i^\beta j^\beta(\beta - 1) = 0.$$

This is a quadratic in $s^\beta$. Since $f$ is positively infinite at $s = j$ and $s = i$ the quadratic must have at least one real root in $(j, i)$ and consequently both roots must be real.

This implies that $f'(s)$ vanishes at at most four real points, of which at most two can lie in the interval $(j, i)$. It follows that there exists only one turning point in $(j, i)$ and that this point is a minimum. Let the minimum occur at $s = s^* \in (j, i)$ and let $l$ be the largest integer such that $l \leqq s^*$. Splitting the summation over $k = j + 1$ to $k = i - 1$ into two summations

$$\sum_{k=j+1}^{l} \frac{k^{\beta-1}}{(i^\beta - k^\beta)^\alpha (k^\beta - j^\beta)^\gamma} + \sum_{k=l+1}^{i-1} \frac{k^{\beta-1}}{(i^\beta - k^\beta)^\alpha (k^\beta - j^\beta)^\gamma}$$

and interpreting the first sum as the forward rectangular rule and the second as the backward rectangular rule, it may be seen that

$$\sum_{k=j+1}^{i-1} \frac{k^{\beta-1}}{(i^\beta - k^\beta)^\alpha (k^\beta - j^\beta)^\gamma} < \int_j^i \frac{s^{\beta-1}}{(i^\beta - s^\beta)^\alpha (s^\beta - j^\beta)^\gamma} ds$$

(see fig. 1 where the shading represents the summation).

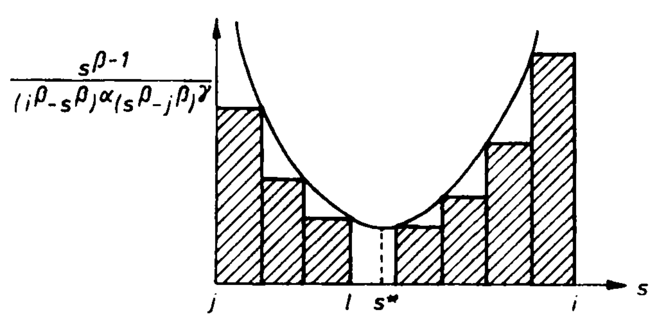


Fig. 1

Theorem 6.1: *Let* $x_i$, $0 \leqq i \leqq N$, *be a sequence of non-negative real numbers satisfying*

$$x_i \leqq \psi_i + M h^{1-\alpha} \sum_{j=0}^{i-1} \frac{x_j}{(i-j)^\alpha}, \qquad 0 \leqq i \leqq N, \tag{6.1}$$

*where* $0 < \alpha < 1$, $M > 0$ *is bounded independently of* $h$, *and* $\psi_i$, $0 \leqq i \leqq N$, *is a monotonic increasing sequence of non-negative real numbers. Then*

$$x_i \leqq \psi_i E_{1-\alpha}(M\Gamma(1-\alpha)(ih)^{1-\alpha}), \qquad 0 \leqq i \leqq N. \tag{6.2}$$

Proof: For $0 \leqq i \leqq N$

$$h \sum_{j=0}^{i-1} k_{ij} = Mh^{1-\alpha} \sum_{j=0}^{i-1} \frac{1}{(i-j)^\alpha} \leqq Mh^{1-\alpha} \sum_{j=0}^{i-1} \int_j^{j+1} \frac{ds}{(i-s)^\alpha} \leqq \frac{MT^{1-\alpha}}{1-\alpha}$$

where $Nh \leqq T$. Therefore $h \sum_{j=0}^{i-1} k_{ij}$ is bounded independently of $h$ for each $i$, $0 \leqq i \leqq N$.

Let $\varrho \in N$ be chosen so that $\frac{\varrho - 1}{\varrho} \leqq \alpha \leqq \frac{\varrho}{\varrho + 1}$. It can be shown by induction using lemma 6.1 with $\beta = 1$ that the discrete iterated kernels $k_{ij}^{(n)}$ of $k_{ij}$ where $k_{ij} = M(h(i-j))^{-\alpha}$ satisfy

$$0 \leqq k_{ij}^{(n)} \leqq \frac{(M\Gamma(1-\alpha))^n h^{(n-1)-n\alpha}}{\Gamma(n(1-\alpha))(i-j)^{n\alpha-(n-1)}}.$$

Hence

$$k_{ij}^{(\varrho+1)} \leqq \frac{(M\Gamma(1-\alpha))^{\varrho+1} T^{\varrho-(\varrho+1)\alpha}}{\Gamma((\varrho+1)(1-\alpha))} = M'$$

where $M'$ is a positive constant bounded independently of $h$. Thus DI—DIII are satisfied with $\mu = \varrho + 1$, and invoking theorem 5.1

$$x_i \leqq y_i, \quad 0 \leqq i \leqq N,$$

where

$$y_i = \psi_i + Mh^{1-\alpha} \sum_{j=0}^{i-1} \frac{y_j}{(i-j)^\alpha}, \; 0 \leqq i \leqq N.$$

From theorem 4.1

$$y_i = \sum_{n=1}^{\infty} ((\psi^h)^n)_i, \quad 0 \leqq i \leqq N,$$

and using induction it is straightforward to demonstrate that for all $n \geqq 1$

$$0 \leqq ((\psi^h)^n)_i \leqq \psi_i \frac{(M\Gamma(1-\alpha))^{n-1} (ih)^{(n-1)(1-\alpha)}}{\Gamma((n-1)(1-\alpha)+1)}, \quad 0 \leqq i \leqq N.$$

Hence

$$x_i \leqq \psi_i E_{1-\alpha}(M\Gamma(1-\alpha)(ih)^{1-\alpha}), \quad 0 \leqq i \leqq N.$$

Note that

$$||x^h||_\infty \leqq C||\psi^h||_\infty$$

where $C = E_{1-\alpha}(M\Gamma(1-\alpha)\,T^{1-\alpha})$ is bounded independently of $h$.

MCKEE [22] has also considered inequalities of the form (6.1). The same inequality has been considered here not only because it shall be employed to prove convergence of product integration methods but also because it provides a useful example of the more general discrete Gronwall inequality derived in section 5. Moreover the bound (6.2) is both sharper and more elegant than that derived by MCKEE.

Example 6.1: Consider the weakly singular integro-differential equation discussed in section 3, namely

$$y'(t) = q(t) + f(t, y(t)) + c \int_0^t \frac{y'(s)}{(t-s)^\alpha} ds, \quad 0 \leqq t \leqq T, \tag{6.3}$$

where $0 \leqq \alpha < 1$.

An Euler-type method combined with the simplest product integration method yields the following approximating equation

$$\psi^h(y^h) = 0, \quad \psi^h: C(\Omega^h) \to C(\Omega^h)$$

where

$$(\psi^h(y^h))_i = \begin{cases} y_0 - \tilde{y}_0, \\ \frac{y_i - y_{i-1}}{h} - q(t_i) - f(t_i, y_i) - c \sum_{j=0}^{i-1} w_{ij}(y_{j+1} - y_j), \quad 1 \leqq i \leqq N, \end{cases} \tag{6.4}$$

where $t_i = ih$, $0 \leqq i \leqq N$, $Nh = T$, $y_i$ denotes an approximation to $y(t_i)$, $\tilde{y}_0$ is a given starting value and

$$w_{ij} = \frac{1}{h} \int_{t_j}^{t_{j+1}} \frac{ds}{(t_i - s)^\alpha}, \quad 0 \leqq j < i \leqq N.$$

Summing the equation

$$y_i - y_{i-1} = hq(t_i) + hf(t_i, y_i) + ch \sum_{j=0}^{i-1} w_{ij}(y_{j+1} - y_j)$$

over $i = 1$ to $l$ and interchanging the order of summation

$$y_l = \left(1 - \frac{ct_l^{1-\alpha}}{1-\alpha}\right) y_0 + h \sum_{i=1}^{l} q(t_i) + h \sum_{i=1}^{l} f(t_i, y_i) + ch \sum_{j=1}^{l} w_{lj-1} y_j,$$

which is the discrete analogue of equation (3.2).

The true solution $y(t_i)$ satisfies the perturbed equation

$$\frac{y(t_i) - y(t_{i-1})}{h} - q(t_i) - f(t_i, y(t_i)) - c \sum_{j=0}^{i-1} w_{ij}(y(t_{j+1}) - y(t_j)) = T_i, \quad 1 \leqq i \leqq N.$$

That is

$$T_i = \frac{y(t_i) - y(t_{i-1})}{h} - y'(t_i) + c \sum_{j=0}^{i-1} \int_{t_j}^{t_{j+1}} \left( y'(s) - \frac{y(t_{j+1}) - y(t_j)}{h} \right) \frac{1}{(t_i - s)^\alpha} \, ds .$$

In general the solution $y(t)$ of (6.3) will have a discontinuous second derivative at the origin, but will possess continuous derivatives away from the origin (see BRUNNER [5]). It can then be shown that

$$|T_i| \leqq Mh^{1-\alpha} + O(h)$$

for some $M$ independent of $h$. Details may be found in SCOTT [26].

Letting $x_i = |y(t_i) - y_i|$ it follows, assuming the starting value is accurate of order 1, that

$$x_i \leqq \left(1 - \frac{ct_i^{1-\alpha}}{1-\alpha}\right) C_0 h + h \sum_{j=1}^{i} |f(t_j, y(t_j)) - f(t_j, y_j)| + ch \sum_{j=1}^{i} w_{ij-1} x_j + h \sum_{j=1}^{i} |T_j| \leqq$$

$$\leqq Ch + Mh^{1-\alpha} + Lh \sum_{j=1}^{i} x_j + ch \sum_{j=1}^{i} w_{ij-1} x_j$$

for some $C$, $M$ independent of $h$.

It is straightforward to show that for $1 \leqq j \leqq i$

$$0 < w_{ij-1} \leqq \frac{(h(i-j))^{-\alpha}}{(1-\alpha)}, \qquad (0^{-\alpha} \equiv 1) .$$

Therefore

$$x_i \leqq Ch + Mh^{1-\alpha} + \hat{L} h x_i + \hat{M} h^{1-\alpha} \sum_{j=0}^{i-1} \frac{x_j}{(i-j)^\alpha}$$

where $\hat{L} = L + \frac{ch^{-\alpha}}{1-\alpha}$ and $\hat{M} = \frac{c}{1-\alpha} + LT^\alpha$ is independent of $h$.

Provided $h\hat{L} < 1$,

$$x_i \leqq C'h + M'h^{1-\alpha} + \hat{M}'h^{1-\alpha} \sum_{j=0}^{i-1} \frac{x_j}{(i-j)^\alpha}, \qquad 0 \leqq i \leqq N ,$$

where $C' = C/(1 - \hat{L}h)$, $M' = M/(1 - \hat{L}h)$, $\hat{M}' = \hat{M}/(1 - \hat{L}h)$ are bounded independently on $h$.

Invoking theorem 6.1 with $\psi_i = C'h + M'h^{1-\alpha}$,

$$x_i \leqq (C'h + M'h^{1-\alpha}) \, E_{1-\alpha}(\hat{M}'\Gamma(1-\alpha) \, t_i^{1-\alpha}) , \qquad 0 \leqq i \leqq N .$$

This proves convergence of order at least $1 - \alpha$.

If $y \in C^2[0, T]$ then $|T_i| \leqq C_i h$, $1 \leqq i \leqq N$, and the method is then convergent of order at least 1.

Example 6.2: As a further example to illustrate an application of theorem 6.1 consider the second kind Volterra integral equation

$$y(t) = g(t) + \int_0^t \frac{G(t, s, y(s))}{(t-s)^\alpha} \, ds , \qquad 0 \leqq t \leqq T ,$$

where $0 < \alpha < 1$. KERSHAW [18] suggests the following scheme:

$$\varphi^h(y^h) = 0 , \qquad \varphi^h : C(\Omega^h) \to C(\Omega^h) ,$$

where

$$(\varphi^h(y^h))_i = \begin{cases} y_0 - g(0) \\ y_i - g_i(t_i) - h \sum_{j=0}^{i} w_{ij} G(t_i, t_j, y_j) , \qquad 1 \leqq i \leqq N , \end{cases} \tag{6.5}$$

with

$$w_{i0} = \frac{1}{h^2} \int_0^{t_1} \frac{(t_1 - s)}{(t_i - s)^\alpha} \, ds , \qquad w_{ii} = \frac{1}{h^2} \int_{t_{i-1}}^{t_i} \frac{(s - t_{i-1})}{(t_i - s)^\alpha} \, ds , \qquad 1 \leqq i \leqq N ,$$

$$w_{ij} = \frac{1}{h^2} \int_{t_j}^{t_{j+1}} \frac{(t_{j+1} - s)}{(t_i - s)^\alpha} \, ds + \frac{1}{h^2} \int_{t_{j-1}}^{t_j} \frac{(s - t_{j-1})}{(t_i - s)^\alpha} \, ds , \qquad 1 \leqq j \leqq i - 1 , \qquad 2 \leqq i \leqq N .$$

It can again be shown that for some $M$, independent of $h$,

$$0 \leqq w_{ij} \leqq M(h(i-j))^{-\alpha} , \qquad 0 \leqq j \leqq i \leqq N ,$$

where $0^{-\alpha} \equiv 1$.

The true solution $y$ satisfies the perturbed equation

$$y(t_i) - g(t_i) - h \sum_{j=0}^{i} w_{ij} G(t_i, t_j, y(t_j)) = T_i , \qquad 1 \leqq i \leqq N .$$

That is,

$$T_i = \sum_{j=0}^{i-1} \int_{t_j}^{t_{j+1}} \left[ \frac{(t_{j+1} - s)}{h} \left( G(t_i, s, y(s)) - G(t_i, t_j, y(t_j)) \right) + \frac{(s - t_j)}{h} \left( G(t_i, s, y(s)) - G(t_i, t_{j+1}, y(t_{j+1})) \right) \right] \frac{1}{(t_i - s)^\alpha} \, ds .$$

Provided $G(t, s, y)$ is (uniformly) Lipschitz continuous in $y$

$$|T_i| \leqq C_i h^s + O(h^{s+1})$$

where

$$s = \begin{cases} 1 - \alpha, & \text{if } y \text{ has discontinuous derivatives at the origin} \\ 2, & \text{if } y \text{ is smooth on } [0, T]. \end{cases}$$

Therefore, $x_i = |y(t_i) - y_i|$ satisfies

$$x_i \leqq Lh \sum_{j=0}^{i} w_{ij} x_j + C_i h^s + O(h^{s+1})$$

and for $h$ sufficiently small

$$x_i \leqq L' h^{1-\alpha} \sum_{j=0}^{i-1} \frac{x_j}{(i-j)^\alpha} + C' h^s + O(h^{s+1})$$

where $L'$, $C'$ are independent of $h$.

Convergence of order $s$ follows on applying theorem 6.1.

The above convergence analysis of the discretization method (6.5) is more straightforward than that given by KERSHAW [18]. Furthermore, it is clear that the analysis may be extended to general product integration methods.

## 7. Further generalization of Gronwall's inequality

Generalizations of theorems 3.1 and 6.1 are given for Volterra integral equations with an Abel's type singularity of the form

$$y(t) = g(t) + \int_0^t \frac{G(t, s, y(s))}{(t^\beta - s^\beta)^\alpha} \, ds, \qquad 0 \leqq t \leqq T, \tag{7.1}$$

where $0 < \alpha < 1$, $\beta \geqq 1$ and $G(t, s, y)$ satisfies the Lipschitz condition

$$|G(t, s, y_1) - G(t, s, y_2)| \leqq Ls^\sigma |y_1 - y_2|$$

for some non-negative constants $L$, $\sigma$ with $\sigma \geqq \beta - 1$.

A practical example of equation (8.1) with $\beta \neq 1$ is given by LIGHTHILL [19] and by CHAMBRÉ and ACRIVOS [7] where $\alpha = \frac{2}{3}$, $\beta = \frac{3}{2}$ and $\sigma = 1$.

Theorem 7.1: *Let the function $x$ be continuous and non-negative on $[0, T]$. If*

$$x(t) \leqq \psi(t) + M \int_0^t \frac{s^\sigma x(s)}{(t^\beta - s^\beta)^\alpha} \, ds, \qquad 0 \leqq t \leqq T, \tag{7.2}$$

*where $0 < \alpha < 1$, $1 \leqq \beta \leqq \sigma + 1$, $\sigma \geqq 0$, $M$ is a positive constant and $\psi(t)$ is a continuous, non-negative monotonic increasing function on $[0, T]$, then*

$$x(t) \leqq \psi(t) \sum_{n=0}^{\infty} \left( \frac{Mt^{\sigma+1-\alpha\beta}}{\beta} \right)^n \hat{B}_n(\alpha, \beta, \sigma), \qquad 0 \leqq t \leqq T, \tag{7.3}$$

$$\hat{B}_n(\alpha, \beta, \sigma) = \begin{cases} 1, & n = 0, \\ \prod_{l=1}^{n} B\left(\frac{l}{\beta}(\sigma + 1 - \alpha\beta) + \alpha, (1 - \alpha)\right), & n \geqq 1. \end{cases} \tag{7.4}$$

*In the special case $\sigma + 1 - \beta = 0$ (7.3) reduces to*

$$x(t) \leqq \psi(t) \, E_{1-\alpha}\left( \frac{M\Gamma(1-\alpha)}{\beta} \, t^{\beta(1-\alpha)} \right), \qquad 0 \leqq t \leqq T. \tag{7.5}$$

Theorem 7.2: *Let $x_i$, $0 \leqq i \leqq N$, be a sequence of non-negative real numbers. If*

$$x_i \leqq \psi_i + Mh^{\sigma+1-\alpha\beta} \sum_{j=0}^{i-1}{}' \frac{j^\sigma x_j}{(i^\beta - j^\beta)^\alpha}, \qquad 0 \leqq i \leqq N, \tag{7.6}$$

*where $0 < \alpha < 1$, $1 \leqq \beta \leqq \sigma + 1$, $\sigma \geqq 0$, $M$ is a positive constant, and $\psi_i$, $0 \leqq i \leqq N$, is a monotonic increasing sequence of non-negative real numbers, then*

$$x_i \leqq \psi_i \sum_{n=0}^{\infty} \left( \frac{M(ih)^{\sigma+1-\alpha\beta}}{\beta} \right)^n \hat{B}_n(\alpha, \beta, \sigma), \qquad 0 \leqq i \leqq N. \tag{7.7}$$

*When $\sigma + 1 - \beta = 0$,*

$$x_i \leqq \psi_i E_{1-\alpha}\left( \frac{M\Gamma(1-\alpha)}{\beta} \, (ih)^{\beta(1-\alpha)} \right), \qquad 0 \leqq i \leqq N. \tag{7.8}$$

The proofs of theorems 7.1 and 7.2 employ similar arguments to those already presented and so are omitted (SCOTT [26]).

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