Generalized reducible quadrature methods for Volterra integral and integro-differential equations

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Abstract: A general class of convergent methods for the numerical solution of ordinary differential equations is employed to obtain a class of convergent generalized reducible quadrature methods for Volterra integral equations of the second kind and Volterra integro-differential equations.

Keywords: Reducible quadrature, Volterra equations, convergence.

1. Introduction

Consider the second kind Volterra integral equation

$$y(t) = g(t) + \int_0^t G(t, s, y(s)) \, ds, \quad 0 \leqslant t \leqslant T, \tag{1.1}$$

and the Volterra integro-differential equation

$$y'(t) = F\left(t, y(t), \int_0^t G(t, s, y(s)) \, ds\right), \quad 0 \leqslant t \leqslant T, \tag{1.2}$$

with $y(0)$ given.

In the subsequent discussion it will be assumed that the functions g, G, F are sufficiently smooth to guarantee the existence of a (unique) smooth solution y.

An essential part in the discretization of (1.1) and (1.2) at $t = t_i$ is the approximation of the Volterra integral

$$z(t_i) = \int_0^{t_i} G(t_i, s, y(s)) \, ds$$

by numerical quadrature

$$z_i = h \sum_{j=0}^{i} w_{i,j} G(t_i, t_j, y_j).$$

Here, h denotes the stepsize, $t_i = ih$, $0 \leqslant i \leqslant n$, $nh = T$, and y_i, z_i denote numerical approximations to $y(t_i)$, $z(t_i)$, respectively. The w_{ij} are the weights of the quadrature rule.

Suppose that in the case $G(t, s, y) = G(s, y)$, z_i can be written as a linear combination of $z_{i-1}, z_{i-2}, \ldots, z_{i-k}$, $G(t_i, y_i)$, $G(t_{i-1}, y_{i-1}), \ldots, G(t_{i-k}, y_{i-k})$, with coefficients independent of h. The resulting relation can then be written as

$$\sum_{j=0}^{k} a_j z_{i-j} = h \sum_{j=0}^{k} b_j G(t_{i-j}, y_{i-j}),$$

and the quadrature formula is said to be reducible to the linear multistep method for ordinary differential equations characterised by (ρ, σ), where

$$\rho(\zeta) := \sum_{j=0}^{k} a_j \zeta^{k-j}, \qquad \sigma(\zeta) := \sum_{j=0}^{k} b_j \zeta^{k-j}. \tag{1.3}$$

Matthys [4] first introduced a class of reducible quadrature rules with the aim of proving A-stability results for numerical methods for (1.2). Reducible quadrature rules were discussed further by Wolkenfelt [10,11], who explicitly constructed such rules for (1.1) and (1.2), and presented convergence and stability results.

In his thesis, Wolkenfelt [10, p. 97] remarks "The question whether generalizations of our results are possible if we employ cyclic linear multistep methods, multistep Runge–Kutta methods or other methods for solving ordinary differential equations is still open.... If such generalizations are possible, it is evident that we have a powerful tool for constructing and analysing, in a unified way, numerical methods for solving Volterra type equations."

In this paper a class of multistage multistep methods for the numerical solution of ordinary differential equations will be employed to construct generalized reducible quadrature methods for solving (1.1) and (1.2). To unify the analysis a general convergence theorem for discretization methods will be employed to present convergence results.

2. Preliminaries

The solution $y \in X$ of a differential or integral equation is to be approximated on a (finite) interval $[0, T]$ using a discretization method by $y^h \in X^h$. The set of values of h and the discrete space X^h to be used in the following analysis must first be introduced.

Let T, h_0 be given with $0 < h_0 \leqslant T$ and $T/h_0 = n_0$, a positive integer. Define $J := \{ h : h = T/n, \ n \in \mathbb{N}, \ n \geqslant n_0 \}$, and given a positive integer m, independent of h, set $N = n - \delta + 1$ where $\delta = \min(2, m)$.

Then define the approximating space X^h as follows:

$$X^h := \left\{ x^h : x^h = (x_0, x_1, \ldots, x_N)^{\mathrm{T}}, \ x_i = (x_{i,\tau}) \in \mathbb{R}^m, \ 0 \leqslant i \leqslant N \right\}$$

with norm

$$\| x^h \|_\infty = \max_{0 \leqslant i \leqslant N} | x_i |_m, \qquad | x_i |_m = \max_{1 \leqslant \tau \leqslant m} | x_{i,\tau} |.$$

In the subsequent discussion, for $x^h \in X^h$, $| x^h |_m$ will denote the vector of norms

$$| x^h |_m = (| x_0 |_m, | x_1 |_m, \ldots, | x_N |_m)^{\mathrm{T}}.$$

It is clear that $| x^h |_m \in \mathbb{R}^{N+1}$ and $(| x^h |_m)_i = | x_i |_m$, $0 \leqslant i \leqslant N$.

Definition 2.1. An operator H^h: $X^h \to X^h$ will be said to be a discrete Volterra operator if for each i, $0 \le i \le N$, $(H^h(y^h))_i$ is independent of $(y^h)_j$ for all $j > i$.

Definition 2.2. A discretization method will be said to have a discrete fundamental form if it is expressible in the form

$$y^h = g^h + H^h(y^h) \tag{2.1}$$

where g^h is known, and will depend upon the starting values of the method, and H^h: $X^h \to X^h$ is a nonlinear discrete Volterra operator satisfying

$$\left(\mid H^h(y_1^h) - H^h(y_2^h) \mid_m \right)_i \le \left(K^h \mid y_1^h - y_2^h \mid_m \right)_i, \tag{2.2}$$

for all y_1^h, $y_2^h \in X^h$ and each i, $0 \le i \le N$, where K^h: $\mathbb{R}^{N+1} \to \mathbb{R}^{N+1}$ is a discrete Volterra operator given by

$$\left(K^h \mid y^h \mid_m \right)_i = hM \sum_{j=0}^{i} \left(\mid y^h \mid_m \right)_j, \quad 0 \le i \le N, \tag{2.3}$$

for some M, independent of h.

When $m = 1$ this is a particular case of the discrete fundamental form introduced by Dixon and McKee [1]; see also Scott [8].

Using the Banach Fixed Point Theorem it may be shown that for all $h \in J$ satisfying $hM < 1$ the discrete fundamental form has a unique solution $y^h \in X^h$.

In the following r^h: $X \to X^h$ will denote a linear operator and will be defined by

$$\left((r^h y)(t) \right)_{i,\tau} = y(t_{i,\tau}), \quad 1 \le \tau \le m, \quad 0 \le i \le N,$$

where $\{ t_{i,\tau} \} \in [0, T]$, $1 \le \tau \le m$, $0 \le i \le N$, is a given set of points associated with the discretization.

Definition 2.3. Let $y \in X$ be the solution of the differential or integral equation which is to be solved by a discretization method expressible in discrete fundamental form. For $h \in J$ the consistency error of the discrete fundamental form is defined to be

$$\theta^h = r^h y - g^h - H^h(r^h y). \tag{2.4}$$

The discrete fundamental form is said to be consistent of order s if for some positive constant C, independent of h,

$$\| \theta^h \|_\infty \le Ch^s + O(h^{s+1}). \tag{2.5}$$

This allows the following convergence result to be presented.

Theorem 2.1. *Let* $y \in X$ *be the solution of the equation which is to be solved numerically. Let* $y^h \in X^h$ *be the solution of a discretization which is expressible in discrete fundamental form. If the discrete fundamental form is consistent of order s, then for all $h \in J$ sufficiently small*

$$\| r^h y - y^h \|_\infty \le Ch^s + O(h^{s+1}), \tag{2.6}$$

where C is a positive constant independent of h.

Proof. The solution $y \in X$ of the underlying equation satisfies the perturbed fundamental form

$$r^h y = g^h + H^h(r^h y) + \theta^h,$$

where θ^h is the consistency error.

Subtracting (2.1) and using (2.2), (2.3)

$$\left(\mid r^h y - y^h \mid_m \right)_i \leqslant \left(\mid \theta^h \mid_m \right)_i + hM \sum_{j=0}^{i} \left(\mid r^h y - y^h \mid_m \right)_j, \quad 0 \leqslant i \leqslant N.$$

Setting $x_i = (\mid r^h y - y^h \mid_m)_i$ and $\phi_i = (\mid \theta^h \mid_m)_i / (1 - hM)$ for $hM < 1$, and invoking the standard discrete Gronwall inequality yields

$$\max_i(x_i) \leqslant \max_i(\phi_i) \exp(M'T), \quad 0 \leqslant i \leqslant N,$$

where $M' = M/(1 - hM)$.

The bound (2.6) is immediate (using (2.5)). $\square$

3. Multistage multistep methods

The class of methods for the ordinary differential equation

$$y'(t) = f(t, y(t)), \quad y(0) = y_0, \qquad 0 \leqslant t \leqslant T, \tag{3.1}$$

which is to be employed to derive generalized reducible quadrature methods is now introduced.

The class of simple m-stage k-step methods for solving (3.1) is defined to be those methods expressible in the form

$$h^{-1} \sum_{j=0}^{k} \sum_{\nu=1}^{m} (A_j)_{\tau\nu} y_{i-j,\nu} = \sum_{j=0}^{k} \sum_{\nu=1}^{m} (B_j)_{\tau\nu} f(t_{i-j,\nu}, y_{i-j,\nu}),$$

$$1 \leqslant \tau \leqslant m, \quad k \leqslant i \leqslant N, \tag{3.2}$$

where $\{A_j, B_j\}_{j=0}^{k}$ are $m \times m$ matrices with entries denoted by $(A_j)_{\tau\nu}, (B_j)_{\tau\nu}, 1 \leqslant \tau, \nu \leqslant m$; A_0 is assumed nonsingular and

$$\sum_{j=0}^{k} A_j e = 0, \quad \text{where } e = (1, 1, \ldots, 1)^T. \tag{3.3}$$

Here $y_{i,\tau}$ denotes an approximation to $y(t_{i,\tau})$ where $t_{i,\tau} \in [0, T]$, $1 \leqslant \tau \leqslant m$, $0 \leqslant i \leqslant N$, are given (if $m = 1$, $t_{i,1} = t_i = ih$, $0 \leqslant i \leqslant N$, $Nh = T$). It is assumed that starting values $\hat{y}_{i,\tau}$, $1 \leqslant \tau \leqslant m$, $0 \leqslant i \leqslant k - 1$, have been precomputed.

Associated with the m-stage k-step ODE method (3.2) will be the characteristic polynomials ρ, σ defined by

$$\rho(\zeta) := \det\left(\sum_{j=0}^{k} A_j \zeta^{k-j} \right), \qquad \sigma(\zeta) := \det\left(\sum_{j=0}^{k} B_j \zeta^{k-j} \right). \tag{3.4}$$

The class of m-stage k-step methods is wide and includes linear multistep methods, cyclic linear multistep methods, predictor corrector methods and Runge–Kutta methods (see Stetter [9]).

Example 1. As an example to illustrate the notation consider the following 2-cyclic linear multistep method for (3.1) based on the Simpson scheme and the 2-step Adams–Moulton method.

$$y_{2i} - y_{2i-2} = \tfrac{1}{3}h\big(f(t_{2i}, y_{2i}) + 4f(t_{2i-1}, y_{2i-1}) + f(t_{2i-2}, y_{2i-2})\big), \qquad (3.5a)$$

$$y_{2i+1} - y_{2i} = \tfrac{1}{12}h\big(5f(t_{2i+1}, y_{2i+1}) + 8f(t_{2i}, y_{2i}) - f(t_{2i-1}, y_{2i-1})\big), \qquad (3.5b)$$

$$1 \leqslant i \leqslant N,$$

with given starting values $\hat{y}_0$, $\hat{y}_1$. Here y_i denotes an approximation to $y(t_i)$, $t_i = ih$, $0 \leqslant i \leqslant 2N + 1$, $(2N + 1)h = T$. (See, for example, Donelson and Hansen [2].)

Let $m = 2$, $k = 1$ and set

$$t_{i,1} = 2ih,$$
$$t_{i,2} = (2i + 1)h, \qquad 0 \leqslant i \leqslant N.$$

The scheme (3.5) may then be expressed as a 2-stage 1-step method of the form (3.2) with

$$A_1 = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad A_0 = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix},$$

$$B_1 = \begin{pmatrix} \tfrac{1}{3} & \tfrac{4}{3} \\ 0 & -\tfrac{1}{12} \end{pmatrix}, \qquad B_0 = \begin{pmatrix} \tfrac{1}{3} & 0 \\ \tfrac{8}{12} & \tfrac{5}{12} \end{pmatrix}. \qquad (3.6)$$

It is clear that A_0 is nonsingular and condition (3.3) is satisfied.

This simple example will be employed throughout this paper to illustrate the ideas and results presented.

In matrix notation (3.2) may be expressed in the form

$$h^{-1}A^h y^h = B^h f(y^h) + \hat{g}^h, \qquad (3.7)$$

where

$$y^h = (y_0, y_1, \ldots, y_N)^{\mathrm{T}} \in X^h, \qquad y_i = (y_{i,\tau}) \in \mathbb{R}^m, \quad 0 \leqslant i \leqslant N,$$

$$f(y^h) = (f(y_0), f(y_1)\ldots, f(y_N))^{\mathrm{T}} \in X^h,$$

$$f(y_i) = (f(t_{i,\tau}, y_{i,\tau})) \in \mathbb{R}^m, \quad 0 \leqslant i \leqslant N,$$

and

$$\hat{g}^h = h^{-1}(\hat{y}_0, \hat{y}_1, \ldots, \hat{y}_{k-1}, 0, \ldots, 0)^{\mathrm{T}} \in X^h,$$

$$\hat{y}_i = (\hat{y}_{i,\tau}) \in \mathbb{R}^m, \quad 0 \leqslant i \leqslant k - 1.$$

Here A^h, B^h are $(N + 1)m \times (N + 1)m$ matrices given by

$$A^h = \begin{pmatrix} I_{km} & & & & & & \\ A_k & \ldots & & A_1 & A_0 & & 0 \\ & A_k & \ldots & & A_1 & A_0 & \\ & & \ddots & & & \ddots & \ddots \\ 0 & & & A_k & \ldots & & A_1 & A_0 \end{pmatrix},$$

$$B^h = \begin{pmatrix} 0_{km} & & & & & & \\ B_k & \ldots & & B_1 & B_0 & & 0 \\ & B_k & \ldots & & B_1 & B_0 & \\ & & \ddots & & & \ddots & \ddots \\ & & B_k & \ldots & & B_1 & B_0 \end{pmatrix} \qquad (3.8)$$

where I_{km}, 0_{km} denote the $km \times km$ identity and null matrices respectively.

To express the multistage multistep method in discrete fundamental form premultiply (3.7) by $h(A^h)^{-1}$ to obtain

$$y^h = g^h + H^h(y^h) \tag{3.9}$$

where $g^h = h(A^h)^{-1}\hat{g}^h$ and $H^h(y^h) = h(A^h)^{-1}B^hf(y^h)$. Assuming Lipschitz continuity of $f(t, y)$ with respect to y, H^h: $X^h \to X^h$ is a discrete Volterra operator satisfying (2.2), (2.3) provided

$$\max_{i,j} |((A^h)^{-1}B^h)_{ij}|_m \leqslant M, \tag{3.10}$$

for some M, independent of h. (Here $|((A^h)^{-1}B^h)_{ij}|_m$ denotes the infinity norm of the (i, j)th $m \times m$ matrix element of $(A^h)^{-1}B^h$.)

Since B^h is banded, with bandwidth independent of h, (3.10) holds if there exists M', independent of h, such that

$$\max_{i,j} |((A^h)^{-1})_{ij}|_m \leqslant M'. \tag{3.11}$$

Definition 3.1. The m-stage k-step ODE method (3.2) is zero stable if condition (3.11) holds.
Define

$$S^1 := \{\rho : \rho \text{ is a simple Von Neumann polynomial}\}.$$

It can be shown (see, for example, McKee [5]) that if $m = 1$ (3.11) is satisfied if and only if the characteristic polynomial ρ given at (3.4) satisfies $\rho \in S^1$. In this case (3.11) is equivalent to the usual condition for Dahlquist stability.

If $m > 1$ (3.11) is satisfied if $\rho \in S^1$ but in this case $\rho \in S^1$ is not a necessary condition for zero stability (see Stetter [9] for further details).

For Example 1

$$\rho(\zeta) = \det(A_0\zeta + A_1) = \det\begin{pmatrix} \zeta - 1 & 0 \\ -\zeta & \zeta \end{pmatrix} = \zeta(\zeta - 1).$$

Hence $\rho \in S^1$ and the method (3.5) is zero stable.

From Definition 3.1 it follows that any zero stable m-stage k-step method has a discrete fundamental form given by (3.9).

The consistency error of the fundamental form (3.9) for (3.2) is related to the truncation error of the m-stage k-step method.

Definition 3.2. Let

$$T^h = A^h r^h y - hB^h f(r^h y) - h\hat{g}^h. \tag{3.12}$$

For $1 \leqslant \tau \leqslant m$, $0 \leqslant i \leqslant k - 1$, $(T^h)_{i,\tau}$ is the starting error at $t = t_{i,\tau}$.

The starting errors are of order $s \geqslant 1$ if there exist constants $C_{i\tau}$, $1 \leqslant \tau \leqslant m$, $0 \leqslant i \leqslant k - 1$, independent of h, some of which (but not all) may be zero such that

$$(T^h)_{i,\tau} = C_{i\tau}h^s + O(h^{s+1}).$$

For $1 \leqslant \tau \leqslant m$, $k \leqslant i \leqslant N$, $(T^h)_{i,\tau}$ is the truncation error of the the stage of the m-stage method at $t = t_{i,\tau}$.

The discretization (3.2) is said to be of order $s \geq 1$ if there exist constants $C_{i\tau}$, $1 \leq \tau \leq m$, $k \leq i \leq N$, independent of h, some of which (but not all) may be zero such that

$$(T^h)_{i,\tau} = C_{i\tau} h^{s+1} y^{(s+1)}(t_{i,\tau}) + O(h^{s+2}).$$

From (3.12) and Definition 2.3, the consistency error of the discrete fundamental form (3.9) for (3.2) is given by

$$\theta^h = (A^h)^{-1} T^h. \tag{3.13}$$

Consequently, if the discretization (3.2) is zero stable,

$$\left(|\theta^h|_m \right)_i \leq M' \left(\sum_{j=0}^{k-1} \left(|T^h|_m \right)_j + \sum_{j=k}^{i} \left(|T^h|_m \right)_j \right). \tag{3.14}$$

It follows that for a zero stable scheme if the starting values are of order s and the discretization is of order s then the consistency error of the discrete fundamental form is of order (at least) s. This permits the following convergence result to be deduced from Theorem 2.1.

Theorem 3.1. *Let $f(t, y)$ be Lipschitz continuous in y. Let y be the solution of (3.1) and let y^h be defined by (3.2). If the m-stage k-step method (3.2) is zero stable and of order s, with starting values of order s, then for all $h \in J$ sufficiently small*

$$\| r^h y - y^h \|_\infty \leq C h^s + O(h^{s+1}), \tag{3.15}$$

for some positive C, independent of h.

The above result was obtained using (3.14). However, it may be possible to prove higher order convergence if the consistency error (3.13) is considered directly without employing the upper bound (3.14).

From (3.13) θ^h is the solution of the equations

$$A^h \theta^h = T^h. \tag{3.15}$$

In the past several authors including, for example, Pitcher [7], and McKee and Pitcher [6], have attempted to improve on the bound (3.14) by considering the form of the inverse matrix $(A^h)^{-1}$. This however is difficult and results have only been presented for subclasses of matrices of the form (3.8). The inversion of A^h may clearly be avoided since, because A^h is block lower triangular, with at most k nonzero $m \times m$ matrices in each row, (3.15) may be used to determine $(\theta^h)_0$, $(\theta^h)_1$, $(\theta^h)_2, \ldots$ successively as follows:

$$(\theta^h)_i = (T^h)_i, \quad i = 0, 1, \ldots, k-1,$$

$$A_k(\theta^h)_0 + A_{k-1}(\theta^h)_1 + \cdots + A_0(\theta^h)_k = (T^h)_k,$$

$$A_k(\theta^h)_1 + A_{k-1}(\theta^h)_2 + \cdots + A_0(\theta^h)_{k+1} = (T^h)_{k+1},$$

and so on.

In general $(\theta^h)_n$, for $n \geq k$, is determined from the nth set of m equations at (3.15), that is,

$$A_k(\theta^h)_{n-k} + A_{k-1}(\theta^h)_{n-k+1} + \cdots + A_0(\theta^h)_n = (T^h)_n.$$

Note that it was assumed that A_0 is nonsingular.

By using (3.15) it can be shown that for certain m-stage k-step methods the order of convergence will exceed the order of the discretization (3.2) (provided the starting values are sufficiently accurate).

Corollary 3.1. *Under the hypothesis of Theorem* 3.1, *if the discretization* (3.2) *is zero stable and if the consistency error* θ^h *determined by* (3.15) *satisfies*

$$(\theta^h)_{i,\tau} = C'_{i\tau} h^p + O(h^{p+1}), \qquad 1 \leqslant \tau \leqslant m, \quad k \leqslant i \leqslant N,$$

for some constants $C'_{i\tau}$ (*not all zero*), *bounded independently of* h, *and some* $p \geqslant s$, *then provided the starting values are of order* p,

$$\| r^h y - y^h \|_\infty \leqslant C' h^p + O(h^{p+1}), \tag{3.17}$$

for some positive constant C', *independent of* h.

To illustrate Corollary 3.1 consider Example 1.

Simpson's rule (3.5a) is of order 4, and the Adams–Moulton scheme (3.5b) is of order 3. Provided y is smooth there exists C_1, C_2, $1 \leqslant i \leqslant N$, such that

$$T_{i,1} = C_1 h^5 y^{(5)}(t_{i,1}) + O(h^6), \quad 1 \leqslant i \leqslant N,$$

and

$$T_{i,2} = C_2 h^4 y^{(4)}(t_{i,2}) + O(h^5), \quad 1 \leqslant i \leqslant N,$$

and assume

$$T_{0,\tau} = C_{0\tau} h^4 + O(h^5), \quad \tau = 1, 2.$$

Using (3.6) the nth set of 2 equations in (3.15) becomes

$$\begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \theta_{n-1,1} \\ \theta_{n-1,2} \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \theta_{n,1} \\ \theta_{n,2} \end{pmatrix} = \begin{pmatrix} T_{n,1} \\ T_{n,2} \end{pmatrix}.$$

Consequently,

$$\theta_{n,1} = T_{n,1} + \theta_{n-1,1} = \ldots = \sum_{j=1}^i T_{j,1} + T_{0,1}$$

$$= \tilde{C}_{n1} h^4 + O(h^5).$$

Also,

$$\theta_{n,2} = T_{n,2} + \theta_{n,1} = C_2 h^4 y^{(4)}(t_{n,2}) + \tilde{C}_{n1} h^4 + O(h^5)$$
$$= \tilde{C}_{n2} h^4 + O(h^5).$$

It follows from Corollary 3.1 that convergence of (3.5) is order 4. Note Theorem 3.1 predicts convergence of order 3.

4. Generalized reducible quadrature

Consider the integral

$$z(t) = \int_0^t G(t, s, y(s)) \, ds, \quad 0 \leqslant t \leqslant T. \tag{4.1}$$

Let

$$\chi(t, x) = \int_0^t G(x, s, y(s)) \, ds, \qquad 0 \leq x, \, t \leq T. \tag{4.2}$$

Differentiating, for each $x \in [0, T]$,

$$\frac{d\chi}{dt}(t, x) = G(x, t, y(t)), \qquad \chi(0, x) = 0, \quad 0 \leq t \leq T. \tag{4.3}$$

An application of an m-stage k-step ODE method to (4.3) yields

$$h^{-1}A^h\chi^h(x) = B^h G(x, y^h) + \hat{g}^h(x) \tag{4.4}$$

where, with x fixed,

$$\chi^h(x) = (\chi_0(x), \chi_1(x), \dots, \chi_N(x))^T \in X^h,$$

with $\chi_i(x) = (\chi_{i,\tau}(x)) \in \mathbb{R}^m$, $0 \leq i \leq N$, where $\chi_{i,\tau}(x)$ denotes an approximation to $\chi(t_{i,\tau}, x)$,

$$G(x, y^h) = (G(x, y_0), G(x, y_1), \dots, G(x, y_N))^T \in X^h$$

with $G(x, y_i) = (G(x, t_{i,\tau}, y_{i,\tau})) \in \mathbb{R}^m$, $0 \leq i \leq N$, and $\hat{g}^h = h^{-1}(\hat{\chi}_0(x), \hat{\chi}_1(x), \dots, \hat{\chi}_{k-1}(x),$ $0, \dots, 0)^T \in X^h$ with $\hat{\chi}_i(x) = (\hat{\chi}_{i,\tau}(x)) \in \mathbb{R}^m$, $0 \leq i \leq k - 1$.

It is assumed that starting values $\hat{\chi}_{i,\tau}(t_{i,\tau})$ (approximating $z(t_{i,\tau})$), $1 \leq \tau \leq m$, $0 \leq i \leq k - 1$, are found using some starting quadrature rule, which will be denoted by S_{km}.

Premultiplying (4.4) by $h(A^h)^{-1}$, putting $x = t_{i,\tau}$ and noting that $(\chi^h(t_{i,\tau}))_{i,\tau}$ denotes an approximation to $\chi(t_{i,\tau}, t_{i,\tau}) = z(t_{i,\tau})$, suggests the following discretization of (4.1):

$$z_{i,\tau} = h \sum_{j=0}^{i} \sum_{\nu=1}^{m} (W_{ij})_{\tau\nu} G(t_{i,\tau}, t_{j,\nu}, y_{j,\nu}), \qquad 1 \leq \tau \leq m, \quad k \leq i \leq N, \tag{4.5}$$

where $(W_{ij})_{\tau\nu}$ denotes the (τ, ν)th entry in the (i, j)th $m \times m$ matrix element of W^h where

$$W^h = (A^h)^{-1}\tilde{B}^h, \tag{4.6}$$

with $\tilde{B}^h$ given by (3.8) with 0_{km} replaced by S_{km}.

The quadrature rules (4.5) constructed by means of (4.6) will be called generalized reducible quadrature and will be denoted by $[S_{km}; (\rho, \sigma)]$. If the discretization (3.2) is a linear multistep method then (4.5) is the reducible quadrature rule used by Wolkenfelt [10].

The matrix of weights W^h is determined not by inverting A^h but by solving

$$A^h W^h = \tilde{B}^h$$

for W^h. Partition W^h into the form

$$W^h = \begin{bmatrix} S_{km} & 0 \\ \omega_k & \Gamma \end{bmatrix}. \tag{4.7}$$

Only the weights in the matrix ω_k depend on the entries of S_{km}, and it is straightforward to observe that Γ is an m-block isoclinal matrix, that is,

$$\Gamma = \begin{pmatrix} \Gamma_0 & & & \\ \Gamma_1 & \Gamma_0 & & 0 \\ \Gamma_2 & \Gamma_1 & \Gamma_0 & \\ \Gamma_3 & \Gamma_2 & \Gamma_1 & \Gamma_0 \\ \vdots & & & & \ddots \end{pmatrix} \tag{4.8}$$

where the sequence $\{\Gamma_n\}_{n=0}^{\infty}$ satisfies

$$
\begin{aligned}
A_0\Gamma_0 &= B_0, \\
A_0\Gamma_1 + A_1\Gamma_0 &= B_1, \\
A_0\Gamma_k + A_1\Gamma_{k-1} + \cdots + A_k\Gamma_0 &= B_k, \\
A_0\Gamma_n + A_1\Gamma_{n-1} + \cdots + A_k\Gamma_{n-k} &= 0, \quad n \geqslant k+1.
\end{aligned}
\tag{4.9}
$$

Hence to find W^h it is sufficient to determine ω_k and then to generate the sequence $\{\Gamma_n\}$ using (4.9) to find Γ.

As an illustration consider employing Example 1 to yield a reducible quadrature rule. Using (3.6) equations (4.9) give

$$
\Gamma_0 = \begin{pmatrix} \frac{1}{3} & 0 \\ 1 & \frac{5}{12} \end{pmatrix}, \quad
\Gamma_1 = \begin{pmatrix} \frac{2}{3} & \frac{4}{3} \\ \frac{2}{3} & \frac{5}{4} \end{pmatrix}, \quad
\Gamma_n = \begin{pmatrix} \frac{2}{3} & \frac{4}{3} \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}, \quad n \geqslant 2.
$$

Now consider the order of the generalized reducible rule (4.5).
The solution $\chi(t, x)$ of (4.3) satisfies

$$
A^h r^h \chi(x) = h B^h G(x, r^h y) + h \hat{g}^h(x) + T^h(x),
$$

where, for $1 \leqslant \tau \leqslant m$, $k \leqslant i \leqslant N$, $(T^h(x))_{i,\tau} = T_{i,\tau}(x)$ denotes the truncation error of the m-stage k-step ODE method at $t = t_{i,\tau}$. Hence

$$
z(t_{i,\tau}) = h \sum_{j=0}^{i} \sum_{\nu=1}^{m} (W_{ij})_{\tau\nu} G(t_{i,\tau}, t_{j,\nu}, y(t_{j,\nu})) + ((A^h)^{-1} T^h(t_{i,\tau}))_{i,\tau},
$$

$$
1 \leqslant \tau \leqslant m, \quad k \leqslant i \leqslant N,
\tag{4.10}
$$

where, for $1 \leqslant \tau \leqslant m$, $0 \leqslant i \leqslant k-1$, $(T^h(t_{i,\tau}))_{i,\tau}$ denotes the error in the starting rule at $t = t_{i,\tau}$, and for $1 \leqslant \mu, \tau \leqslant m$, $k \leqslant i, l \leqslant N$, $(T^h(t_{i,\tau}))_{l,\mu} = T_{l,\mu}(t_{i,\tau})$.

If the consistency error of the discrete fundamental form (3.9) for the discretization (3.2) is of order s, the generalized reducible quadrature rule $[S_{km}; (\rho, \sigma)]$ will be of order s.

For example, when the Simpson–Adams–Moulton scheme of Example 1 is employed to generate a reducible quadrature rule, it follows from Section 3 that the resulting quadrature rule is of order 4, provided the starting values are of order 4.

5. Convergence results for generalized reducible quadrature

5.1. Volterra integral equations of the second kind.

Applying a generalized reducible quadrature rule to (1.1) yields

$$
y^h = g^h + h W^h(y^h),
\tag{5.1}
$$

where

$$
g^h = (\hat{y}_0, \hat{y}_1, \ldots, \hat{y}_{k-1}, g_k, \ldots, g_N)^{\mathsf{T}} \in X^h,
$$

with $\hat{y}_{i,\tau} \in \mathbb{R}^m$, $0 \leqslant i \leqslant k - 1$, vectors of precomputed starting values,

$$g_i = \left(g(t_{i,\tau})\right) \in \mathbb{R}^m, \quad k \leqslant i \leqslant N,$$

and

$$\left(W^h(y^h)\right)_{i,\tau} = \begin{cases} 0, & 1 \leqslant \tau \leqslant m, \quad 0 \leqslant i \leqslant k - 1, \\ \sum_{j=0}^{i} \sum_{\nu=1}^{m} \left(W_{ij}\right)_{\tau\nu} G\left(t_{i,\tau}, t_{j,\nu}, y_{j,\nu}\right), & 1 \leqslant \tau \leqslant m, \quad k \leqslant i \leqslant N. \end{cases} \tag{5.2}$$

Equation (5.1) is of the form

$$y^h = g^h + H^h(y^h), \tag{5.3}$$

where H^h: $X^h \to X^h$ is a discrete Volterra operator satisfying (2.2) with

$$\left(K^h \mid y^h \mid_m\right)_i = hL \max_{i,j} \mid (W^h)_{ij} \mid_m \sum_{j=0}^{i} \left(\mid y^h \mid_m\right)_j, \quad 0 \leqslant i \leqslant N,$$

where L is the Lipschitz constant for $G(t, s, y)$ with respect to y. Recalling (4.6) it follows that (5.3) is the discrete fundamental form for the quadrature method (5.1) provided the underlying m-stage k-step ODE method is zero stable. The following convergence result is now immediate.

Theorem 5.1. *Let $G(t, s, y)$ be Lipschitz continuous in y. Let y be the solution of* (1.1) *and let y^h be defined by* (5.1). *If the generalized reducible quadrature $[S_{km}; (\rho, \sigma)]$ is of order s and the starting values are of order $\hat{s}$, then for $h \in J$ sufficiently small*

$$\| r^h y - y^h \|_\infty \leqslant Ch^{s^*} + O(h^{s^*+1}), \tag{5.4}$$

for some positive constant C independent of h and $s^* = \min(s, \hat{s})$.

5.2. Volterra integro-differential equations

Consider the Volterra integro-differential equation

$$y'(t) = F(t, y(t), z(t)), \tag{5.5a}$$

$$z(t) = \int_0^t G(t, s, y(s)) \, ds, \quad 0 \leqslant t \leqslant T, \tag{5.5b}$$

with $y(0)$ given.

Applying an m-stage $\bar{k}$-step method $(\bar{\rho}, \bar{\sigma})$ to (5.5a) and a generalized reducible quadrature rule $[S_{km}; (\rho, \sigma)]$ to (5.5b) yields

$$h^{-1}\bar{A}^h y^h = \bar{B}^h F(y^h, z^h) + \bar{g}^h, \tag{5.6a}$$

$$\left(z^h\right)_i = \begin{cases} 0, & 0 \leqslant i \leqslant k - 1, \\ h\left(W^h(y^h)\right)_i, & k \leqslant i \leqslant N, \end{cases} \tag{5.6b}$$

where

$$F(y^h, z^h) = \left(F(y_0, z_0), F(y_1, z_1), \ldots, F(y_N, z_N)\right)^{\mathrm{T}} \in X^h,$$

$$F(y_i, z_i) = \left(F(t_{i,\tau}, y_{i,\tau}, z_{i,\tau})\right) \in \mathbb{R}^m, \quad 0 \leqslant i \leqslant N,$$

$$\bar{g}^h = h^{-1}(\hat{y}_0, \hat{y}_1, \ldots, \hat{y}_{k+\bar{k}-1}, 0, \ldots, 0)^{\mathrm{T}} \in X^h,$$

where $\hat{y}_i \in \mathbb{R}^m$, $0 \leqslant i \leqslant k + \bar{k} - 1$, are given vectors of starting values. Here $\bar{A}^h$, $\bar{B}^h$ are of the form (3.8) with I_{km}, 0_{km} replaced by $I_{(k+\bar{k})m}$, $0_{(k+\bar{k})m}$, respectively.

No generality is lost by assuming that both the (ρ, σ)- and $(\bar{\rho}, \bar{\sigma})$-multistage multistep methods are m-stage methods for, if not, the methods can be rewritten with $m = \text{lcm}(m_1, m_2)$, where m_1, m_2 denote the number of stages of the (ρ, σ) and $(\bar{\rho}, \bar{\sigma})$-multistage methods respectively.

Premultiplying (5.6a) by $h(\bar{A}^h)^{-1}$ gives

$$y^h = h(\bar{A}^h)^{-1}\bar{g}^h + H^h(y^h), \tag{5.7}$$

where

$$\left(| H^h(y_1^h) - H^h(y_2^h) |_m \right)_i$$

$$\leqslant h \max_{i,j} \left| ((\bar{A}^h)^{-1}\bar{B}^h)_{ij} \right|_m \sum_{j=k+\bar{k}}^{i} \left(| F(y_1^h, z_1^h) - F(y_2^h, z_2^h) |_m \right)_j.$$

Assuming Lipschitz continuity of $F(t, y, z)$ with respect to the second and third variables

$$\left(| F(y_1^h, z_1^h) - F(y_2^h, z_2^h) |_m \right)_j \leqslant L_1 \left(| y_1^h - y_2^h |_m \right)_j + L_2 \left(| z_1^h - z_2^h |_m \right)_j.$$

From (5.6b), for $j \geqslant k$,

$$\left(| z_1^h - z_2^h |_m \right)_j \leqslant hL \max_{i,j} | (W^h)_{ij} |_m \sum_{l=k}^{j} \left(| y_1^h - y_2^h |_m \right)_l.$$

Provided the $(\bar{\rho}, \bar{\sigma})$- and (ρ, σ)-multistage multistep methods are zero stable, interchanging the order of double summation gives

$$\left(| H^h(y_1^h) - H^h(y_2^h) |_m \right)_i \leqslant hM \sum_{j=0}^{i} (L_1 + L_2 Lh(i-j+1))\left(| y_1^h - y_2^h |_m \right)_j,$$

where M is independent of h. Therefore, since $L_1 + L_2 Lh(i-j+1) \leqslant L'$, where L' is a constant independent of h, (5.7) is the discrete fundamental form for (5.6).

Theorem 5.2. *Let $G(t, s, y)$ be Lipschitz continuous in y and let $F(t, y, z)$ be Lipschitz continuous in y and z. Let y be the solution of (1.2) and let y^h be defined by (5.6). Let the $(\bar{\rho}, \bar{\sigma})$-multistage multistep method be zero stable and let the consistency error of its associated fundamental form be of order $\bar{s}$. Let the generalized reducible quadrature rule $[S_{km}: (\rho, \sigma)]$ be of order s, and let the starting values be of order $\hat{s}$. Then for all $h \in J$ sufficiently small*

$$\| r^h y - y^h \|_\infty \leqslant Ch^{s^*} + O(h^{s^*+1}), \tag{5.7}$$

for some constant C independent of h and $s^* = \min(\hat{s}, s, \bar{s})$.

Proof. Define $\bar{\phi}^h$, ϕ^h as follows

$$r^h y = h(\bar{A}^h)^{-1}(\bar{B}^h F(r^h y, r^h z) + \bar{g}^h) + \bar{\phi}^h,$$

$$(r^h z)_i = h(W^h(r^h y) + \phi^h)_i, \quad k \leqslant i \leqslant N.$$

Setting $(r^h\tilde{z})_i = (r^h z - \phi^h)_i$, $k \leqslant i \leqslant N$, and defining $(r^h\tilde{z})_i = 0$, $(hW^h(r^h y))_i = 0$, $0 \leqslant i \leqslant k - 1$, it follows that

$$r^h y = h(\overline{A}^h)^{-1}\overline{B}^h F(r^h y, hW^h(r^h y)) + h(\overline{A}^h)^{-1}\overline{g}^h$$

$$+ h(\overline{A}^h)^{-1}\overline{B}^h \{ F(r^h y, r^h z) - F(r^h y, r^h\tilde{z}) \} + \overline{\phi}^h.$$

That is,

$$r^h y = h(\overline{A}^h)^{-1}\overline{g}^h + H^h(r^h y) + \theta^h,$$

where the consistency error θ^h is given by

$$\theta^h = \overline{\phi}^h + h(\overline{A}^h)^{-1}\overline{B}^h \{ F(r^h y, r^h z) - F(r^h y, r^h\tilde{z}) \}.$$

Using the zero stability of the $(\bar{\rho}, \bar{\sigma})$-multistage multistep method

$$\left(|\theta^h|_m \right)_i \leqslant \left(|\overline{\phi}^h|_m \right)_i + hML \sum_{j=k}^{i} \left(|r^h z - r^h\tilde{z}|_m \right)_j,$$

and since $(r^h\tilde{z} - r^h z)_i = (\phi^h)_i$, $k \leqslant i \leqslant N$, it may be deduced that

$$\| \theta^h \|_\infty \leqslant \| \overline{\phi}^h \|_\infty + \overline{C} \max_{k \leqslant i \leqslant N} \left(|\phi^h|_m \right)_i,$$

where $\overline{C}$ is independent of h. Thus

$$\| \theta^h \|_\infty \leqslant Ch^{s^*} + O(h^{s^*+1}), \quad s^* = \min(\hat{s}, s, \bar{s}),$$

and the bound (5.7) follows using Theorem 2.1.

6. Numerical results

The expected rates of convergence of the generalized reducible quadrature method for Volterra integral equations of the second kind and Volterra integro-differential equations were tested by employing Example 1 to generate the quadrature weights. In the case of a Volterra integro-differential equation the $(\bar{\rho}, \bar{\sigma})$ multistage multistep method was taken to be the Simpson–Adams–Moulton cyclic scheme of Example 1.

The generalized reducible quadrature rule generated using Example 1 is of order 4 (assuming order 4 starting values). Therefore, Theorems 5.1 and 5.2 predict that, provided the starting values are chosen to be of order at least 4, convergence will be of order 4.

The following two test equations were used:

Problem 6.1 (renewal equation from Feller [3]).

$$y(t) = \tfrac{1}{2}t^2 \exp(-t) + \tfrac{1}{2}\int_0^t (t - s)^2 \exp(s - t)y(s) \, ds, \quad 0 \leqslant t \leqslant 6,$$

with exact solution $y(t) = \tfrac{1}{3} - \tfrac{1}{3}\exp(-\tfrac{3}{2}t)\{\cos(\tfrac{1}{2}\sqrt{3}\,t) + \sqrt{3}\sin(\tfrac{1}{2}\sqrt{3}\,t)\}$.

Problem 6.2.

$$y'(t) = 1 - t \exp(-t^2) + y(t) - 2\int_0^t ts \exp(-y^2(s)) \, ds, \quad 0 \leqslant t \leqslant 2,$$

$$y(0) = 0,$$

with exact solution $y(t) = t$. (See also Wolkenfelt [10].)

All necessary starting values were computed from the exact solutions.

Tables 1 and 2 list the number of correct digits (defined by $-\log_{10}$ (absolute error)) at the end point of the range of the integration for a sequence of stepsizes. Note that for a method of order p and h sufficiently small it would be expected theoretically that halving the stepsize will yield an increase of $0.3 * p$ in the number of correct digits ($0.3 \simeq \log_{10} 2$). The results for Problems 6.1 and 6.2 employing the reducible quadrature generated by Example 1 will be denoted by M1 and [M1; M1], respectively.

For comparison, for Problem 6.1, the results are also listed in Table 1 for the 4th-order backward differentiation reducible quadrature used by Wolkenfelt [10]; these results will be denoted by BD4. In addition are listed the results for the pth order Adams–Moulton reducible quadrature with $p = 3$, 4; these results will be denoted by AM3, AM4. Note that AM3 corresponds to employing (3.5b) on its own to generate a quadrature rule, without using (3.5a).

For Problem 6.2. the results are given in Table 2 for the methods

[BD4; BD4]: $(\bar{\rho}, \bar{\sigma})$ is the 4th order backward differentiation formula, $W^h = [S_{km}; (\rho, \sigma)]$ where (ρ, σ) is also the 4th order backward differentiation formula.

[BD4; AM4]: $(\bar{\rho}, \bar{\sigma})$ is the 4th order backward differentiation formula, $W^h = [S_{km}; (\rho, \sigma)]$ where (ρ, σ) is the 4th order Adams–Moulton formula.

The methods M1 and [M1; M1] are clearly convergent of order 4 for Problems 6.1 and 6.2 respectively. By comparing M1 and AM3 it is clearly advantageous to use (3.5a) and (3.5b) together to generate a reducible quadrature rule rather than just employing the Adams–Moulton formula (3.5b). Note that the M1 method is more accurate than the BD4 and AM4 methods, and the [M1; M1] method is more accurate than the [BD4; BD4] and [BD4; AM4] methods.

Table 1

h^{-1}	M1	BD4	AM3	AM4
4	3.7	2.9	3.3	3.6
8	4.9	3.9	4.1	4.7
16	6.0	5.1	5.0	5.9
32	7.2	6.2	5.9	7.1
64	8.4	7.4	6.8	8.3

Table 2

h^{-1}	[M1; M1]	[BD4; BD4]	[BD4; AM4]
10	4.1	2.8	3.6
20	5.3	3.9	4.8
40	6.6	5.1	6.0

7. Concluding remarks

In this paper a general class of methods for ordinary differential equations has been employed to derive generalized reducible quadrature methods for Volterra integral equations of the second kind and Volterra integro-differential equations. It was previously observed by Wolkenfelt et al. [13] that, formally, any integration method for ordinary differential equations could be applied to Volterra integral equations of the second kind. However, it was thought that convergence of these methods would require special consideration and they state "in fact, it appears that convergence is not trivially implied by the convergence of methods for ordinary differential equations. However, in cases where the integration method can be identified with a direct solution method for Volterra integral equations, we may apply the convergence conditions holding for these direct methods. When such an identification is not possible then convergence must be established by other means".

In this paper it has been shown by using a general convergence theorem that the generalized $[S_{km}; (\rho, \sigma)]$ reducible quadrature applied to (1.1) will converge under the same conditions as those used in Section 3 to prove convergence of the underlying (ρ, σ)-multistage multistep method for ordinary differential equations. Furthermore, the general convergence theorem allows convergence of the generalized reducible quadrature method for Volterra integro-differential equations to be proved in a straightforward manner.

The class of multistage multistep methods may also be employed to generalize the class of multilag and modified multilag methods proposed by Wolkenfelt [10,12], and convergence may again be proved using the general convergence theorem. Details are given in Scott [8].

Acknowledgements

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