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Robust Algebraic Multilevel Domain Decomposition Preconditioner for General Sparse Matrices

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Collaborators

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- Tyrone Rees (STFC, RAL)
- Jennifer Scott (STFC, RAL)
- Pierre-Henri Tournier (CNRS, LJLL, Sorbonne Uni.)

Outline

Introduction to (Algebraic) Additive Schwarz

Adaptive Coarse Spaces
- Some Existing CS
- New CS
- Normal Eqs.
- More CS

Summary

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Motivation

Solving sparse linear systems is omnipresent in scientific computing.

$$Ax = b.$$

$A \in \mathbb{K}^{n \times n}$ very large, *sparse*, and ill conditioned.

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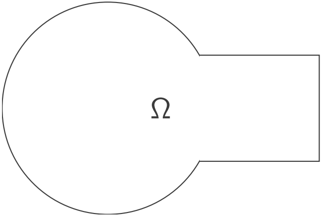
- **Scalability**: more computing resources yields faster solver
- **Black box:** prefer non-intrusive solvers. Only provide A and b
- **Easy setup:** Few knowledge on linear solvers. Not worry how to set it up perfectly. As minimal parameters as possible. (Type: Hermitian, saddle-point, etc; Accuracy; Max iter)

Overlapping DD

Solve the Poisson equation in Ω



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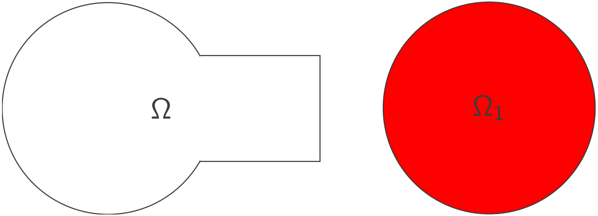

$$R_1 A R_1^T x = R_1 b$$

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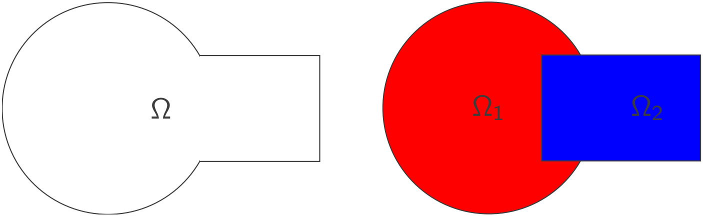

$$R_2 A R_2^T x = R_2 b$$

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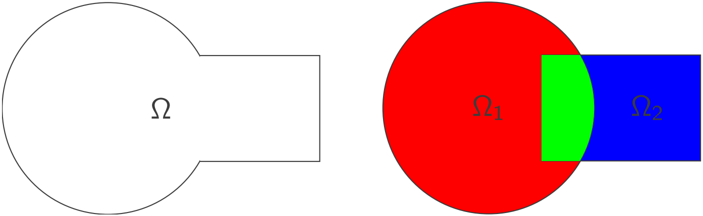

$$R_1^T(R_1AR_1^T)^{-1}R_1+R_2^T(R_2AR_2^T)^{-1}R_2$$

Iterate updating the solution values.

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Overlapping DD

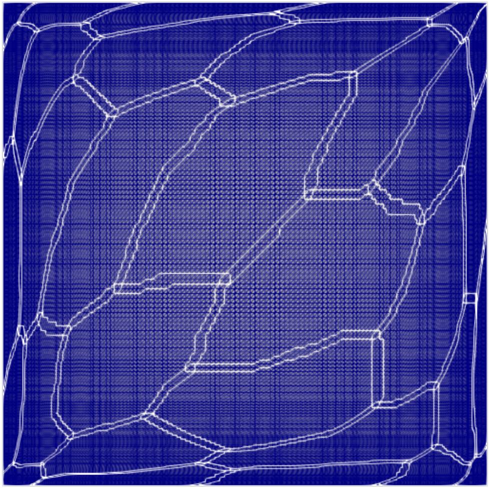

Figure: Semistructured mesh decomposed into 32 overlapping subdomains

Ingredients: Overlapping Subdomain

The sparsity graph of A has n nodes.
N non-overlapping sudbomains $\{\Omega_{Ii}\}_{1\leq i\leq N}$: N disjoint subsets of $\Omega = [\![1, n]\!]$. $n_{Ii} = \#\Omega_{Ii}$

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Ingredients: Restriction and Partition of Unity

Restriction to subset nodes:

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$$\sum_{i=1}^{N} R_i^T D_i R_i = I_n$$

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$$D_1 = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 0 \end{pmatrix} \qquad D_2 = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 0 \end{pmatrix}$$

Ingredients: Local Problems

$$A_{ii} = R_i A R_i^T .$$

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Define $P_i = I([R_{Ii}, R_{\Gamma i}, R_{ci}], :)$,

$$P_i A P_i^\top = \begin{pmatrix} A_{Ii,Ii} & A_{Ii,\Gamma i} & \\ A_{\Gamma i,Ii} & A_{\Gamma i,\Gamma i} & A_{\Gamma i,ci} \\ & A_{ci,\Gamma i} & A_{ci,ci} \end{pmatrix}$$

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δ-Overlap

Through the sparsity graph, define $\Omega_{\Gamma_{1:\delta}i}$ Define $P_i = I([R_{Ii}, R_{\Gamma_{1:\delta-1}i}, R_{\Gamma_{\delta}i}, R_{ci}], :)$,

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$R_i = [R_{Ii}, R_{\Gamma_{1:\delta-1} i}, R_{\Gamma_{\delta} i}]$.

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One-Level Schwarz

Four stages:

1. Restrict
2. Solve locally
3. Augment
4. Update

$$M_1^{-1} = \sum_{i=1}^{N} R_i^T A_{ii}^{-1} R_i.$$

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One-Level Schwarz Not Scalable

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N	2	4	8	16	32	64
It	42	53	66	74	84	97

Table: 2D Poisson on 300×300 mesh. Metis partitioning.

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Need a second level (coarse space correction) to maintain robustness

$$M_2^{-1} = R_0^H A_{00}^{-1} R_0 + \sum_{i=1}^{N} R_i^T A_{ii}^{-1} R_i,$$

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where $A_{00} = R_0 A R_0^H$

Adaptive Coarse Spaces (for Overlapping Schwarz) I

PDE based (Two-level)

- A coarse space construction based on local Dirichlet-to-Neumann maps [Nataf et al., 2011]
- Abstract robust coarse spaces for systems of PDEs via generalized eigenproblems in the overlaps [Spillane et al., 2014]
- SHEM: an optimal coarse space for RAS and Its multiscale approximation [Gander and Loneland, 2017]
- Adaptive GDSW coarse spaces of reduced dimension for overlapping Schwarz methods [Heinlein et al., 2020]
- A multilevel Schwarz preconditioner based on a hierarchy of robust coarse spaces [Al Daas et al., 2021]
- A comparison of coarse spaces for Helmholtz problems in the high frequency regime [Bootland et al., 2021]
- Multilevel spectral domain decomposition [Bastian et al., 2022]
- A fully algebraic and robust two-level Schwarz method based on optimal local approximation spaces [Heinlein and Smetana, 2022]

Adaptive Coarse Spaces (for Overlapping Schwarz) II

Fully Algebraic

- A class of efficient locally constructed preconditioners based on coarse spaces [Al Daas and Grigori, 2019]
- Fully algebraic domain decomposition preconditioners with adaptive spectral bounds [Gouarin and Spillane, 2021]
- A Robust Algebraic Domain Decomposition Preconditioner for Sparse Normal Equations [Al Daas et al., 2022b]
- A robust algebraic multilevel domain decomposition preconditioner for sparse symmetric positive definite matrices [Al Daas and Jolivet, 2022]
- Efficient algebraic two-level Schwarz preconditioner for sparse matrices [Al Daas et al., 2022a]

Interface-to-Interior Operator

$$P_i = I([R_{Ii}, R_{\Gamma_{1:\delta-1}i}, R_{\Gamma_\delta i}, R_{ci}], :),$$

$$P_i A P_i^\top = \begin{pmatrix} A_{Ii,Ii} & A_{Ii,\Gamma_{1:\delta-1}i} & & \\ A_{\Gamma_{1:\delta-1}i,Ii} & A_{\Gamma_{1:\delta-1}i,\Gamma_{1:\delta-1}i} & A_{\Gamma_{1:\delta-1}i,\Gamma_\delta i} & \\ & A_{\Gamma_\delta i,\Gamma_{1:\delta-1}i} & A_{\Gamma_\delta i,\Gamma_\delta i} & A_{\Gamma_\delta i,ci} \\ & & A_{ci,\Gamma_\delta i} & A_{ci,ci} \end{pmatrix}$$

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Consider $T_i : x_{\delta i} \in \Omega_{\Gamma_\delta i} \mapsto x_{Ii} \in \Omega_{Ii}$, the restriction of the corresponding local solution.

$$\begin{pmatrix} A_{Ii,Ii} & A_{Ii,\Gamma_{1:\delta-1}i} & \\ A_{\Gamma_{1:\delta-1}i,Ii} & A_{\Gamma_{1:\delta-1}i,\Gamma_{1:\delta-1}i} & A_{\Gamma_{1:\delta-1}i,\Gamma_\delta i} \\ & & I \end{pmatrix} \begin{pmatrix} x_{Ii} \\ x_{\Gamma_{1:\delta-1}i} \\ x_{\delta i} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ x_{\delta i} \end{pmatrix}$$

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$$T_i(x_{\delta i}) = x_{Ii} = \left(\begin{pmatrix} A_{Ii,Ii} & A_{Ii,\Gamma_{1:\delta-1}i} \\ A_{\Gamma_{1:\delta-1}i,Ii} & A_{\Gamma_{1:\delta-1}i,\Gamma_{1:\delta-1}i} \end{pmatrix} \begin{pmatrix} 0 \\ -A_{\Gamma_{1:\delta-1}i,\Gamma_\delta i} x_{\delta i} \end{pmatrix} \right)_{Ii}$$

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Theorem

To be submitted [HAD, Jolivet, Nataf, Tournier]

Theorem

Set $R_0^H = [R_{I1}Z_1, \ldots, R_{IN}Z_N]$, *where* $Z_i = Im(T_i)$

$$M_2^{-1} = R_0^H A_{00}^{-1} R_0 + \sum_{i=1}^{N} R_i^T A_{ii}^{-1} R_i.$$

If A is HPD:

$$\kappa \left(M_2^{-1} A \right) = C$$

C depends only on the largest number of neighbouring overlapping subdomains

SVD Interface-to-Interior Operator

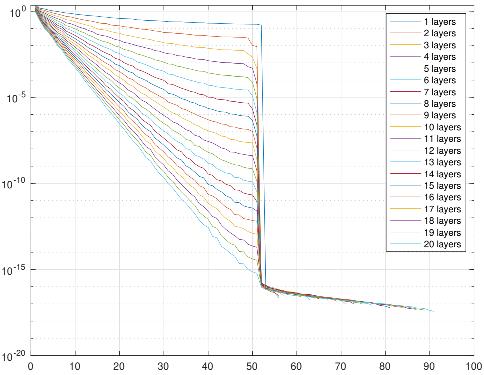

Figure: $N = 32$. Subdomain 1. Singular values of the interface-to-interior operator for a Poisson equation.

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SVD Interface-to-Interior Operator

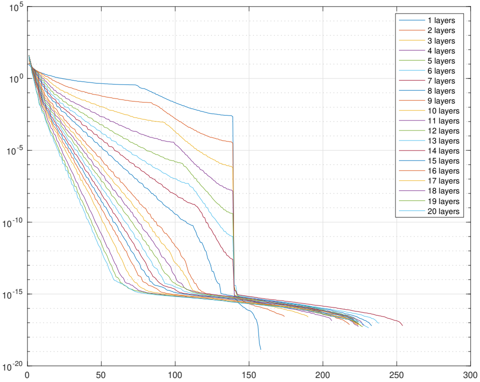

Figure: $N = 32$. Subdomain 1. Singular values of the interface-to-interior operator for the matrix $A^T A$, where A is the *Rucci1* matrix (SSMC).

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Theorem

To be submitted [HAD, Jolivet, Nataf, Tournier]

Theorem

Set $R_0^H = [R_{I1}Z_1, \ldots, R_{IN}Z_N]$, *where* $Z_i = tSVD(Im(T_i), \varepsilon)$

$$M_2^{-1} = R_0^H A_{00}^{-1} R_0 + \sum_{i=1}^{N} R_i^T A_{ii}^{-1} R_i.$$

If A *is HPD:*

$$\kappa\left(M_2^{-1}A\right) = C\left(1 + \kappa\left(M_1^{-1}A\right)\varepsilon\right)$$

C depends only on the largest number of neighbouring overlapping subdomains

Numerical experiments

N	4	8	16	32
Diffusion	14 (160)	12 (320)	11 (640)	8 (1280)
Adv-Diff	14 (160)	13 (320)	14 (640)	12 (1280)
Stokes	47 (320)	42 (640)	43 (1280)	49 (2559)
Biharmonic	51 (240)	55 (480)	34 (960)	22 (1920)
Elasticity	50 (320)	37 (640)	36 (1267)	28 (2529)

Table: Strong scaling on variety of problems

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Numerical experiments

N	8	32
Diffusion	10 (160)	13 (640)
Adv-Diff	12 (160)	13 (640)
Stokes	32 (614)	49 (2555)
Biharmonic	11 (639)	15 (2560)
Elasticity	18 (554)	28 (2529)

Table: Weak scaling on variety of problems

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Nornal Equations

[HAD, Jolivet, Scott. SISC 22']

Theorem

For a sparse A with $A = B^H B$, or $A = B^H diag(g) B$, $g \geq 0$

$$\kappa\left(M_2^{-1} A\right) = C\left(1 + \tau\right)$$

where $\tau > 0$ is a user-specified.

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PDE-CO

Solve

$$\min_y \|y - \hat{y}\|^2_{\Omega_1} + \beta\|u\|^2_{\Omega_2} \quad \text{subject to } \mathcal{L}y = u \text{ in } \Omega$$

The resulting matrix

$$\begin{pmatrix} M & & K^* \\ & \beta R & L^* \\ K & L & \end{pmatrix}$$

Mass lumping yields an equivalent diagonal matrix W to the (1:2,1:2)-block. $\widetilde{S} = J^* J$, where $J^* = [KL]W^{-1/2}$.

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Poisson PDE-CO

IFISS: Grid $2^8 \times 2^8$, $\beta = 0.01$, Q_2-FE, matrix length $\approx 200K$.

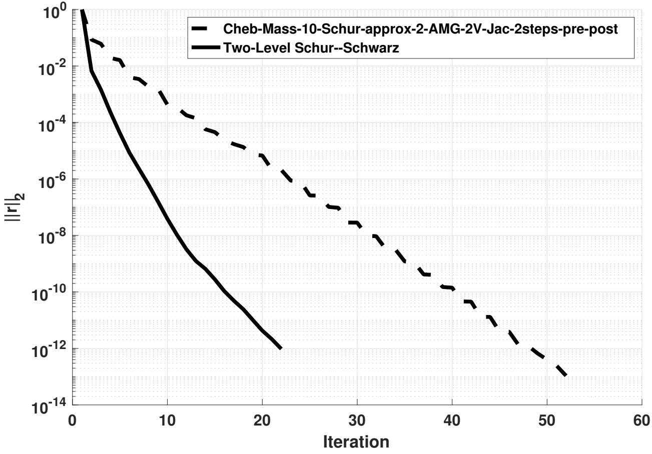

Figure: Residual history

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Helmholtz PDE-CO

Test case inspired by Kouri et al. 21'
Grid 160×160, $\beta = 10^{-5}$, P_1-FE, matrix length $\approx 50K$.

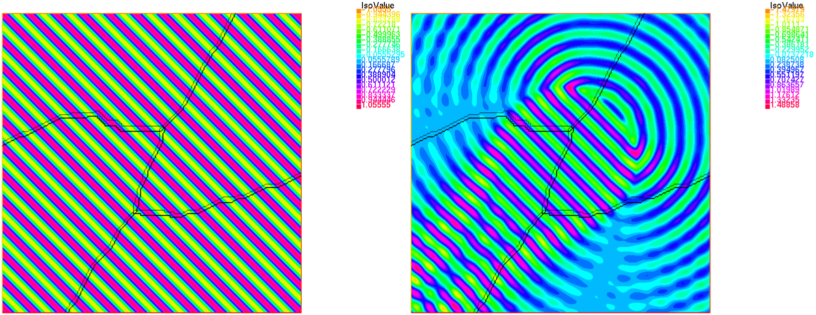

Figure: State (real part): Desired (left), solution (right)

Helmholtz PDE-CO

Test case inspired by Kouri et al. 21'
Grid 160×160, $\beta = 10^{-5}$, P_1-FE, matrix length $\approx 50K$.

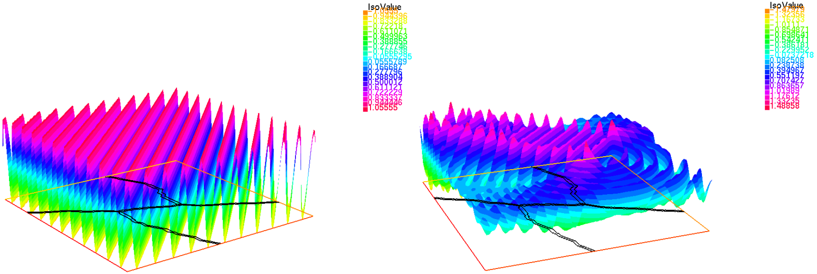

Figure: 3D view of the state (real part)

Helmholtz PDE-CO

Test case inspired by Kouri et al. 21'
Grid 160×160, $\beta = 10^{-5}$, P_1-FE, matrix length $\approx 50K$.

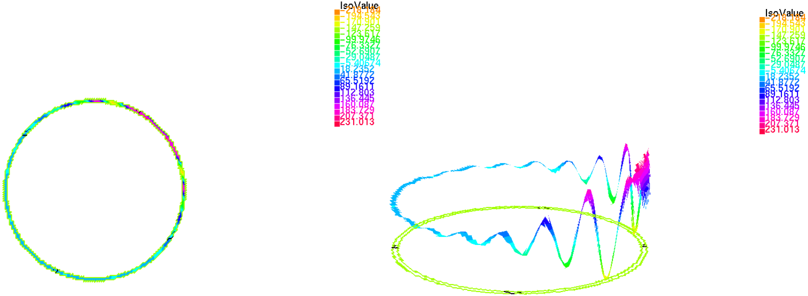

Figure: Control (real part)

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Helmholtz PDE-CO

Test case inspired by Kouri et al. 21'
Grid 160×160, $\beta = 10^{-5}$, P_1-FE, matrix length $\approx 50K$.

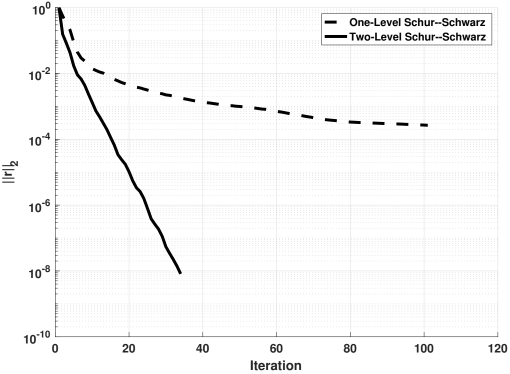

Figure: Residual history

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Diagonally Dominant HPD

[HAD, Jolivet, Rees. SISC 23']

Theorem

For a sparse A HPD

$$\kappa\left(M_2^{-1}A\right) = C\left(1+\tau\right)$$

where $\tau > 0$ is a user-specified.

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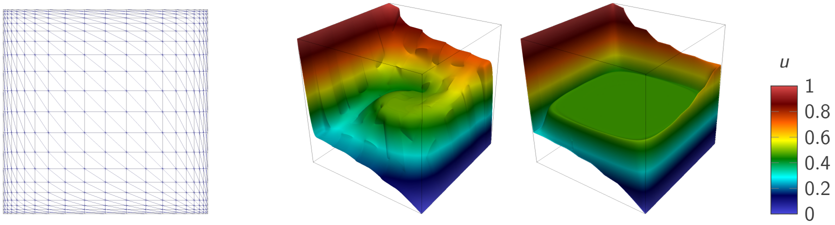
Highly Non Symmetric

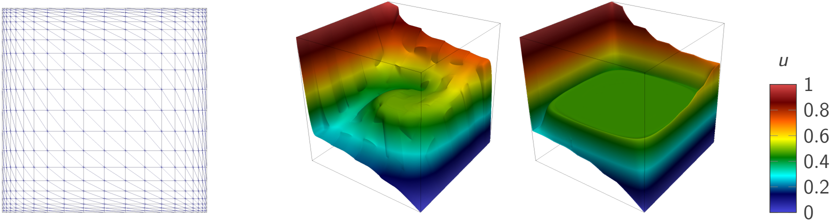
$$\nabla \cdot (Vu) - \nu \nabla \cdot (\kappa \nabla u) = 0 \text{ in } \Omega \quad u = 0 \text{ on } \Gamma_0 \quad u = 1 \text{ on } \Gamma_1$$



(a) Mesh (b) $\nu = 10^{-2}$ (c) $\nu = 10^{-4}$

Highly Non Symmetric

$$\nabla \cdot (Vu) - \nu \nabla \cdot (\kappa \nabla u) = 0 \text{ in } \Omega \quad u = 0 \text{ on } \Gamma_0 \quad u = 1 \text{ on } \Gamma_1$$



(a) Mesh (b) $\nu = 10^{-2}$ (c) $\nu = 10^{-4}$

Prec	Dimension	k	N	n	ν = 1	10^{-1}	10^{-2}	10^{-3}	10^{-4}
M_2^{-1}	2	1	1,024	$6.3 \cdot 10^6$	23 (52,875)	20 (52,872)	19 (52,759)	20 (47,497)	21 (28,235)
	3	2	4,096	$8.1 \cdot 10^6$	18 ($1.8 \cdot 10^5$)	14 ($1.8 \cdot 10^5$)	11 ($1.6 \cdot 10^5$)	16 (97,657)	29 (76,853)
GAMG	2	1	1,024	$6.3 \cdot 10^6$	42	48	88	†	†
	3	2	4,096	$8.1 \cdot 10^6$	40	38	65	†	†
…erAMG	2	1	1,024	$6.3 \cdot 10^6$	50	49	19	7	†
	3	2	4,096	$8.1 \cdot 10^6$	12	9	7	†	†

Summary & Perspectives

Summary:

- Algebraic DD provides a simple way to construct preconditioners that are effective, efficient, black-box and easy to set up
- Provable: Diagonally weighted normal equations matrix (Schur complement); HPD; Diagonally dominant HPD
- All preconditioner are accessible in PETSc PCHPDDM

Perspectives:

- Extension to general Schur complement $B^H G^{-1} B$
- Extend theory to non-Hermitian matrices

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Thank you for your attention!

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Bibliographies I

Al Daas, H. and Grigori, L. (2019).
A class of efficient locally constructed preconditioners based on coarse spaces.
SIAM Journal on Matrix Analysis and Applications, 40(1):66–91.

Al Daas, H., Grigori, L., Jolivet, P., and Tournier, P.-H. (2021).
A multilevel Schwarz preconditioner based on a hierarchy of robust coarse spaces.
SIAM Journal on Scientific Computing, 43(3):A1907–A1928.

Al Daas, H. and Jolivet, P. (2022).
A robust algebraic multilevel domain decomposition preconditioner for sparse symmetric positive definite matrices.
SIAM Journal on Scientific Computing, 44(4):A2582–A2598.

Al Daas, H., Jolivet, P., and Rees, T. (2022a).
Efficient algebraic two-level schwarz preconditioner for sparse matrices.

Al Daas, H., Jolivet, P., and Scott, J. A. (2022b).
A robust algebraic domain decomposition preconditioner for sparse normal equations.
SIAM Journal on Scientific Computing, 44(3):A1047–A1068.

Bastian, P., Scheichl, R., Seelinger, L., and Strehlow, A. (2022).
Multilevel spectral domain decomposition.
SIAM Journal on Scientific Computing, pages S1–S26.

Bootland, N., Dolean, V., Jolivet, P., and Tournier, P.-H. (2021).
A comparison of coarse spaces for Helmholtz problems in the high frequency regime.
Computers & Mathematics with Applications, 98:239–253.

Bibliographies II

Gander, M. J. and Loneland, A. (2017).
SHEM: an optimal coarse space for RAS and its multiscale approximation.
In Lee, C.-O., Cai, X.-C., Keyes, D. E., Kim, H. H., Klawonn, A., Park, E.-J., and Widlund, O. B., editors, *Domain Decomposition Methods in Science and Engineering XXIII*, pages 313–321, Cham. Springer International Publishing.

Gouarin, L. and Spillane, N. (2021).
Fully algebraic domain decomposition preconditioners with adaptive spectral bounds.
Preprint.

Heinlein, A., Klawonn, A., Knepper, J., Rheinbach, O., and Widlund, O. B. (2020).
Adaptive GDSW coarse spaces of reduced dimension for overlapping Schwarz methods.
Technical report, Universität zu Köln.

Heinlein, A. and Smetana, K. (2022).
A fully algebraic and robust two-level schwarz method based on optimal local approximation spaces.

Nataf, F., Xiang, H., Dolean, V., and Spillane, N. (2011).
A coarse space construction based on local Dirichlet-to-Neumann maps.
SIAM Journal on Scientific Computing, 33(4):1623–1642.

Spillane, N., Dolean, V., Hauret, P., Nataf, F., Pechstein, C., and Scheichl, R. (2014).
Abstract robust coarse spaces for systems of PDEs via generalized eigenproblems in the overlaps.
Numerische Mathematik, 126(4):741–770.

$$\nabla \cdot (Vu) - \nu \nabla \cdot (\kappa \nabla u) = 0 \text{ in } \Omega \quad u = 0 \text{ on } \Gamma_0 \quad u = 1 \text{ on } \Gamma_1$$

$$V(x,y) = \begin{pmatrix} x(1-x)(2y-1) \\ -y(1-y)(2x-1) \end{pmatrix} \quad \text{or} \quad V(x,y,z) = \begin{pmatrix} 2x(1-x)(2y-1)z \\ -y(1-y)(2x-1) \\ -z(1-z)(2x-1)(2y-1) \end{pmatrix},$$

Two to multi-level [H.A., P.J., L.G.,P.-H.T. SISC '21]

$$A_{00} = R_0 A R_0^H$$

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Two to multi-level [H.A., P.J., L.G.,P.-H.T. SISC '21]

$$A_{00} = R_0 A R_0^H$$

$$u^H \sum_{i=1}^{N} \widetilde{A}_i u \leq k_m u^H A u$$

Two to multi-level [H.A., P.J., L.G.,P.-H.T. SISC '21]

$$A_{00} = R_0 A R_0^H$$

$$u^H \sum_{i=1}^{N} \widetilde{A}_i u \leq k_m u^H A u$$

$$(R_0 v)^H \sum_{i=1}^{N} \widetilde{A}_i (R_0 v) \leq k_m (R_0 v)^H A (R_0 v)$$

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$$v^H \sum_{i=1}^{N} (R_0^H \widetilde{A}_i R_0) v \leq k_m v^H (R_0^H A R_0) v = k_m v^H A_{00} v$$

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Two to multi-level [H.A., P.J., L.G.,P.-H.T. SISC '21]

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$$v^H \sum_{i=1}^{N} (R_0^H \widetilde{A}_i R_0) v \leq k_m v^H (R_0^H A R_0) v = k_m v^H A_{00} v$$

$$v^H \sum_{j=1}^{N_2} \underbrace{\left(\sum_{i \in G_j} (R_0^H \widetilde{A}_i R_0) \right)}_{\widetilde{A}_{00,j}} v \leq k_m v^H (R_0^H A R_0) v = k_m v^H A_{00} v$$

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GenEO FEM [Dolean et al '15]

$$a(u, v) = \sum_{K \in \mathcal{T}} \int_K uv \to A$$

$$\widetilde{a}(u, v) = \sum_{K \in \mathcal{T}_i} \int_K uv \to \widetilde{A}_i$$

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